A motivic formalism in representation theory

Shane KELLY

(joint with Jens Niklas EBERHARDT)

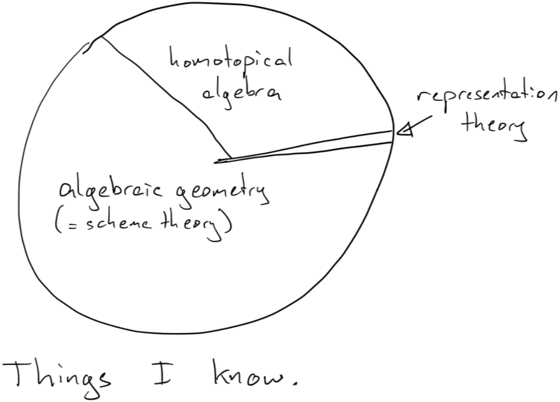
Slides: www.math.titech.ac.jp/$\sim$ shanekelly/MSJSep19.pdf

20th Sep.2019, Kanazawa

Overview

1. General representation theory background
2. The modular category $\mathcal{O}$
3. Geometry
4. Motives
5. A toy application

An apology.


homotopical algebra
representation theory
algebraic geometry
(= scheme theory)
Things I know.

If you know more representation theory than me:

1. G is a connected reductive linear algebraic group,
2. B is a Borel subgroup,
3. T is a split maximal torus.

We always work over an algebraically closed field $k = \overline{k}$, but G, B, T are defined over $\mathbb{Z}$. Sometimes $G(\mathbb{C})$ and $G(\overline{\mathbb{F}}_p)$ appear in the same equation.

If you know less (or equal) representation theory than me:

1. $G = SL_n = \{n \times n \text{ matricies } M \text{ s.t. } \det M = 1\}$,
2. $B =$ upper triangular matricies of SL_n,
3. $T =$ diagonal matricies of SL_n.

We want to study (algebraic) *representations* of G:

$$\rho : G \times V \to G$$

V a finite dimensional k-vector space,
$\rho(g, -) : V \to V$ is linear $\forall\, g \in G$,
$\rho(g_1 g_2, v) = \rho(g_1, \rho(g_2, v))$,
algebraic: ρ is defined using polynomials.

Example

$V = \{f(x, y) = \alpha x^n + \beta x^{n-1} y + \cdots + \omega y^n : \alpha, \beta, \ldots, \omega \in k\}$
$=$ homogeneous polynomials in x, y of degree n,
$G = SL_2$ acts by

$$\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right], \quad f(x, y) \mapsto f(dx - by, ay - cx)$$

Def. A *character* is a group homomorphism $\lambda : T \to k^*$.

Example

Given a *character* $\lambda : T \to k^*$, extend to $B \to k^*$, define $\mathcal{O}(\lambda) =$ sheaf of λ-invariant functions,

$$\nabla(\lambda) = \Gamma(G/B, \mathcal{O}(\lambda))$$

with the action induced by G acting on G/B.

Note: the previous slide was the case $G = SL_2$, and

$$\lambda : \left[\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right] \mapsto a^n$$

of this construction.

Definition

The $\nabla(\lambda)$ are called *costandard* representations.

Strategy: If there is a subspace $W \subseteq V$ such that $\rho(g, W) \subseteq W$ for all $g \in G$, then $\rho : G \times V \to V$ is built from $G \times W \to W$ and $G \times (V/W) \to (V/W)$.

Definition

ρ is *irreducible* if there is no nonzero proper subrepresentations.

Theorem (Jordan-Hölder)

Every representation can be built from a unique (up to reordering) tuple of irreducible representations.

Facts:

1. If char.$k = 0$, then irreducible $\iff$ costandard.
2. If char.$k = p > 0$, then irreducible $\not\iff$ costandard.
3. In general, $\forall$ costandard representation $\nabla(\lambda)$ $\exists!$ irreducible subrepresentation $L(\lambda) \subseteq \nabla(\lambda)$ ($\leftarrow$ definition of $L(\lambda)$).
4. All irreducible representations are of the form $L(\lambda)$.

Example

Recall,

$$\rho_n : SL_2 \times \left\{ \begin{array}{c} \text{homogeneous} \\ \text{degree n polynomials} \end{array} \right\} \to \left\{ \begin{array}{c} \text{homogeneous} \\ \text{degree n poly.} \end{array} \right\}$$

$$\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right], \quad f(x, y) \mapsto f(dx - by, ay - cx)$$

Because $(u + v)^p = u^p + v^p$ the vector space

$$\{\alpha x^p + \omega y^p : \alpha, \omega \in k\}$$

is fixed by SL_2.
So the costandard representation ρ_p is not irreducible (but $\rho_0, \rho_1, \ldots, \rho_{p-1}$ are irreducible).

Question

Given two characters $\lambda, \mu : T \to k^*$, how many copies of $L(\lambda)$ are used to build $\nabla(\mu)$?

Definition

$[\nabla(\nu) : L(\lambda)]$ is the number of copies of $L(\lambda)$ used to build $\nabla(\mu)$.

The modular category $\mathcal{O}$

Consider $Rep_G =$ the category of all representations.

Problems:

1. Infinitely many irreducible representations.
2. No projective objects.

Replace Rep_G with: the *modular category* $\mathcal{O}$.

Nice properties of $\mathcal{O}$:

1. Finite set of irreducibles $\{\overline{L(\lambda_x)}\}_{x \in W}$.
2. Irreducibles are canonically indexed by the Weyl group $W = N_G T / T$. (If $G = SL_n$, then $W \cong Sym_n$).
3. $\mathcal{O}$ is an abelian category with enough projectives.
4. There is a "smallest" projective cover $P(\lambda) \to \overline{L(\lambda)}$ of each irreducible $\overline{L(\lambda)}$ ($\leftarrow$ definition of $P(\lambda)$)

Super sketchy definition for non-experts:

Start with two sets

$$\varnothing \subseteq N \subseteq A \subseteq \{ \text{ characters } T \to k^* \}.$$

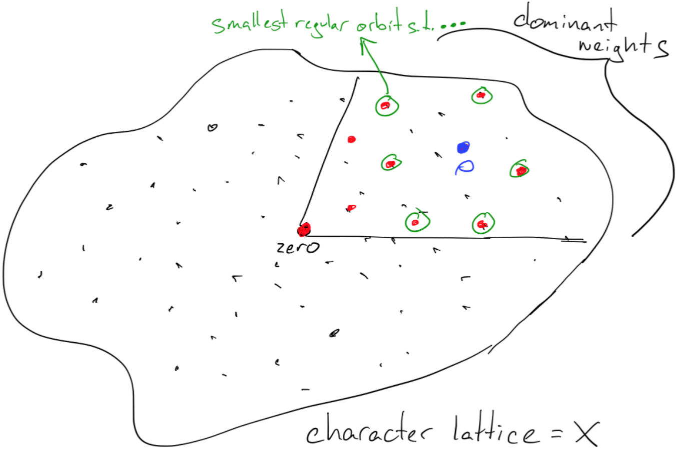
Objects of $\mathcal{O}$ = the set $\mathcal{A}$ of representations that can be built out of $L(\lambda)$ for $\lambda \in A$.

Morphisms: Designed to formally kill any representations that can be built out of $L(\lambda)$ for $\lambda \in N$.

More detailed definition for experts:

-Omitted because I'm underqualified.-

Red = A; Green = $A \setminus N$.



Geometry

Question: Is there a "geometric description" of $\mathcal{O}$?

Consider G/B. It has a canonical decomposition

$$G/B = \cup_{x \in W} BxB/B$$

with $BxB/B \cong \mathbb{A}^{n_x}$ for some $n_x \in \mathbb{Z}$.

Example

If $G = SL_n$, then

$$SL_n/B \cong Fl_n = \{0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_{n-1} \subsetneq k^n\}$$

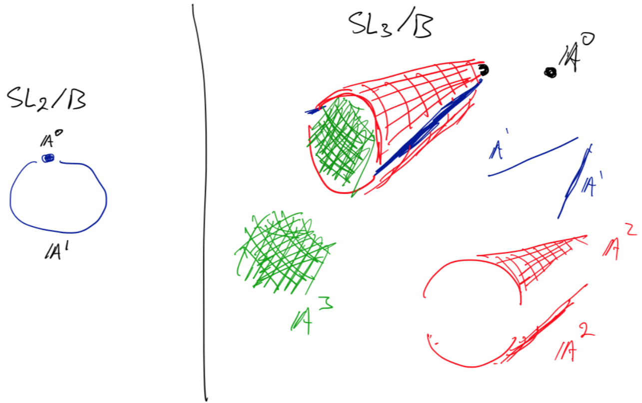
$$[a_{ij}] \mapsto 0 \subsetneq \langle a_{\bullet 1} \rangle \subsetneq \langle a_{\bullet 1}, a_{\bullet 2} \rangle \subsetneq \langle a_{\bullet 1}, a_{\bullet 2}, a_{\bullet 3} \rangle \subsetneq \cdots \subsetneq k^n$$

$SL_n/B = \cup_{x \in Sym_n} BxB/B$ is the costandard decomposition of Fl_n.

Even more specifically,

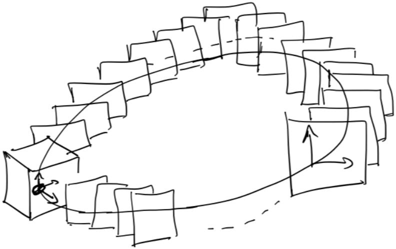
$$SL_2/B \cong \mathbb{P}^1 = \{\infty\} \cup \mathbb{A}^1$$

$$SL_3/B \cong FL_2 = \mathbb{A}^0 \cup \mathbb{A}^1 \cup \mathbb{A}^1 \cup \mathbb{A}^2 \cup \mathbb{A}^2 \cup \mathbb{A}^3$$



What do we mean by “geometric representation of $\mathcal{O}$”?

Given a stratified variety $X = \cup_{s\in S} X_s$, consider sheaves of F-vector spaces on X which are constant on each strata. (sheaf of F-vector spaces “$=$” set of vector spaces continuously parametrised by X)



On $\mathbb{P}^1 = \mathbb{A}^1 \cup \{*\}$, (stratified sheaf) $\leftrightsquigarrow$ $(\phi : V_0 \to V_1)$.
$V_0 =$ vector space at $\{*\}$,
$V_1 =$ vector space at points of $\mathbb{A}^1$,
$\phi =$ information about how V_1 deforms to V_0.

More generally, (stratified sheaf on $X = \cup_{s \in S} X_s$, s.t. $\pi_1(X_s) = \{1\}$, e.g., $X_s \cong \mathbb{A}^{n_s}$) $\leftrightsquigarrow$

- V_s; $s \in S$
- $\phi_{st} : V_s \to V_t$ whenever $X_t \subset \overline{X_s}$.
- Compatibility condition.

More generally, we can consider the bounded derived category of sheaves on $X = \cup_{s \in S} X_s$. For $s \in S$, we have intersection complexes IC_s.

Theorem (Soergel)

$$\bigoplus_i \hom_{D^b((G/B)(\mathbb{C}),\mathbb{C})}(IC_x, IC_y[i]) \cong \hom_{\mathcal{O}_0}(P(x), P(y))$$

- $(G/B)(\mathbb{C})$ = complex variety associated to G/B.
- $D^b((G/B)(\mathbb{C}), \mathbb{C})$ bounded derived category of complexes of sheaves of $\mathbb{C}$-vector spaces on $(G/B)(\mathbb{C})$.
- $\mathcal{O}_0$ = complex Lie algebra inspiration for $\mathcal{O}$.
- $P(x), P(y)$ complex Lie algebra versions of $P(\mu)$.

Remark

The above isomorphism is used in Soergel's proof of the Kazhdan-Lusztig conjecture.

Replacing intersection complexes IC_x with *parity sheaves* [Soergel, Juteau-Mautner-Williamson], we can get an $\overline{\mathbb{F}}_p$-linear version:

$$\bigoplus_i \hom_{D^b((G/B)(\mathbb{C}),\overline{\mathbb{F}}_p)}(E_x, E_y[i]) \cong \hom_{\mathcal{O}}(P(\mu_x), P(\mu_y))$$

Here, $\mathcal{O}$ is the modular category $\mathcal{O}$ from above, and μ_x, μ are the characters corresponding to $x, y \in W$.

Cannot replace $(G/B)(\mathbb{C})$ on the left with $(G/B)(\overline{\mathbb{F}_p})$ because

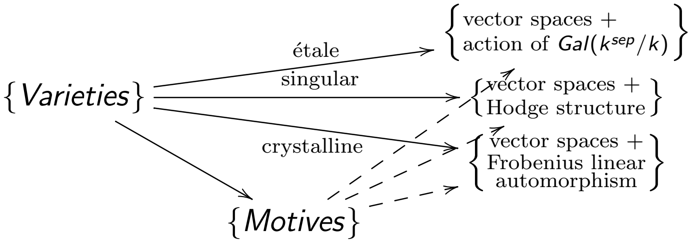
$$H^1_{et}(\mathbb{A}^n_{\overline{\mathbb{F}}_p}, \overline{\mathbb{F}}_p) \neq 0.$$

However, motivic cohomology gives the correct groups

$$H^n_M(\mathbb{A}^n_{\overline{\mathbb{F}}_p}, \overline{\mathbb{F}}_p) = \begin{cases} \overline{\mathbb{F}}_p & n = 0 \\ 0 & n \neq 0 \end{cases}$$

Motives

Idea (Grothendieck 1960's): there are many cohomology theories in algebraic geometry.



Motives should be a "universal" cohomology theory, that contains information about all the other ones.

Notice: targets are vector spaces + some structure.

Motives are *defined* by a universal property.

Like tensor product of modules.

Sometimes one can give a concrete representative of a motive, but its often better to work with them abstractly.

Like tensor product of modules.

However, for conceptual purposes:
a motive is a vector space equipped with some extra structure (what structure exactly depends on the setting).
Cf. Tannakian formalism.

Thanks to work of Ayoub, Cisinski-Deglise, Morel, Voevodsky, . . . , there is a good theory of relative motives (think: motives continuously parametrised by some base) with functors $f_!, f^!, f^*, f_*, \otimes$, hom.

Definition

Let $H(S, \overline{\mathbb{F}}_p)$ be the triangulated category of relative motives over a variety S, with $\overline{\mathbb{F}}_p$-coefficients.

Remark

In [Eberhardt-K.] we construct $H(S, \overline{\mathbb{F}}_p)$ using relatively elementary methods. No algebraic cycles (no intersection theory / moving lemmas), no model categories, no infinity categories. Our description is possible thanks to a theorem of [Geisser-Levine] about Milnor K-theory.

Definition

A motive $M \in H(\overline{\mathbb{F}}_p, \overline{\mathbb{F}}_p)$ is *mixed Tate* if it can be built from the cohomology of projective spaces (by extension, direct summand, direct sum, internal hom).

A relative motive $M \in H(X, \overline{\mathbb{F}}_p)$ is *mixed Tate* if it is constant. I.e., of the form f^*M' for some mixed Tate motive $M' \in H(\overline{\mathbb{F}}_p, \overline{\mathbb{F}}_p)$, where $f : X \to \overline{\mathbb{F}}_p$.

Let $X = \cup_{s \in S} X_s$ be a stratified space, $i_s : X_s \to X$. A motive $M \in H(X, \overline{\mathbb{F}}_p)$ is *stratified mixed Tate* if $i_s^* M \in H(X_s, \overline{\mathbb{F}}_p)$ is mixed Tate, for all $s \in S$.

$MTDer_S(X) :=$ the category of stratified mixed Tate motives.

Example

Observation: Since $H^*_M(\mathbb{A}^n, \overline{\mathbb{F}}_p) = \overline{\mathbb{F}}_p$,
$MTDer(\mathbb{A}^n) \cong D^b(Vec.Sp^{\mathbb{Z}}) \leftarrow$ graded vector spaces.

Theorem (Eberhardt-K.)

Let G be a semisimple simply connected split algebraic group over $\overline{\mathbb{F}}_p$ and $G^\vee$ the Langlands dual group. Then there is an equivalence of categories

$$MTDer_{(B^\vee)}(G^\vee/B^\vee) \cong D^b(\mathcal{O}^{2\mathbb{Z}}(G))$$

between the category of stratified mixed Tate motives on $G^\vee/B^\vee$ and the derived evenly graded modular category $\mathcal{O}^{2\mathbb{Z}}(G)$.

A toy application

For $B \subseteq P \subseteq SL_n$ a parabolic subgroup, consider

$$\Pi_P : MTDer(SL_n/B) \xrightarrow{\pi_*} MTDer(SL_n/P) \xrightarrow{\pi^*} MTDer(SL_n/B)$$

For any stratum $i_s : X_s \to X$ we have the *skyscraper* motive $i_{s!}\overline{\mathbb{F}}_p$.

Given "nice" characters μ, ν, using Π_P and $i_{s!}\overline{\mathbb{F}}_p$ for various P and s, one can construct a relative motive $\mathcal{P}_\mu$ such that for an appropriate stratum X_ν we have

$$\dim i_\nu^* \mathcal{P}_\mu = [\nabla(\nu) : L(\mu)].$$

($=$ number of copies of $L(\mu)$ used to build $\nabla(\mu)$).

Thank-you.