1 Étale cohomology

1.1 From Weil conjectures to $l$-adic cohomology

We began with the question:

**Question 1.** Given a smooth projective variety $X/\mathbb{F}_q$, how many $\mathbb{F}_q^n$-points does $X$ have for each $n$? That is, calculate

$$Z(X, t) = \exp\left( \sum_{n=1}^{\infty} |X(\mathbb{F}_q^n)| \frac{t^n}{n} \right).$$

This lead to the Weil conjectures:

**Theorem 2** (Weil conjectures). If $X$ is a smooth projective variety of dimension $d$ over $\mathbb{F}_q$,

1. (Rationality) $Z(X, t)$ is a rational function of $t$, i.e., it is in $\mathbb{Q}(t) \subseteq \mathbb{Q}((t))$.
2. (Functional equation) There is an integer $e$ such that

$$Z(X, q^{-d}t^{-1}) = \pm q^{ed/2} Z(X, t).$$

3. (Riemann Hypothesis) We can write

$$Z(X, t) = \frac{P_1(t)P_3(t) \cdots P_{2d-1}(t)}{P_0(t)P_2(t) \cdots P_{2d}(t)}$$

with $P_i(t) \in \mathbb{Z}[t]$, and such that the roots of $P_i(t)$ have absolute value $q^{-i/2}$. Moreover, $P_0(t) = 1 - t$ and $P_{2d}(t) = 1 - q^d t$.

4. (Betti numbers) If $X$ comes from a smooth projective variety over $\mathbb{Z}_p$, then

$$\deg P_i(t) = \dim_\mathbb{Q} H^i(X(\mathbb{C}), \mathbb{Q}).$$

The strategy was to develop a cohomology theory

$$H^\bullet : (\text{Varieties}/k)^{op} \rightarrow \text{graded } \mathbb{Q}\text{-vector spaces}$$

for arbitrary varieties over any field $k$, which satisfied the following properties for smooth projective varieties $X$.

1. (Finiteness) $\dim H^\bullet(X)$ is finite, and $H^i(X) = 0$ for $i \notin \{0, 1, \ldots, 2 \dim X\}$.
2. (Poincaré Duality) There is a canonical isomorphism $H^{2\dim X}(X) \cong \mathbb{Q}$ and a natural perfect pairing

$$H^i(X) \times H^{2d-i}(X) \rightarrow \mathbb{Q}$$
3. (Lefschetz Trace Formula)

\[ |X(\mathbb{F}_{q^m})| = \sum_{i=0}^{2 \dim X} (-1)^i \text{Tr}(\phi_i^m) \]

where \( X_{\mathbb{F}_q} = X \times_{\mathbb{F}_q} \mathbb{F}_q \), \( \phi : X_{\mathbb{F}_q} \to X_{\mathbb{F}_q} \) is the Frobenius morphism, and \( \phi_i : H^i(X_{\mathbb{F}_q}) \to H^i(X_{\mathbb{F}_q}) \) is the induced morphism.

4. (Compatibility) If \( k = \mathbb{C} \) then \( H^\bullet(X) \) is isomorphic to singular cohomology.

Then,

\[ (\text{Lefschetz Trace Formula}) \Rightarrow (\text{Rationality}) \]
\[ (\text{Poincaré Duality}) \Rightarrow (\text{Functional equation}) \]
\[ (\text{Compatibility}) \Rightarrow (\text{Betti numbers}) \]

Eigenvalues \( \alpha_{i,j} \) of \( \phi_i| H^i(X_{\mathbb{F}_q}) \) have \( |\alpha_{i,j}| = q^{-i/2} \) \( \Rightarrow (\text{Riemann Hypothesis}) \)

We saw that:

1. (Serre) Due to the existence of supersingular elliptic curves, there cannot be any cohomology theory with the above properties taking values in \( \mathbb{Q} \)-vector spaces.

2. For curves, étale cohomology with \( \mathbb{Z}/l^n \)-coefficients has Poincaré Duality and

\[ \text{rank}_{\mathbb{Z}/l^n} H^i_{\text{et}}(X_{\mathbb{F}_q}, \mathbb{Z}/l^n) = \dim_{\mathbb{Q}} H^i_{\text{sing}}(X(\mathbb{C}), \mathbb{Q}) \]

This leads us to define:

\[ H^i_{\text{et}}(X, \mathbb{Q}_l) := \left( \lim_{\substack{\longleftarrow \\ n \geq 1}} H^i_{\text{et}}(X, \mathbb{Z}/l^n) \right) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l. \quad (1) \]

1.2 Successes of \( l \)-adic cohomology

**Theorem 3.** The \( \mathbb{Q}_l \)-vector spaces \( H^i_{\text{et}}(X, \mathbb{Q}_l) \) satisfy (Finiteness), (Poincaré Duality), (Lefschetz Trace Formula), and (Riemann Hypothesis).

We also wanted to see (but ran out of time) that the \( \mathbb{Z}/l^n \) cohomology groups had a very strong Poincaré Duality formalism.

**Theorem 4.** For any separated finite type morphism between noetherian \( \mathbb{Z}[\frac{1}{l}] \)-schemes \( f : Y \to X \), and object \( E \in D(X_{\text{et}}, \mathbb{Z}/l^n) \) there are adjunctions

\[ (f^*, f_*) : D(Y_{\text{et}}, \mathbb{Z}/l^n) \rightleftarrows D(X_{\text{et}}, \mathbb{Z}/l^n) \]
\[ (f_!, f^!) : D(X_{\text{et}}, \mathbb{Z}/l^n) \rightleftarrows D(Y_{\text{et}}, \mathbb{Z}/l^n) \]
\[ (- \otimes E, \hom(E, -)) : D(X_{\text{et}}, \mathbb{Z}/l^n) \rightleftarrows D(X_{\text{et}}, \mathbb{Z}/l^n) \]

satisfying a number of properties such as a Proper Base Change and Smooth Base Change formulas.
In order to have these functors for sheaves of \( \mathbb{Z}_l \)-modules, some work is needed.

**Definition 5** ([BS, Def.3.5.3]). For a scheme \( X \), define \( \text{Shv}_{\text{et}}(X)^{\mathbb{N}} \) to be the category of \( \mathbb{N} \)-indexed projective systems in \( \text{Shv}_{\text{et}}(X) \). The derived category of this abelian category is denoted by \( D(X^{\mathbb{N}}_{\text{et}}) \).

We write \( D(X_{\text{et}}, (\mathbb{Z}_l)_{\bullet}) \subseteq D(X^{\mathbb{N}}_{\text{et}}) \) for the full subcategory of those objects \((\cdots \to K_2 \to K_1)\) such that \( K_m \in D(X_{\text{et}}, \mathbb{Z}/l^m) \) and \( K_m \otimes_{\mathbb{Z}/l^m} \mathbb{Z}/l^{m-1} \to K_{m-1} \) is a quasi-isomorphism. Here, \( \otimes \) is the left derived tensor product.

**Theorem 6** (Ekedahl). The functors \( f^*, f_*, f^!, f_! \), \( \otimes \), \( \text{hom} \) can be extended to the categories \( D(X_{\text{et}}, (\mathbb{Z}_l)_{\bullet}) \) in a sensible way.

We also had a very nice Galois theory.

**Theorem 7** (Stacks Project, Tags 0BNB, 0BMY, 0BN4). Let \( X \) be a connected scheme, \( x \in X \) a geometric point, \( \text{FEt}_X \) the category of finite étale \( X \)-schemes, and consider the functor

\[
F : \text{FEt}_X \to \text{Set}; \quad Y \mapsto |Y_x|.
\]

The étale fundamental group of \( X \) is the profinite group

\[
\pi_1^{\text{et}}(X, x) = \text{Aut}(F)
\]

and \( F \) induces an equivalence of categories

\[
\text{FEt}_X \cong \text{Fin-} \pi_1^{\text{et}}(X, x)\text{-Set}
\]

with the category of finite sets equipped with a continuous \( \pi_1^{\text{et}}(X, x) \)-action.

There is also a linear version of this. Recall that \( \text{Loc}_X(R) \) is the category of local systems with \( R \)-coefficients. That is, sheaves \( F \) of \( R \)-modules such that for some covering \( \{f_i : U_i \to X\} \), each \( f_i^* F \) is isomorphic to the constant sheaf \( R^n \) for some \( n \). Similar to the case of topological spaces, \( \pi_1 \) determines the category of local systems.

**Proposition 8.** If \( X \) is a connected locally noetherian \( \mathbb{Z}(l) \)-scheme, then there is an equivalence of categories

\[
\mathbb{Q}_l \otimes_{\mathbb{Z}_l} \lim_{\leftarrow n} \text{Loc}_X(\mathbb{Z}/l^n) \cong \left\{ \begin{array}{c}
\text{continuous finite dimensional} \\
\mathbb{Q}_l\text{-linear representations of } \pi_1^{\text{et}}(X)
\end{array} \right\}.
\]

### 1.3 Shortcomings of \( l \)-adic cohomology

All of this is not quite as nice as it could be though.

**Problem 9.**

1. The definition \( H^i_{\text{et}}(X, \mathbb{Q}_l) := \left( \lim_{\leftarrow n \geq 1} H^i_{\text{et}}(X, \mathbb{Z}/l^n) \right) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \) is ad hoc, and not very pleasant to work with.
2. The categories $D(X_{et}, (\mathbb{Z}_l)_*)$ are horrible to work with.

3. The equivalence between local systems and $\pi_1$-representations is no longer true in general if one uses, honest $\mathbb{Q}_l$-local systems instead of the ad hoc $\mathbb{Q}_l \otimes_{\mathbb{Z}_l} \text{Loc}_X(\mathbb{Z}/l^n)$ (cf. [Bhatt-Scholze, Pro-étale topology, Example 7.4.9] for an example due to Deligne).

Question 10. So why can’t we just use sheaves of $\mathbb{Z}_l$-coefficients?

Representability!

Finite coefficients work so well due to the equivalence of categories.

Theorem 11. There is equivalence of categories

$$F\text{Et}(X) \cong \text{Loc}_X(\text{FinSet})$$

between the category of finite étale $X$-schemes and the category of locally constant étale sheaves.

This suggests that we should enlarge the category $\text{Et}(X)$ to include filtered limits.

2 Pro-étale schemes

Definition 12. A morphism $\text{Spec}(B) \to \text{Spec}(A)$ of affine schemes is pro-étale if there exists a cofiltered system $(B_\lambda)_{\lambda \in \Lambda}$ of étale finite presentation $A$-algebras such that $B = \varprojlim B_\lambda$. The system $(B_\lambda)$ is called a presentation for $B$.

Exercise 1. Let $(B_\lambda)_{\lambda \in \Lambda}$ be a cofiltered system of rings. Let $\text{Primes}(C)$ denote the set of prime ideals of a ring $C$, and $\text{Spc}(C)$ the underlying topological space of $\text{Spec}(C)$, i.e., $\text{Spc}(C)$ is $\text{Primes}(C)$ equipped with its Zariski topology.

1. Show that $\text{Primes}(\varprojlim B) = \varprojlim \text{Primes}(B_\lambda)$.

2. Show that for any $f \in B_\lambda$ with image $\overline{f} \in \varprojlim B_\lambda$, the set $D(\overline{f}) \subseteq \text{Primes}(\varprojlim B_\lambda)$ of primes not containing $\overline{f}$ is the preimage of the set $D(f) \subseteq \text{Primes}(B_\lambda)$ of primes not containing $f$, under the canonical map $\pi : \text{Primes}(\varprojlim B_\lambda) \to \text{Primes}(B_\lambda)$. That is, show $D(\overline{f}) = \pi^{-1}(D(f))$.

3. Deduce that $\text{Spc}(\varprojlim B_\lambda) = \varprojlim \text{Spc}(B_\lambda)$.

Exercise 2. Let $k$ be an algebraically closed field. Using Exercise 1, show that for every pro-finite set $S$, there exists a pro-étale $k$-scheme $\text{Spec}(B) \to \text{Spec}(k)$ with $S \cong \text{Spc}(B)$.

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1 A system is cofiltered if (i) it is nonempty, (ii) for every pair of objects $B_\lambda, B_{\lambda'}$ there is a third object $B_{\lambda'}$ and morphisms in the system $B_\lambda \to B_{\lambda'}, B_{\lambda'} \to B_{\lambda''}$, and (iii) for any pair of parallel morphisms in the system $B_\lambda \rightrightarrows B_{\lambda'}$ there exists a morphism in the system $B_{\lambda'} \to B_{\lambda''}$ such that the two compositions are equal.
**Exercise 3.** Let $k$ be a field and $k \subseteq k_{sep}$ a separable closure. Show that the Spec($k_{sep}$) → Spec($k$) is pro-étale.

**Exercise 4.** Suppose that Spec($B$) → Spec($A$), Spec($C$) → Spec($A$) are pro-étale with $B = \lim_{\lambda \in \Lambda} B_\lambda$ and $C = \lim_{\mu \in M} C_\mu$ presentations. Show that Spec($B \times_{\text{Spec}(A)} C$) → Spec($A$) is pro-étale. Hint: consider the system $(B_\lambda \otimes_A C_\mu)_{(\lambda, \mu) \in \Lambda \times M}$.

**Exercise 5.** Recall that if $L/k$ is a (finite) Galois extension, then Spec($L \otimes_k L$) ≅ $\Pi_{\text{Gal}(L/k)}$ Spec($L$). Recall also that an separable closure $k_{sep}/k$ is the union of the finite Galois subextensions $k \subseteq L \subseteq k_{sep}$ and $\text{Gal}(k_{sep}/k) \cong \lim_{\leftarrow k \subseteq L \subseteq k_{sep}} \text{Gal}(L/k)$. Show that

$$\text{Spec}(k_{sep} \otimes_k k_{sep}) \cong \text{Gal}(k_{sep}/k)$$

as topological spaces.

**Exercise 6.** Let $A$ be a ring and $p \in \text{Spec}(A)$ a point. Show that the canonical morphism Spec($A_p$) → Spec($A$) is pro-étale.

**Example 13.** Let $p_n$ be the $n$th prime number (so $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, $p_4 = 7$, $p_5 = 11$, $p_6 = 13$, $p_7 = 17$, ...). For any $n \in \mathbb{N}$, the map

$$X_n := \text{Spec}(\mathbb{Z}[\frac{1}{p_1}, \ldots, \frac{1}{p_n}]) \amalg (\amalg_{i=1}^n \text{Spec}(\mathbb{Z}(p_i))) \to \text{Spec}(\mathbb{Z})$$

is pro-étale. Moreover, there are canonical morphisms $X_{n+1} \to X_n$ induced by the canonical pro-étale morphisms

$$\text{Spec}(\mathbb{Z}[\frac{1}{p_1}, \ldots, \frac{1}{p_n}, \frac{1}{p_{n+1}}]) \text{Spec}(\mathbb{Z}(p_{n+1})) \to \text{Spec}(\mathbb{Z}[\frac{1}{p_1}, \ldots, \frac{1}{p_{n+1}}])$$

Consequently, $X := \lim_{\leftarrow n} X_n$ is a pro-étale Spec($\mathbb{Z}$) scheme. As a set, we have

$$X = \{\eta\} \amalg (\amalg_{n \geq 1} \{\eta_n, p_n\})$$

where $\{\eta_n, p_n\}$ correspond to the points of Spec($\mathbb{Z}(p_n)$), and $\eta$ corresponds to the generic points of the Spec($\mathbb{Z}[\frac{1}{p_1}, \ldots, \frac{1}{p_n}]$)’s. The open sets of $X$ are disjoint unions of sets of the form

$$\{\eta_i\}, \quad \{\eta_i, p_i\}, \quad X \setminus (\amalg_{i=1}^N \{\eta_i, p_i\})$$

In particular, every open covering of $X$ can be refined by one which is a finite family of sets of the above form. These sets’ corresponding rings of functions are

$$\mathbb{Q}, \quad \mathbb{Z}(p_i), \quad \lim_{n \to \infty} \mathbb{Z}[\frac{1}{p_1}, \ldots, \frac{1}{p_n}] \times (\mathbb{Z}(p_{N+1}) \times \mathbb{Z}(p_{N+2}) \times \cdots \times \mathbb{Z}(p_n)).$$

The latter is a subring of $\prod_{i > N} \mathbb{Z}(p_i)$ with $\mathbb{Z}[\frac{1}{p_1}, \ldots, \frac{1}{p_n}]$ embedded diagonally into $\prod_{i > n} \mathbb{Z}(p_i)$. Here is a picture.
Exercise 7. Consider the $X$ from Example 13. Show that for every open covering $\{U_i \to X\}_{i \in I}$ the associated morphism $\amalg U_i \to X$ admits a section.

3 The pro-étale topology

The property in the above example is extremely important.

Definition 14. An object in a site is weakly contractible if for every covering $\{U_i \to X\}$ the morphism $\amalg U_i \to X$ admits a section.

Example 15.

1. Strictly hensel rings are weakly contractible with respect to étale coverings.

2. The scheme $\text{Spec}(B)$ constructed in Exercise 2 is weakly contractible with respect to étale coverings (use the fact that any étale covering of $\text{Spec}(\lim_{\to} B_\lambda)$ is the base change of an étale covering of some $B_\lambda$).

3. The scheme $X$ constructed in Example 13 is weakly contractible with respect to Zariski coverings, but not étale coverings, since none of the residue fields are separably closed.

Lemma 16. If $X$ is a weakly contractible object, then $H^n(X,F) = 0$ for all $i$ and all $F$. More interestingly, the evaluation at $X$ functor $\text{Shv}(C,\text{Ab}) \to \text{Ab}$ is exact.

Proof. To calculate cohomology we choose an injective resolution (or fibrant replacement) $F \to I^\bullet$. By definition, the cohomology sheaves $\Gamma(X,F)$ are zero for $n > 0$. This means that for every $s \in H^n(X,F)$, there exists a covering $\{U_i \to X\}$ such that $s|_{U_i} = 0$ for all $i$. But every covering of $X$ admits a section, and therefore $s = 0$.

Suppose $0 \to F \to G \to H \to 0$ is a short exact sequence. Evaluation on an object is left exact, so it suffices to show that $G(X) \to H(X)$ is surjective. By definition of a surjective morphism of sheaves, for every $s \in H(X)$ there is a covering $\{U_i \to X\}$ such that for each $i$ the section $s|_{U_i}$ is in the image of $G(U_i) \to H(U_i)$. But $\amalg U_i \to X$ admits a section, so $s \in H(X)$ is in the image of $G(X) \to H(X)$. □

Definition 17. A site is locally weakly contractible if every object admits a covering by weakly contractible objects.

Proposition 18. If $C$ is a locally weakly contractible site, then for any system $(\cdots \to F_2 \to F_1)$ of surjective morphisms of sheaves, $R\text{lim}_{n \in \mathbb{N}} F_n = \lim_{n \in \mathbb{N}} F_n$. 
It turns out that if we add pro-étale morphisms to $\text{Et}(X)$, then the new bigger site is locally weakly contractible. Limits are so nice in this new site that it fixes the problems described above.

**Theorem 19.** Let $X$ be a connected noetherian scheme.

1. We have
   $$H^i(X_{\text{proet}}, \mathbb{Q}_l) \cong H^i(X_{\text{et}}, \mathbb{Q}_l)$$
   where the right hand side is the limit Eq.1, and the left hand side is honest sheaf cohomology of $\mathbb{Q}_l$.

2. The six functors of Theorem 4 work for the honest derived categories $D(X_{\text{proet}}, \mathbb{Z}_l)$.

3. If $X = \text{Spec}(k)$ is the spectrum of a field, then the subcategory of quasicompact quasiseparated objects $X^{\text{qcqs}}_{\text{proet}}$ is canonically isomorphic to the category of profinite continuous (not necessarily finite) $\text{Gal}(k^{\text{sep}}/k)$-sets
   $$\text{Spec}(k)^{\text{qcqs}}_{\text{proet}} \cong \text{Pro-Fin-}\text{Gal}(k^{\text{sep}}/k)-\text{Set}.$$  

4. Honest $\mathbb{Q}_l$-local systems on $X$ are equivalent to continuous representations of $\pi^\text{proet}_1(X)$ on finite dimensional $\mathbb{Q}_l$-vector spaces.