1 Étale cohomology

1.1 From Weil conjectures to l-adic cohomology

We began with the question:

Question 1. Given a smooth projective variety $X/\mathbb{F}_q$, how many $\mathbb{F}_{q^n}$-points does X have for each n? That is, calculate

$$Z(X,t) = \exp\left(\sum_{n=1}^{\infty} |X(\mathbb{F}_{q^n})| \frac{t^n}{n}\right).$$

This lead to the Weil conjectures:

Theorem 2 (Weil conjectures)**.** *If X is a smooth projective variety of dimension d over $\mathbb{F}_q$.*

1. *(Rationality) $Z(X,t)$ is a rational function of t, i.e., it is in $\mathbb{Q}(t) \subseteq \mathbb{Q}((t))$.*
2. *(Functional equation) There is an integer e such that*
$$Z(X, q^{-d}t^{-1}) = \pm q^{ed/2} t^e Z(X,t).$$
3. *(Riemann Hypothesis) We can write*
$$Z(X,t) = \frac{P_1(t)P_3(t)\dots P_{2d-1}(t)}{P_0(t)P_2(t)\dots P_{2d}(t)}$$
with $P_i(t) \in \mathbb{Z}[t]$, and such that the roots of $P_i(t)$ have absolute value $q^{-i/2}$. Moreover, $P_0(t) = 1-t$ and $P_{2d}(t) = 1 - q^d t$.
4. *(Betti numbers) If X comes from a smooth projective variety over $\mathbb{Z}_{(p)}$, then*
$$\deg P_i(t) = \dim_{\mathbb{Q}} H^i(X(\mathbb{C}), \mathbb{Q}).$$

The strategy was to develop a cohomology theory

$$H^\bullet : (\text{Varieties}/k)^{op} \to \text{graded } \mathbb{Q}\text{-vector spaces}$$

for arbitrary varieties over any field k, which satisfied the following properties for smooth projective varieties X.

1. (Finiteness) $\dim H^\bullet(X)$ is finite, and $H^i(X) = 0$ for $i \notin \{0, 1, \dots, 2\dim X\}$.
2. (Poincaré Duality) There is a canonical isomorphism $H^{2\dim X}(X) \cong \mathbb{Q}$ and a natural perfect pairing
$$H^i(X) \times H^{2d-i}(X) \to \mathbb{Q}$$

3. (Lefschetz Trace Formula)

$$|X(\mathbb{F}_{q^m})| = \sum_{i=0}^{2\dim X} (-1)^i \operatorname{Tr}(\phi_i^m)$$

where $X_{\overline{\mathbb{F}}_q} = X \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q$, $\phi : X_{\overline{\mathbb{F}}_q} \to X_{\overline{\mathbb{F}}_q}$ is the Frobenius morphism, and $\phi_i : H^i(X_{\overline{\mathbb{F}}_q}) \to H^i(X_{\overline{\mathbb{F}}_q})$ is the induced morphism.

4. (Compatibility) If $k = \mathbb{C}$ then $H^\bullet(X)$ is isomorphic to singular cohomology.

Then,

$$\begin{aligned}
\text{(Lefschetz Trace Formula)} &\Rightarrow \text{(Rationality)} \\
\text{(Poincaré Duality)} &\Rightarrow \text{(Functional equation)} \\
\text{(Compatibility)} &\Rightarrow \text{(Betti numbers)} \\
\text{Eigenvalues } \alpha_{i,j} \text{ of } \phi_i | H^i(X_{\overline{\mathbb{F}}_q}) \text{ have } |\alpha_{i,j}| = q^{-i/2} &\Rightarrow \text{(Riemann Hypothesis)}
\end{aligned}$$

We saw that:

1. (Serre) Due to the existence of supersingular elliptic curves, there cannot be any cohomology theory with the above properties taking values in $\mathbb{Q}$-vector spaces.

2. For curves, étale cohomology with $\mathbb{Z}/l^n$-coefficients has Poincaré Duality and

$$\operatorname{rank}_{\mathbb{Z}/l^n} H^i_{\mathsf{et}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Z}/l^n) = \dim_{\mathbb{Q}} H^i_{\mathsf{sing}}(X(\mathbb{C}), \mathbb{Q})$$

This leads us to define:

$$H^i_{\mathsf{et}}(X, \mathbb{Q}_l) := \left(\varprojlim_{n \geq 1} H^i_{\mathsf{et}}(X, \mathbb{Z}/l^n) \right) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l. \tag{1}$$

1.2 Successes of l-adic cohomology

Theorem 3. *The $\mathbb{Q}_l$-vector spaces $H^i_{\mathsf{et}}(X, \mathbb{Q}_l)$ satisfy (Finiteness), (Poincaré Duality), (Lefschetz Trace Formula), and (Riemann Hypothesis).*

We also wanted to see (but ran out of time) that the $\mathbb{Z}/l^n$ cohomology groups had a very strong Poincaré Duality formalism.

Theorem 4. *For any separated finite type morphism between noetherian $\mathbb{Z}[\frac{1}{l}]$-schemes $f : Y \to X$, and object $E \in D(X_{\mathsf{et}}, \mathbb{Z}/l^n)$ there are adjunctions*

$$\begin{aligned}
(f^*, f_*) &: D(Y_{\mathsf{et}}, \mathbb{Z}/l^n) \rightleftarrows D(X_{\mathsf{et}}, \mathbb{Z}/l^n) \\
(f_!, f^!) &: D(X_{\mathsf{et}}, \mathbb{Z}/l^n) \rightleftarrows D(Y_{\mathsf{et}}, \mathbb{Z}/l^n) \\
(- \otimes E, \underline{\hom}(E, -)) &: D(X_{\mathsf{et}}, \mathbb{Z}/l^n) \rightleftarrows D(X_{\mathsf{et}}, \mathbb{Z}/l^n)
\end{aligned}$$

satisfying a number of properties such as a Proper Base Change and Smooth Base Change formulas.

In order to have these functors for sheaves of $\mathbb{Z}_l$-modules, some work is needed.

Definition 5 ([BS, Def.3.5.3])**.** *For a scheme X, define $\mathsf{Shv}_{\mathsf{et}}(X)^{\mathbb{N}}$ to be the category of $\mathbb{N}$-indexed projective systems in $\mathsf{Shv}_{\mathsf{et}}(X)$. The derived category of this abelian category is denoted by $D(X_{\mathsf{et}}^{\mathbb{N}})$.*

We write $D(X_{\mathsf{et}}, (\mathbb{Z}_l)_\bullet) \subseteq D(X_{\mathsf{et}}^{\mathbb{N}})$ for the full subcategory of those objects $(\cdots \to K_2 \to K_1)$ such that $K_m \in D(X_{\mathsf{et}}, \mathbb{Z}/l^m)$ and $K_m \otimes_{\mathbb{Z}/l^m} \mathbb{Z}/l^{m-1} \to K_{m-1}$ is a quasi-isomorphism. Here, $\otimes$ is the left derived tensor product.

Theorem 6 (Ekedahl)**.** *The functors $f^*, f_*, f_!, f^!, \otimes, \underline{\mathrm{hom}}$ can be extended to the categories $D(X_{\mathsf{et}}, (\mathbb{Z}_l)_\bullet)$ in a sensible way.*

We also had a very nice Galois theory.

Theorem 7 (Stacks Project, Tags 0BNB, 0BMY,0BN4)**.** *Let X be a connected scheme, $\overline{x} \in X$ a geometric point, FEt_X the category of finite étale X-schemes, and consider the functor*

$$F : \mathrm{FEt}_X \to \mathrm{Set}; \qquad Y \mapsto |Y_{\overline{x}}|.$$

The étale fundamental group *of X is the profinite group*

$$\pi_1^{\mathsf{et}}(X, \overline{x}) = \mathrm{Aut}(F)$$

and F induces an equivalence of categories

$$\mathrm{FEt}_X \cong \mathrm{Fin}\text{-}\,\pi_1^{\mathsf{et}}(X, \overline{x})\,\text{-Set}$$

with the category of finite sets equipped with a continuous $\pi_1^{\mathsf{et}}(X, \overline{x})$-action.

There is also a linear version of this. Recall that $\mathrm{Loc}_X(R)$ is the category of *local systems* with R-coefficients. That is, sheaves F of R-modules such that for some covering $\{f_i : U_i \to X\}$, each $f_i^* F$ is isomorphic to the constant sheaf R^n for some n. Similar to the case of topological spaces, π_1 determines the category of local systems.

Proposition 8. *If X is a connected locally noetherian $\mathbb{Z}_{(l)}$-scheme, then there is an equivalence of categories*

$$\mathbb{Q}_l \otimes_{\mathbb{Z}_l} \varprojlim \mathrm{Loc}_X(\mathbb{Z}/l^n) \cong \left\{ \begin{array}{c} \textit{continuous finite dimensional} \\ \mathbb{Q}_l\textit{-linear representations of } \pi_1^{et}(X) \end{array} \right\}.$$

1.3 Shortcomings of l-adic cohomology

All of this is not quite as nice as it could be though.

Problem 9.

1. The definition $H^i_{\mathsf{et}}(X, \mathbb{Q}_l) := \left(\varprojlim_{n \geq 1} H^i_{\mathsf{et}}(X, \mathbb{Z}/l^n) \right) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ is a ad hoc, and not very pleasant to work with.

2. The categories $D(X_{\mathsf{et}}, (\mathbb{Z}_l)_\bullet)$ are horrible to work with.

3. The equivalence between local systems and π_1-representations is no longer true in general if one uses, honest $\mathbb{Q}_l$-local systems instead of the ad hoc $\mathbb{Q}_l \otimes_{\mathbb{Z}_l} \mathrm{Loc}_X(\mathbb{Z}/l^n)$ (cf. [Bhatt-Scholze, Pro-étale topology, Example 7.4.9] for an example due to Deligne).

Question 10. So why can't we just use sheaves of $\mathbb{Z}_l$-coefficients?

$$\boxed{\text{Representability!}}$$

Finite coefficients work so well due to the equivalence of categories.

Theorem 11. *There is equivalence of categories*

$$\mathrm{FEt}(X) \cong \mathrm{Loc}_X(\mathrm{FinSet})$$

between the category of finite étale X-schemes and the category of locally constant étale sheaves.

This suggests that we should enlarge the category $\mathrm{Et}(X)$ to include filtered limits.

2 Pro-étale schemes

Definition 12. *A morphism* $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *of affine schemes is* pro-étale *if there exists a cofiltered*[1] *system $(B_\lambda)_{\lambda\in\Lambda}$ of étale finite presentation A-algebras such that $B = \varinjlim B_\lambda$. The system (B_λ) is called a* presentation *for B.*

Exercise 1. Let $(B_\lambda)_{\lambda\in\Lambda}$ be a cofiltered system of rings. Let $\mathrm{Primes}(C)$ denote the set of prime ideals of a ring C, and $\mathrm{Spc}(C)$ the underlying topological space of $\mathrm{Spec}(C)$, i.e., $\mathrm{Spc}(C)$ is $\mathrm{Primes}(C)$ equipped with its Zariski topology.

1. Show that $\mathrm{Primes}(\varinjlim B) = \varprojlim \mathrm{Primes}(B_\lambda)$.

2. Show that for any $f \in B_\lambda$ with image $\overline{f} \in \varinjlim B_\lambda$, the set $D(\overline{f}) \subseteq \mathrm{Primes}(\varinjlim B_\lambda)$ of primes not containing $\overline{f}$ is the preimage of the set $D(f) \subseteq \mathrm{Primes}(B_\lambda)$ of primes not containing f, under the canonical map $\pi : \mathrm{Primes}(\varinjlim B_\lambda) \to \mathrm{Primes}(B_\lambda)$. That is, show $D(\overline{f}) = \pi^{-1}(D(f))$.

3. Deduce that $\mathrm{Spc}(\varinjlim B_\lambda) = \varprojlim \mathrm{Spc}(B_\lambda)$.

Exercise 2. Let k be an algebraically closed field. Using Exercise 1, show that for every pro-finite set S, there exists a pro-étale k-scheme $\mathrm{Spec}(B) \to \mathrm{Spec}(k)$ with $S \cong \mathrm{Spc}(B)$.

[1] A system is cofiltered if (i) it is nonempty, (ii) for every pair of objects $B_\lambda, B_{\lambda'}$ there is a third object $B_{\lambda''}$ and morphisms in the system $B_\lambda \to B_{\lambda''}, B_{\lambda'} \to B_{\lambda''}$, and (iii) for any pair of parallel morphisms in the system $B_\lambda \rightrightarrows B_{\lambda'}$ there exists a morphism in the system $B_{\lambda'} \to B_{\lambda''}$ such that the two compositions are equal.

Exercise 3. Let k be a field and $k \subseteq k^{sep}$ a separable closure. Show that the $\mathrm{Spec}(k^{sep}) \to \mathrm{Spec}(k)$ is pro-étale.

Exercise 4. Suppose that $\mathrm{Spec}(B) \to \mathrm{Spec}(A), \mathrm{Spec}(C) \to \mathrm{Spec}(A)$ are pro-étale with $B = \varinjlim_{\lambda\in\Lambda} B_\lambda$ and $C = \varinjlim_{\mu \in M} C_\mu$ presentations. Show that $\mathrm{Spec}(B) \times_{\mathrm{Spec}(A)} \mathrm{Spec}(C) \to \mathrm{Spec}(A)$ is pro-étale. Hint: consider the system $(B_\lambda \otimes_A C_\mu)_{(\lambda,\mu)\in\Lambda\times M}$.

Exercise 5. Recall that if L/k is a (finite) Galois extension, then $\mathrm{Spec}(L \otimes_k L) \cong \amalg_{Gal(L/k)} \mathrm{Spec}(L)$. Recall also that an separable closure k^{sep}/k is the union of the finite Galois subextensions $k \subseteq L \subseteq k^{sep}$ and $Gal(k^{sep}/k) \cong \varprojlim_{k\subseteq L\subseteq k^{sep}} Gal(L/k)$. Show that

$$\mathrm{Spc}(k^{sep} \otimes_k k^{sep}) \cong Gal(k^{sep}/k)$$

as *topological spaces.*

Exercise 6. Let A be a ring and $\mathfrak{p} \in \mathrm{Spec}(A)$ a point. Show that the canonical morphism $\mathrm{Spec}(A_\mathfrak{p}) \to \mathrm{Spec}(A)$ is pro-étale.

Example 13. Let p_n be the nth prime number (so $p_1 = 2, p_2 = 3, p_3 = 5, p_4 = 7, p_5 = 11, p_6 = 13, p_7 = 17, \dots$). For any $n \in \mathbb{N}$, the map

$$X_n := \mathrm{Spec}(\mathbb{Z}[\tfrac{1}{p_1}, \dots, \tfrac{1}{p_n}]) \amalg (\sqcup_{i=1}^n \mathrm{Spec}(\mathbb{Z}_{(p_i)})) \to \mathrm{Spec}(\mathbb{Z})$$

is pro-étale. Moreover, there are canonical morphisms $X_{n+1} \to X_n$ induced by the canonical pro-étale morphisms

$$\mathrm{Spec}(\mathbb{Z}[\tfrac{1}{p_1}, \dots, \tfrac{1}{p_n}, \tfrac{1}{p_{n+1}}]) \amalg \mathrm{Spec}(\mathbb{Z}_{p_{n+1}}) \to \mathrm{Spec}(\mathbb{Z}[\tfrac{1}{p_1}, \dots, \tfrac{1}{p_n}]).$$

Consequently, $X := \varprojlim X_n$ is a pro-étale $\mathrm{Spec}(\mathbb{Z})$ scheme. As a set, we have

$$X = \{\eta\} \amalg (\sqcup_{n\geq 1}\{\eta_i, \mathfrak{p}_i\})$$

where $\{\eta_i, \mathfrak{p}_i\}$ correspond to the points of $\mathrm{Spec}(\mathbb{Z}_{(p_i)})$, and η corresponds to the generic points of the $\mathrm{Spec}(\mathbb{Z}[\frac{1}{p_1}, \dots, \frac{1}{p_n}])$'s. The open sets of X are disjoint unions of sets of the form

$$\{\eta_i\}, \qquad \{\eta_i, \mathfrak{p}_i\}, \qquad X \setminus (\sqcup_{i=1}^N \{\eta_i, \mathfrak{p}_i\}).$$

In particular, every open covering of X can be refined by one which is a finite family of sets of the above form. These sets' corresponding rings of functions are

$$\mathbb{Q}, \qquad \mathbb{Z}_{(p_i)}, \qquad \varinjlim_{n\to\infty} \mathbb{Z}[\tfrac{1}{p_1}, \dots, \tfrac{1}{p_n}] \times (\mathbb{Z}_{(p_N)} \times \mathbb{Z}_{(p_{N+1})} \times \dots \times \mathbb{Z}_{(p_n)}).$$

The latter is a subring of $\prod_{i>N} \mathbb{Z}_{(p_i)}$ with $\mathbb{Z}[\frac{1}{p_1}, \dots, \frac{1}{p_n}]$ embedded diagonally into $\prod_{i>n} \mathbb{Z}_{(p_i)}$. Here is a picture.

Exercise 7. Consider the X from Example 13. Show that for every open covering $\{U_i \to X\}_{i \in I}$ the associated morphism $\amalg U_i \to X$ admits a section.

3 The pro-étale topology

The property in the above example is extremely important.

Definition 14. *An object in a site is* weakly contractible *if for every covering $\{U_i \to X\}$ the morphism $\amalg U_i \to X$ admits a section.*

Example 15.

1. Strictly hensel rings are weakly contractible with respect to étale coverings.
2. The scheme $\mathrm{Spec}(B)$ construcred in Exercise 2 is weakly contractible with respect to étale coverings (use the fact that any étale covering of $\mathrm{Spec}(\varinjlim B_\lambda)$ is the base change of an étale covering of some B_λ).
3. The scheme X constructed in Example 13 is weakly contractible with respect to Zariski coverings, but not étale coverings, since none of the residue fields are separably closed.

Lemma 16. *If X is a weakly contractible object, then $H^n(X, F) = 0$ for all i and all F. More interestingly, the evaluation at X functor $\mathsf{Shv}(C, \mathrm{Ab}) \to \mathrm{Ab}$ is exact.*

Proof. To calculate cohomology we choose an injective resolution (or fibrant replacement) $F \to I^\bullet$. By definition, the cohomology *sheaves* $a\underline{H}^n(-, I^\bullet)$ are zero for $n > 0$. This means that for every $s \in H^n(X, F)$, there exists a covering $\{U_i \to X\}$ such that $s|_{U_i} = 0$ for all i. But every covering of X admits a section, and there fore $s = 0$.

Suppose $0 \to F \to G \to H \to 0$ is a short exact sequence. Evaluation on an object is left exact, so it suffices to show that $G(X) \to H(X)$ is surjective. By definition of a surjective morphism of sheaves, for every $s \in H(X)$ there is a covering $\{U_i \to X\}$ such that for each i the section $s|_{U_i}$ is in the image of $G(U_i) \to H(U_i)$. But $\amalg U_i \to X$ admits a section, so $s \in H(X)$ is in the image of $G(X) \to H(X)$. □

Definition 17. *A site is* locally weakly contractible *if every object admits a covering by weakly contractible objects.*

Proposition 18. *If C is a locally weakly contractible site, then for any system $(\cdots \to F_2 \to F_1)$ of surjective morphisms of sheaves, $R\lim_{n \in \mathbb{N}} F_n = \lim_{n \in \mathbb{N}} F_n$.*

It turns out that if we add pro-étale morphisms to $\mathrm{Et}(X)$, then the new bigger site is locally weakly contractible. Limits are so nice in this new site that it fixes the problems described above.

Theorem 19. *Let X be a connected noetherian scheme.*

1. *We have*
$$H^i(X_{\mathsf{proet}}, \mathbb{Q}_l) \cong H^i(X_{\mathsf{et}}, \mathbb{Q}_l)$$
where the right hand side is the limit Eq.(1), and the left hand side is honest sheaf cohomology of $\mathbb{Q}_l$.

2. *The six functors of Theorem 4 work for the honest derived categories $D(X_{\mathsf{proet}}, \mathbb{Z}_l)$.*

3. *If $X = \mathrm{Spec}(k)$ is the spectrum of a field, then the subcategory of quasicompact quasiseparated objects $X^{qcqs}_{\mathsf{proet}}$ is canonically isomorphic to the category of profinite continuous (not necessarily finite) $\mathrm{Gal}(k^{sep}/k)$-sets*
$$\mathrm{Spec}(k)^{qcqs}_{\mathsf{proet}} \cong \textit{Pro-Fin-}\,\mathrm{Gal}(k^{sep}/k)\textit{-Set}.$$

4. *Honest $\mathbb{Q}_l$-local systems on X are equivalent to continuous representations of $\pi_1^{\mathsf{proet}}(X)$ on finite dimensional $\mathbb{Q}_l$-vector spaces.*