References are:

[Szamuely] "Galois Groups and Fundamental Groups"

[SGA1] Grothendieck, et al. "Revêtements étales et groupe fondamental"

[Stacks project] The Stacks Project, https://stacks.math.columbia.edu/

1 Motivation

If X is a topological space and $x \in X$ a point, then the fundamental group is defined as

$$\pi_1(X, x) = \frac{\hom((S^1, 0), (X, x))}{\hom((S^1 \times [0,1], \{0\} \times [0,1]), (X, x))}$$

the set of (pointed) morphisms from the circle

$$S^1 \cong [0,1]/\{0,1\} \cong \{z \in \mathbb{C} : |z| = 1\}$$

modulo homotopy.

$-----$ picture of a homotopy $---$

This measures how many "holes of dimension 1" are in X.

Example 1.

1. $\pi_1(S^1 \times \cdots \times S^1) \cong \mathbb{Z}^n$.
2. $\pi_1(S^2) \cong 0$.
3. $\pi_1(\mathbb{C} \setminus \{x_1, \ldots, x_n\}) \cong F_n$, the free group with n generators.
4. $\pi_1(M_g) \cong \langle a_1, b_1, \ldots, a_g, b_g : [a_1, b_1] \cdot \cdots \cdot [a_g, b_g] = 1 \rangle$, where M_g is a compact orientable genus g surface.

This definition does not work algebraically. We could try replacing $[0,1]$ with $\mathbb{A}^1$, or $\{z \in \mathbb{C} : |z| = 1\}$ with $\mathbb{C} \setminus \{0\}$, but $S^1 \times S^1$ is the underlying topological space of a compact elliptic curve E over $\mathbb{C}$, but every morphism from $\mathbb{G}_m$ or $\mathbb{A}^1$ to E is constant.

However, if the topological space X is locally contractible, there is another way to define the fundamental group.

Proposition 2 ([Szamuely, Thm.2.3.7])**.** *Suppose X is a connected, locally simply connected topological space, and $Y \to X$ is a local homeomorphism with Y contractible. Then $\pi_1(X)$ is isomorphic to* $\mathrm{Aut}(Y/X)$.

$-----$ picture of helix covering the circle $---$

The fibres of $Y \to X$ are infinite in many cases, so we cannot hope to have such a Y algebraically. However, we can hope to get its finite quotients.

Definition 3. *A local homeomorphism $X' \to X$ of connected topological spaces with finite fibres of size n is called* Galois *if* $\# \operatorname{Aut}(X'/X) = n$.

Proposition 4. *Suppose X is a connected, locally simply connected topological space. For every Galois cover $X' \to X$ there exists a normal subgroup $N \subseteq \pi_1(X) \cong \operatorname{Aut}(Y/X)$ such that $Y/N \cong X'$ and $\operatorname{Aut}(X'/X) \cong \pi_1(X)/N$. In particular,*

$$\pi_1(X)^\vee \cong \varprojlim_{X'/X} \operatorname{Aut}(X'/X)$$

Example 5. If $X = S^1 = \{z \in \mathbb{C} : |z| = 1\}$, then $Y \cong \mathbb{R}$ with the map $\exp(2\pi i) : \mathbb{R} \to S^1; t \mapsto e^{2\pi i t}$. Then to a normal subgroup $n\mathbb{Z} \subset \mathbb{Z}$ is associated the Galois covering $S^1 \to S^1; z \mapsto z^n$, with automorphism group $z \mapsto \omega z$ where $\omega = e^{2\pi i/n}$ is a primitive nth root of unity.

$- - - - -$ picture of helix covering the circle $- --$

So this gives a good candidate for an algebraic definition of a (pro-finite) fundamental group.

In fact, there is an even stronger relationship between $\pi_1(X)$ and local homeomorphisms.

Theorem 6 ([Szamuely, Thm.2.3.4)**.** *Let X be a connected, locally simply connected topological space and $x \in X$ a point. Then*

$$(f : X' \to X) \qquad \mapsto \qquad f^{-1}(x)$$

induces an equivalence of categories between the category of local homeomorphisms $X' \to X$ and the category of left $\pi_1(X, x)$-sets.

$- - - - -$ draw action of $\pi_1(S^1)$ on a fibre of a finite covering $- --$

Of course, $\pi_1(X)$ can be recovered from the category $\pi_1(X)$-Set as the largest transitive $\pi_1(X)$-set. Under the above equivalence, this corresponds to the universal cover $Y \to X$.

Remark 7. There are a number of other sources of categories which are equivalent to finite G-sets for some group G.

1. The category of finite G-sets for some group G.
2. Let k be a field. Then the category of finite products of finite separable extensions of k is equivalent to the category of finite $Gal(k^{sep}/k)$-sets.
3. Let X be a connected compact Riemann surface. Then the category of compact Riemann surfaces equipped with a holomorphic map onto X is equivalent to the category of finite $Gal(\mathcal{M}(X)^{sep}/\mathcal{M}(X))$-sets, where $\mathcal{M}(X)$ is the field of meromorphic functions on X.

We are going to study the structure of such categories.

2 Finite étale morphisms

Definition 8. *Let* FEt_X *denote the category of finite étale morphisms to* X.

Exercise 1. Let A be a connected ring (i.e., the only elements e of A satisfying $e^2 = e$ are 0 and 1), and $\phi : \prod_{i=1}^n A \to \prod_{i=1}^m A$ a homomorphism of A-algebras. Show that ϕ is of the form $(\phi(a_1, \dots, a_n))_j = a_{f(j)}$ for some function $f : \{1, \dots, m\} \to \{1, \dots, n\}$. In other words, for each projection $\pi_j : \prod_{i=1}^m A \to A; (a_1, \dots, a_m) \mapsto a_j$, show that the composition $\pi_j \circ \phi$ is also a projection $\pi_{f(j)} : \prod_{i=1}^n A \to A; (a_1, \dots, a_m) \mapsto a_{f(j)}$. Hint: consider the idempotents $(0, \dots, 0, 1, 0, \dots, 0)$.

Exercise 2. Let Spec(A) be the spectrum of a strictly hensel local ring. Show that

$$\mathrm{FinSet} \to \mathrm{FEt}_{\mathrm{Spec}(A)}; \qquad T \mapsto \prod_{t \in T} A$$

is an equivalence of categories from the category FinSet of finite sets using Exercise 1, and Lec.2 Exer.12 (which says that this functor is essentially surjective).

Exercise 3. Let C be a category. Show that the following are equivalent.

1. C has all finite limits.
2. C has a terminal object and all fibre products.
3. C has all finite products and equalisers.

Hint $2 \Rightarrow 3$.[1] Hint $3 \Rightarrow 1$.[2]

Theorem 9 (Stacks Project, Tags 0BNB, 0BMY)**.** *Let* X *be a connected scheme,* $\overline{x} \in X$ *a geometric point,* $C = \mathrm{FEt}_X$, *and consider the functor*

$$F : C \to \mathrm{Set}; \qquad Y \mapsto |Y_{\overline{x}}|.$$

1. *The category* C *has all finite limits and finite colimits.*
2. *Every object of* C *is a finite (possibly empty) coproduct of connected objects.*
3. $F(Y)$ *is finite for all* $Y \in C$.
4. (a) F *preserves all finite limits and colimits.*
   (b) *A morphism* f *is an isomorphism if and only if* $F(f)$ *is an isomorphism.*

[1] If $f, g : Y \rightrightarrows X$ are two parallel morphisms, consider $(f, g) : Y \times Y \to X \times X$ and the diagonal $X \to X \times X$.

[2] If $X : I \to C$ is a finite diagram, consider the "source" and "target" morphisms $\prod_{Ob(I)} X_i \rightrightarrows \prod_{Mor(I)} X_f$ from the product indexed by the objects of I and the product indexed by the morphisms of I.

Proof of Theorem 9. (1) By Exercise 3, the category FEt_X has all finite limits since it has fibre products and the terminal object X. By the dual of Exercise 3, to have all finite colimits, it suffices to see that it has coequalisers (since it clearly has finite coproducts). Via $\mathcal{A} \mapsto \underline{\mathrm{Spec}}(\mathcal{A})$, finite X-schemes correspond to finite $\mathcal{O}_X$-algebras. The category of finite $\mathcal{O}_X$-algebras has equalisers, so it suffices to show that if $\phi, \psi : \mathcal{A} \rightrightarrows \mathcal{A}'$ are parallel morphisms between finite $\mathcal{O}_X$-algebras such that $\underline{\mathrm{Spec}}(\mathcal{A}), \underline{\mathrm{Spec}}(\mathcal{A}')$ are étale, then $\underline{\mathrm{Spec}}(\mathrm{Eq}(\phi, \psi))$ is also étale. We saw in Lecture 2, that a morphism is étale if and only if its pullback to each strict henselisation $\mathcal{O}^{sh}_{X,x}$ is étale. Since $\mathrm{Spec}(\mathcal{O}^{sh}_{X,x}) \to X$ is flat, the pullback is an exact functor, and in particular preserves equalisers of algebras. So we can assume that X is the spectrum of a strictly hensel local ring A. But then $\mathcal{A} \cong \prod_{i=1}^n A, \mathcal{A}' \cong \prod_{i=1}^m A$, and ϕ, ψ are induced by maps $f, g : \{1, \dots, m\} \to \{1, \dots, n\}$. Let $T = \mathrm{Coeq}(f, g)$. Then one checks that $\prod_{t \in T} A \cong \mathrm{Eq}(\phi, \psi)$, cf. Exercise 2.

(2) and (3) are clear.

(4a) Preserving finite limits is clear, as is finite coproducts. For colimits, note that F factors as $\mathrm{FEt}_X \to \mathrm{FEt}_{\mathrm{Spec}(\mathcal{O}^{sh}_{X,\overline{x}})} \to \mathrm{FEt}_{\overline{x}}$. The first functor is seen to be exact (considering finite étale X-schemes as finite $\mathcal{O}_X$-algebras) since $\mathrm{Spec}(\mathcal{O}^{sh}_{X,x}) \to X$ is flat. The second functor is an equivalence by Exercise 2, and therefore also exact.

(4b) Suppose that $f : Y \to Y'$ is a morphism in FEt_X such that $F(f)$ is an isomorphism. Since F commutes with finite coproducts, and every object of FEt_X is a disjoint union of connected objects, we can assume that Y' is connected. But then f is surjective, and therefore finite étale, and in particular, $f_*\mathcal{O}_Y$ is a finite locally free $\mathcal{O}_{Y'}$-algebra, and it suffices to show that $f_*\mathcal{O}_Y$ has rank one. But this follows from $F(f)$ being an isomorphism. □

3 Galois categories

Definition 10. *A category C equipped with a functor $F : C \to \mathrm{Set}$ satisfying the properties of Theorem 9 is called a* Galois category. *The automorphism group $Aut(F)$ is called the* fundamental group.

Remark 11. The automorphism group is canonically a subgroup

$$Aut(F) \subseteq \prod_{Y \in Ob(C)} Aut(F(Y)).$$

Since each $F(Y)$ is finite, the right hand side is canonically equipped with the pro-finite topology. We give $Aut(F)$ the induced topology (also profinite).

Example 12. The categories in Remark 7 are all Galois categories.

The main theorem of Galois categories is the following.

Theorem 13 (Stacks Project, Tag 0BN4)**.** *Suppose that $F : C \to \mathrm{Set}$ is a Galois category. Then the canonical functor*

$$C \to \textit{Fin-}\mathrm{Aut}(F)\textit{-Set}; \qquad Y \mapsto F(Y)$$

is an equivalence of categories.

Some ideas in the proof. Faithfullness: See Exercise 4.

Fullness: This uses the fact that F preserves decompositions into connected components. A morphism $\phi : F(X) \to F(Y)$ can be identified with its graph $\Gamma_\phi \subseteq F(X) \times F(Y) \cong F(X \times Y)$, a sum of connected components. This corresponds to a sum of connected components of $X \times Y$, which one easily shows is the graph of a morphism f satisfying $F(\phi) = f$.

F preserves connected components: This uses the very important fact that F is pro-representable. That is, there is a filtered inverse system $X : I \to C$ such that for any $Y \in C$, we have $F(Y) \cong \varinjlim_{i \in I} \hom(X_i, Y)$. The motivation for the system is the filtered system $\{G/N : N \text{ normal, cofinite}^3\}$ in G-set of finite quotients of $G = \mathrm{Aut}(F)$. More concretely, one defines an object Y to be *Galois* if $|\mathrm{Aut}(Y)| = |F(Y)|$. Then choose a representative X_i of every isomorphism class of Galois objects, choose for each i some $s_i \in F(X_i)$, and define a morphism $i \to j$ in I to be a morphism $X_i \to X_j$ which sends s_i to s_j. With a bit of work one shows that $\mathrm{Aut}(F)$ is isomorphic to $\varprojlim \mathrm{Aut}(X_i)$, and that every $Y \in C$ is dominated by a Galois object $X' \twoheadrightarrow Y$. It follows that $\mathrm{Aut}(F)$ acts transitively on $F(Y)$ whenever Y is connected, or in other words, F preserves connected objects.

Essentially surjective: This is, in essence, done using Galois descent. Any finite $\mathrm{Aut}(F)$-set is isomorphic to the set of cosets $\mathrm{Aut}(F)/H$ for some cofinite subgroup H. Using the profinite topology on $\mathrm{Aut}(F)$, with a little bit of work, one finds a cofinite normal subgroup $N \subseteq H$ corresponding to some Galois object $Y \in C$. Then $\mathrm{Aut}(F)/H$ is a categorical quotient of $\mathrm{Aut}(F)/N$, and since C has finite colimits and F preserves them, there is a corresponding categorical quotient X' of Y with $F(X') \cong \mathrm{Aut}(F)/H$. □

Exercise 4. Using the fact that F preserves equalisers and reflects isomorphisms, show that F is faithful.

Definition 14. *Let X be a connected scheme and $\overline{x} \to X$ a geometric point. The* fundamental group *of X is defined as*

$$\pi_1^{et}(X) = Aut(F : \mathrm{FEt}_X \to \mathrm{Set}).$$

Remark 15. As seen in the proof of Theorem 13, an alternative way to define the fundamental group is

$$\pi_1^{et}(X) = \varprojlim_{\overline{x} \to X' \to X} Aut(X'_{\overline{x}})$$

where the inverse limit is over those finite étale $X' \to X$ such that we have $|X'_{\overline{x}}| = |Aut(X'/X)|$.

[3] Cofinite means the quotient is finite.

4 Examples

Exercise 5. Suppose that k is a field. Using the description in Remark 15 show that $\pi_1^{et}(k) \cong Gal(k^{sep}/k)$.

Remark 16. Note that for any group G, every connected objects in Fin-G-Set is isomorphic to a set of cosets G/H for some cofinite subgroup H. On the other hand, for any field k, each connected object in $\mathrm{FEt}(k)$ is isomorphic to a finite separable field extension L/k. Hence, in the case of $X = \mathrm{Spec}(k)$, Theorems 13 and 9 contain the classical Galois correspondence.

$$\left\{ \begin{array}{c} \text{finite separable} \\ \text{field extensions of } k \end{array} \right\} \cong \left\{ \begin{array}{c} \text{connected objects} \\ \text{in } \mathrm{FEt}(k) \end{array} \right\}$$
$$\cong \left\{ \begin{array}{c} \text{connected objects} \\ \text{in Fin-}Gal(k^{sep}/k)\text{-Set} \end{array} \right\} \cong \left\{ \begin{array}{c} \text{cofinite subgroups} \\ \text{of } Gal(k^{sep}/k) \end{array} \right\}$$

Exercise 6. Using the fact that every connected finite étale morphism to $\mathrm{Spec}(\mathbb{C}[t,t^{-1}])$ is of the form $\mathrm{Spec}(\mathbb{C}[t,t^{-1}]) \to \mathrm{Spec}(\mathbb{C}[t,t^{-1}]); t \mapsto t^n$, show that
$$\pi_1^{et}(\mathrm{Spec}(\mathbb{C}[t,t^{-1}])) \cong \widehat{\mathbb{Z}}.$$

Exercise 7. Suppose X is a smooth variety over $\mathbb{C}$, and $X(\mathbb{C})$ the topological space of its associated complex analytic manifold. Note that for any local homeomorphism $M \to X(\mathbb{C})$ there is an induced structure of smooth complex analytic manifold on M. In fact:

Theorem 17 (Riemann Existence Theorem)**.** *The functor* $-(\mathbb{C})$ *from* FEt_X *to finite local homeomorphisms* $X'(\mathbb{C}) \to X(\mathbb{C})$ *is an equivalence of categories.*

Using these facts, and Theorem 6 show that
$$\pi_1^{et}(X) \cong \pi_1(X(\mathbb{C}))^\vee.$$

5 Local systems

The equivalence $\mathrm{FEt}_X \cong \pi_1(X)$-set has a linear version.

Definition 18. *Let R be a ring. A* local system *(with R-coefficients) on a scheme X (resp. topological space) is an étale (resp. usual) sheaf of R-modules F such that there exists a covering $\{f_i : U_i \to X\}_{i \in I}$ for which each $f_i^* F \cong R^n$ for some n. The category of local systems is written $Loc_X(R)$.*

Remark 19. This is the underived version of objects which are locally constant *with perfect values* from last week.

Remark 20. The functor $\mathrm{FEt}_X \to \mathsf{Shv}_{\mathsf{et}}(X); Y \mapsto \hom_X(-,Y)$ induces an equivalence
$$\mathrm{FEt}_X \cong Loc_X(\mathrm{FinSet})$$

between FEt_X and the subcategory $Loc_X(\mathrm{FinSet})$ of locally constant sheaves on $\mathrm{Et}(X)$ with finite fibres. From this point of view, Theorem 13 is an equivalence

$$Loc_X(\mathrm{FinSet}) \cong \mathrm{Fin}\text{-}\,\mathrm{Aut}(F)\text{-Set}$$

In this section we are "R-linearising" this equivalence. For the topological version of this, replace FEt_X with finite local homeomorphism.

Proposition 21. *Suppose that X is a path connected, locally simply connected topological space, and K a field. Then there is an equivalence of categories,*

$$Loc_X(K) \cong \left\{ \begin{array}{c} \textit{finite dimensional} \\ K\textit{-linear representations of } \pi_1(X) \end{array} \right\}.$$

Remark 22. Let $Y \to X$ be the universal covering space of X so that $\pi_1(X) \cong \mathrm{Aut}(Y/X)$, let $\phi : \mathrm{Aut}(Y/X) \to GL_n(K)$ be a group homomorphism, and consider the constant sheaf $\widetilde{F} = K^n$ on Y. This constant sheaf sends an open $V \subseteq Y$ to the product $\prod_{\pi_0(V)} K^n$ indexed by the set of connected components $\pi_0(V)$ of V. Notice that if $V = U \times_X Y$ for some $U \subseteq X$, then $\mathrm{Aut}(Y/X)$ permutes the components in $\pi_0(V)$, and therefore acts on $\widetilde{F}(V)$. On the other hand, via $\phi : \mathrm{Aut}(Y/X) \to GL_n(K)$ and the diagonal action of $GL_n(K)$ on $\prod_{\pi_0(V)} K^n$. We define a sheaf on X by sending $U \subseteq X$ to

$$F(U) = \{s \in \widetilde{F}(U \times_X Y) : g(s) = \phi(g)(s)\ \forall g \in \mathrm{Aut}(Y/X)\}.$$

For normal connected schemes, this proposition also works in algebraic geometry.

Proposition 23. *If X is a connected locally noetherian $\mathbb{Z}_{(l)}$-scheme, then there is an equivalence of categories*

$$Loc_X(\mathbb{Q}_l) \cong \left\{ \begin{array}{c} \textit{continuous finite dimensional} \\ \mathbb{Q}_l\textit{-linear representations of } \pi_1^{et}(X) \end{array} \right\}.$$

Remark 24. Note that we now require the representations to be continuous.

Remark 25. The notation is being abused here. By $Loc_X(\mathbb{Q}_l)$ we actually mean sheaves of the form $\mathbb{Q}_l \otimes_{\mathbb{Z}_l} \varprojlim(\cdots \to F_2 \to F_1)$ such that $F_n \in Loc_X(\mathbb{Z}/l^n)$, and each $F_n \otimes_{\mathbb{Z}/l^n} \mathbb{Z}/l^{n-1} \to F_{n-1}$ is an isomorphism. So each F_n must admit some trivialising cover, but we do not require a single cover which trivialises all F_n at once. For example, the sheaf $\mu_{l^\infty} := \varprojlim \mu_{l^n}$ of l^n roots of unity (for all n at once) is a $\mathbb{Z}_l$-local system, but (over $\mathbb{F}_p$ for example) is not trivialised by an étale covering.

However, the representations must be continuous:

Example 26 (See Grétar Amazeen's Master's Thesis[4] Examples 5.62, 6.38 for more details)**.** Let $X = \mathbb{P}^1/\{0, \infty\}$ be the projective line (over an algebraically

[4] `http://page.mi.fu-berlin.de/lei/finalversion%20(1).pdf`

closed field) with 0 and ∞ identified. Let $Y = \dots {}_0 \sqcup_\infty \mathbb{P}^1{}_0 \sqcup_\infty \mathbb{P}^1{}_0 \sqcup_\infty \dots$ be an infinite chain of $\mathbb{P}^1$'s joining ∞ of each $\mathbb{P}^1$ to the 0 of the next one, and $f : Y \to X$ the canonical morphism. Consider the trivial rank on local system $\mathbb{Q}_l$ on Y, and define an equivariant automorphism $\mathbb{Q}_l \to \mathbb{Q}_l$ using multiplication by l.

$- - - -$ draw picture $- - - -$

Then this descends to a local system on X. However, it cannot come from a representation of $\pi_1^{et}(X) \cong \widehat{\mathbb{Z}}$, because by compactness, such representations correspond to homorphisms $\pi_1^{et}(X) \to \mathbb{Z}_l^*$.

Next quarter we will see how this is fixed using the pro-étale topology, in a way where we can also take the naïve definition of $Loc_X(\mathbb{Q}_l)$, that does not use limits.