In this lecture we show how the category $\mathsf{Shv}_{\mathsf{et}}(X)$ of étale sheaves on a scheme X can be reconstructed from $\mathsf{Shv}_{\mathsf{et}}(Z)$ sheaves on a closed subscheme $Z \subseteq X$ and $\mathsf{Shv}_{\mathsf{et}}(U)$ sheaves on its open complement $U = X - Z$, see Theorem 16.

For easy of exposition, all presheaves will be presheaves of abelian groups,[1] and all sites small.[2] All schemes are assumed to be quasi-compact quasi-separated (e.g., Noetherian), and we recall that étale morphisms are by definition locally of finite presentation (this means finite type if working only with Noetherian schemes).[3]

1 Presheaf adjunctions

Definition 1. *Suppose that $\pi : C' \to C$ is a functor. We denote the functor induced by composition as*

$$\pi_p : \mathsf{PreShv}(C) \to \mathsf{PreShv}(C'); \qquad F \mapsto F \circ \pi.$$

Exercise 1. If $\pi' : C'' \to C'$ and $\pi : C' \to C$ are two functors, show that $(\pi \circ \pi')_p = \pi_p \circ \pi'_p$.

Definition 2. *Give a presheaf $F \in \mathsf{PreShv}(C')$ and $X \in C$ define*

$$(\pi^p F)(X) = \varinjlim_{X \to \pi(Y)} F(Y)$$

where the colimit is over the comma category $(X \downarrow \pi)$ whose objects are morphisms $X \to \pi(Y)$ in C, and $\hom(X{\to}\pi(Y), X{\to}\pi(Y')) = \{f : Y \to Y'$ s.t. the triangle $\begin{smallmatrix} \pi(Y) & \longrightarrow & \pi(Y') \\ \nwarrow & & \nearrow \\ & X & \end{smallmatrix}$ commutes $\}$.

Remark 3. There is also a right adjoint to π_p defined in an analogous way, but we will not use it.

Exercise 2. Using the universal property of the colimit, show that a morphism $X \to X'$ in C induces a morphism $(\pi^p F)(X') \to (\pi^p F)(X)$, and that this makes $\pi^p F$ into a presheaf on C.

Exercise 3 (Advanced)**.** Given an object $W \in C$ we write $h_W(-) = \hom_C(-, W)$ for the presheaf represented by W.

1. Show that $\pi^p h_Y = h_{\pi Y}$ for any $Y \in C'$.

2. Show that there is canonical isomorphism $\hom(\pi^p h_Y, G) \cong \hom(h_Y, \pi_p G)$. Note: the right side is isomorphic to $(\pi_p G)(Y)$.

[1] Sheaves of sets work just as well.

[2] This is to ensure that the colimits defining π^p are well-defined. In practice, there are many functors between large categories for which these colimits are still well-defined.

[3] These finiteness assumptions ensure that $\mathrm{Et}(X)$ are small (enough) categories (so that the left adjoints π^p existence is guaranteed), but otherwise are basically only used in the proof of Proposition 12.

3. Show that for any presheaf $F \in \mathsf{PreShv}(C')$, we have $F \cong \varinjlim h_Y$ where $h_Y = \hom_{C'}(-, Y)$ is the presheaf represented by Y, and the colimit is over the category $\int_C F$ whose objects are pairs (Y, s) with $Y \in C'$ and $s \in F$ and morphisms $(Y, s) \to (Y', s')$ are morphisms $Y \to Y'$ of C' such that $F(Y') \to F(Y)$ sends s' to s.

4. Show that π^p preserves any colimits of presheaves.

5. Deduce that for any $F \in \mathsf{PreShv}(C')$ (not just representable presheaves) there is a canonical isomorphism $\hom_{\mathsf{PreShv}(C)}(\pi^p F, G) \cong \hom_{\mathsf{PreShv}(C')}(F, \pi_p G)$.

Corollary 4. *The pair* (π^p, π_p) *is an adjunction* $\mathsf{PreShv}(C) \rightleftarrows \mathsf{PreShv}(C')$.

Exercise 4. Using Exercise 1, Corollary 4, and the uniqueness properties of adjunctions show that $\pi'^p \circ \pi^p = (\pi \circ \pi')^p$.

Exercise 5. Suppose that $f : Y \to X$ is a morphism of topological spaces, and let $\pi : \mathrm{Op}(X) \to \mathrm{Op}(Y); U \mapsto f^{-1}U$ be the induced functor between the categories of open subsets of X, Y. Show that π_p is the usual push-forward $\mathsf{PreShv}(Y) \to \mathsf{PreShv}(X)$ and π^p is the usual inverse image of presheaves functor $\mathsf{PreShv}(X) \to \mathsf{PreShv}(Y)$.

Exercise 6. Suppose that the category C has a final object X, and let $\pi : * \to C$ be the functor from the category with one morphism which sends the unique object to X. Show that π_p is the global sections functor $F \mapsto F(X)$, and π^p is the constant presheaf functor $(\pi^p A)(U) = A$ for $A \in \mathrm{Ab} = \mathsf{PreShv}(*)$.

Exercise 7. Let $Y \to X$ be an étale morphism of schemes, and consider the functors

$$\pi : \mathrm{Et}(X) \to \mathrm{Et}(Y); \qquad U \mapsto Y \times_X U \tag{1}$$

and

$$\gamma : \mathrm{Et}(Y) \to \mathrm{Et}(X); \qquad (V \to Y) \mapsto (V \to Y \to X) \tag{2}$$

Show that (γ, π) is an adjunction. Show that $\gamma_p = \pi^p$.

2 Sheaf adjunctions

Definition 5. *Suppose that* C', C *are sites, i.e., categories equipped with Grothendieck topologies. A functor* $\pi : C' \to C$ *is called* continuous *if for every sheaf* F *on* C*, the presheaf* $\pi_p F$ *is a sheaf.*

Exercise 8. Suppose $\pi : C' \to C$ sends fibre products to fibre products. Show that π is continuous if it sends covers to covers.

Example 6. If $Y \to X$ is a morphism topological spaces then the induced morphism of sites $\mathrm{Op}(X) \to \mathrm{Op}(Y)$ is continuous.

Example 7. If $f : Y \to X$ is a morphism of schemes, then π from Equation 1 is continuous. If f is an étale morphism of schemes then γ from Equation 2 is also continuous.

Definition 8. *Suppose $\pi : C' \to C$ is a continuous morphism of sites. The induced functor on sheaves is denoted*

$$\pi_* : \mathsf{Shv}(C) \to \mathsf{Shv}(C').$$

The composition of π^p with sheafification $a : \mathsf{PreShv}(C) \to \mathsf{Shv}(C)$ is denoted

$$\pi^* = a \circ \pi^p : \mathsf{Shv}(C') \to \mathsf{Shv}(C).$$

Exercise 9. Suppose we are in the situation of Definition 8. Using the fact that sheafification $a : \mathsf{PreShv}(C) \to \mathsf{Shv}(C)$ is the left adjoint to the canonical inclusion $\iota : \mathsf{Shv}(C) \to \mathsf{PreShv}(C)$, show that

$$\pi^* : \mathsf{Shv}(C) \rightleftarrows \mathsf{Shv}(C') : \pi_*$$

is an adjunction.

Exercise 10. Using Exercise 1 and Exercise 4, show that if C, C', C'' are equipped with Grothendieck topologies, and π, π' are continuous, then $(\pi \circ \pi')_* = \pi_* \circ \pi'_*$ and $\pi'^* \circ \pi^* = (\pi \circ \pi')^*$.

Definition 9. *If $f : Y \to X$ is a morphism of schemes, and π the pullback functor from Equation (1), we write*

$$f^* := \pi^*, \qquad f_* := \pi_*.$$

If f is étale, so π has a left adjoint γ from Equation 2 then we write

$$f_! := \gamma^*$$

Note that since $\gamma_* = \pi^*$, cf. Exercise 7, in addition to the adjunction (f^*, f_*), we have another adjunction $(f_!, f^*)$.

Lemma 10. *Let $f : Y \to X$ be a morphism of schemes. Then f^* preserves exact sequences.*

Proof. It automatically commutes with colimits because it is a left adjoint. On the other hand, π^p commutes with finite limits because limits of presheaves are calculated object wise, and π^p is defined using filtered colimits, which commute with finite limits. To deduce that $f^* = \pi^*$ commutes with finite limits from π^p commuting, we just recall that sheafification is exact, so $\pi^* = a \circ \pi^p$ is a composition of two functors which commute with finite limits. □

Lemma 11. *Let $f : Y \to X$ and $X' \to X$ be morphisms of schemes. Let h'_X denote the étale sheaf of sets $h_{X'} = \hom_X(-, X') \in \mathsf{Shv}_{\mathsf{et}}(X)$, and similarly, $h_{Y \times_X X'} = \hom_Y(-, Y \times_X X') \in \mathsf{Shv}_{\mathsf{et}}(Y)$. We have*

$$f^* h_{X'} = h_{Y \times_X X'}.$$

Proof. By Yoneda, it suffices to produce isomorphisms

$$\hom_{\mathsf{Shv}}(f^* h_{X'}, F) \cong \hom_{\mathsf{Shv}}(h_{Y \times_X X'}, F)$$

for each $F \in \mathsf{Shv}_{\mathsf{et}}(Y)$, which are natural in F. But we have

$$\begin{aligned}
\hom_{\mathsf{Shv}}(f^* h_{X'}, F) &\cong \hom_{\mathsf{Shv}}(h_{X'}, f_* F) && \text{adjunction} \\
&\cong (f_* F)(X') && \text{Yoneda} \\
&\cong F(Y \times_X X') && \text{definition} \\
&\cong \hom_{\mathsf{Shv}}(h_{Y \times_X X'}, F) && \text{Yoneda.}
\end{aligned}$$

□

Exercise 11. Using the same argument as in Lemma 11 show that if $f : Y \to X$ is an étale morphism of schemes, and $Y' \to Y$ any morphism then

$$f_! h_{Y'} = h_{Y'}$$

where the left Y' is considered as a Y-scheme, and the right one as an X-scheme.

3 Immersions

Exercise 12. Suppose that $j : U \to X$ is an open immersion. Show that in this case, $\gamma : \mathrm{Et}(U) \to \mathrm{Et}(X)$ from Equation 2 is the inclusion of a *full* subcategory. Show that since this subcategory is full, the functor $j^* : \mathsf{Shv}(X) \to \mathsf{Shv}(U)$ is none-other-than the restriction functor

$$j^* F = F|_{\mathrm{Et}(U)}.$$

Show that $j_! : \mathsf{Shv}(U) \to \mathsf{Shv}(X)$ is "extension by zero" in the sense that for any $H \in \mathsf{Shv}(U)$,

$$(j_! H)(V) = \begin{cases} H(V) & V \in \mathrm{Et}(U) \\ 0 & V \notin \mathrm{Et}(U) \end{cases}$$

Show that

$$(j_* H)(V) = H(V) \text{ if } V \in \mathrm{Et}(U),$$

but give an example where $V \notin \mathrm{Et}(U)$, and $(j_* H)(V) \neq 0$.

Deduce that

$$j^* j_! = \mathrm{id} = j^* j_*.$$

Exercise 13. Let X be a scheme, and $\iota : \overline{x} \to X$ a geometric point. Show that

$$\iota^* F = F_{\overline{x}}$$

is the stalk of F at $\overline{x}$, where we implicitly use the identification $\mathsf{Shv}(\overline{x}) \cong \mathrm{Ab}$.

Proposition 12 (Milne Cor.II.3.5)**.** *Let $i : Z \to X$ be the inclusion of a closed immersion, $\overline{x} \to X$ a geometric point, and $G \in \mathsf{Shv}(Z)$. Then*

$$(i_* G)_{\overline{x}} = \begin{cases} G_{\overline{x}} & \mathrm{im}(\overline{x}) \in Z \\ 0 & \mathrm{im}(\overline{x}) \notin Z \end{cases}$$

If $j : U \to X$ is the open complement of Z, then we have $j^* i_* = 0$.

Remark 13. We are abusing the notation a bit here in the case $\mathrm{im}(\overline{x}) \in Z$. When writing $(i_* G)_{\overline{x}}$, we are considering $\overline{x}$ as a geometric point of X, so the colimit is over factorisations through $\mathrm{Et}(X)$. But when writing $G_{\overline{x}}$, we are considering $\overline{x}$ as a geometric point of Z, and so the colimit is over factorisations through $\mathrm{Et}(Z)$.

Easy parts of the proof (Omitted from the lecture). The second claim follows from the first claim, since a sheaf is zero if and only if all its stalks are zero, and the stalks of $j^* i_*$ are all zero by Exercises 10, 12, and 13.

Certainly, if $\mathrm{im}(\overline{x}) \notin Z$, then $(i_* G)_{\overline{x}} = \varinjlim_{\overline{x} \to V \to X} G(V) = 0$, since each $\overline{x} \to V \to X$ is refinable by some $\overline{x} \to V' \to X$ with $Z \times_X V' = \varnothing$ (e.g., $V' = U \times_X V$), and for such V' we have $(i_* G)(V') = G(Z \times_X V') = G(\varnothing) = 0$.

The difficult part is to show that for any $\overline{x} \to V \to Z$ with $V \in \mathrm{Et}(Z)$, there is some $\overline{x} \to V' \to X$ and a factorisation $\overline{x} \to Z \times_X V' \to V \to Z$. If we know this, then the system defining $(i_* G)_{\overline{x}}$ is cofinal in the system defining $G_{\overline{x}}$, and the colimits will be the same.

The proof of this claim is omitted. See Milne Thm.II.3.2(b) for details. Really, check it out. Its a very neat argument using properties of limit schemes from EGA, in particular, EGA IV, Part 3, Cor.8.13.2. □

Exercise 14. Suppose that $F \in \mathsf{Shv}_{\mathsf{et}}(U)$ is a constant sheaf (that is, there is an abelian group such that for each connected $V \in \mathrm{Et}(U)$, we have $F(V) = A$). If $i : Z \to X$ is a nowhere dense closed immersion with open complement $j : U \to X$, using the fact that étale morphisms send generic points to generic points, show that $j_* F$ and $i^* j_* F$ are constant sheaves on X and Z respectively.

4 The localistion sequences

Definition 14. *Let $i : Z \to X$ be a closed immersion of schemes, and $j : U \to X$ the open complement. Define $T(X)$ to be the category*[4] *whose objects are triples (F_1, F_2, ϕ) consisting of two objects $F_1 \in \mathsf{Shv}(Z), F_2 \in \mathsf{Shv}(U)$, and a morphism $\phi : F_1 \to i^* j_* F_2$. Morphisms $(F_1, F_2, \phi) \to (F_1', F_2', \phi')$ are pairs of morphisms*

[4]This is just the comma category $(\mathsf{Shv}(Z) \downarrow i^* j_*)$.

$(F_1 \overset{\psi_1}{\to} F_1', F_2 \overset{\psi_2}{\to} F_2')$ *such that the square commutes.*

$$\begin{array}{ccc} F_1 & \xrightarrow{\phi} & i^*j_*F_2 \\ {\scriptstyle \psi_1}\downarrow & & \downarrow{\scriptstyle i^*j_*\psi_2} \\ F_1' & \xrightarrow[\phi']{} & i^*j_*F_2' \end{array}$$

We define a functor $t : \mathsf{Shv}(X) \to T(X)$ *by*

$$F \mapsto (i^*F, \qquad j^*F, \qquad i^*(F \overset{\eta}{\to} j_*j^*F))$$

where $\eta : \mathrm{id} \to j_*j^*$ *is the unit of the adjunction* (j^*, j_*).

Remark 15. Given $V \in \mathrm{Et}(X)$, we have $j_*j^*F(V) = F(U \times_X V)$, and η is the canonical morphism $F(V) \to F(U \times_X V)$ induced by the canonical morphism $U \times_X V \to V$.

Theorem 16 (Milne Thm.II.3.10)**.** *The functor* $t : \mathsf{Shv}(X) \to T(X)$ *is an equivalence of categories.*

Proof. Given a triple (F_1, F_2, ϕ) in $T(X)$ define

$$s(F_1, F_2, \phi) := \ker\left(i_*F_1 \oplus j_*F_2 \overset{i_*\phi \,+\, \eta}{\longrightarrow} i_*i^*j_*F_2\right).$$

Here, $\eta : \mathrm{id} \to i_*i^*$ is the unit of the adjunction (i^*, i_*). Notice that every morphism of $T(X)$ induces a morphism in $\mathsf{Shv}(X)$ in a way that defines a functor

$$s : T(X) \to \mathsf{Shv}(X).$$

So it suffices to check that $st \cong \mathrm{id}$ and $ts \cong \mathrm{id}$. Consider stF. By definition, this is

$$stF = \ker\left(i_*i^*F \oplus j_*j^*F \overset{i_*\phi \,+\, \eta}{\longrightarrow} i_*i^*j_*j^*F\right).$$

This comes equipped with a canonical morphism $F \to stF$. This morphism is an isomorphism if and only if the sequence

$$0 \to F \to i_*i^*F \oplus j_*j^*F \overset{i_*\phi \,+\, \eta}{\longrightarrow} i_*i^*j_*j^*F$$

is exact. One can check exactness on stalks, so consider a geometric point $\overline{x} \to X$. If $\mathrm{im}(\overline{x}) \in U = X \setminus Z$ then our sequence becomes

$$0 \to F_{\overline{x}} \to 0 \oplus F_{\overline{x}} \longrightarrow 0.$$

If $\mathrm{im}(\overline{x}) \in Z = X \setminus U$ then our sequence becomes

$$0 \to F_{\overline{x}} \to F_{\overline{x}} \oplus (j_*j^*F)_{\overline{x}} \longrightarrow (j_*j^*F)_{\overline{x}}.$$

Hence, we have confirmed exactness, and $F \xrightarrow{\sim} stF$.

Now consider $ts(F_1, F_2, \phi)$. We have

$$\begin{aligned} i^* s(F_1, F_2, \phi) &= i^* \ker\Big(i_* F_1 \oplus j_* F_2 \longrightarrow i_* i^* j_* F_2 \Big) \\ &= \ker\Big(i^* i_* F_1 \oplus i^* j_* F_2 \longrightarrow i^* i_* i^* j_* F_2 \Big) \\ &= \ker\Big(F_1 \oplus i^* j_* F_2 \longrightarrow i^* j_* F_2 \Big) \\ &= F_1 \end{aligned}$$

One similarly checks that $j^* s(F_1, F_2, \phi) \cong F_2$, and that the canonical morphism $i^* s(F_1, F_2, \phi) \to i^* j_* j^* s(F_1, F_2, \phi)$ is none-other-than ϕ, under these identifications. Hence, $ts(F_1, F_2, \phi) = (F_1, F_2, \phi)$. $\square$

Theorem 17 (Milne Prop.II.3.14)**.** *Its possible to define six functors*

$$\mathsf{Shv}_{\mathsf{et}}(Z) \underset{\overset{\longleftarrow}{i^!}}{\overset{\overset{i^*}{\longleftarrow}}{\xrightarrow{i_*}}} \mathsf{Shv}_{\mathsf{et}}(X) \underset{\overset{\longleftarrow}{j_*}}{\overset{\overset{j_!}{\longleftarrow}}{\xrightarrow{j^*}}} \mathsf{Shv}_{\mathsf{et}}(U)$$

such that under the identification $\mathsf{Shv}_{\mathsf{et}}(X) \cong T(X)$*, they correspond to:*

$$\begin{array}{rcl} F_1 & \leftarrow\!\shortmid & (F_1, F_2, \phi) \\ F_1 & \mapsto & (F_1, 0, 0) \\ \ker(\phi) & \leftarrow\!\shortmid & (F_1, F_2, \phi) \end{array} \qquad \begin{array}{rcl} (0, F_2, 0) & \leftarrow\!\shortmid & F_2 \\ (F_1, F_2, \phi) & \mapsto & F_2 \\ (i^* j_* F_2, F_2, \mathrm{id}) & \leftarrow\!\shortmid & F_2 \end{array}$$

1. *Each functor is left adjoint to the one below it.*
2. *The functors $i^*, i_*, j^*, j_!$ preserve exact sequences; $j_*, i^!$ preserve monomorphisms.*
3. *The composites $i^* j_!, i^! j_!, i^! j_*, j^* i_*$ are zero.*
4. *The functors i_*, j_* are fully faithful.*
5. *The functors $j_*, j^*, i^!, i_*$ map injective objects to injective objects.*

Remark 18. Heuristicaly, j_* "fills in the gaps" over Z in a canonical way, and $i^!$ isolates the part of F which cannot be recovered from $F|_{\mathrm{Et}(U)}$ by this "filling in the gaps" process.

Exercise 15 (Not advanced)**.** Prove Theorem 17 using what we have seen so far. Note that if a functor has a left adjoint preserving monomorphisms then it preserves injectives.

Exercise 16 (Not advanced)**.** In the situation of Theorem 17 show that there are short exact sequence

$$0 \to j_! j^* \to \mathrm{id} \to i_* i^* \to 0$$

$$0 \to i_* i^! \to \mathrm{id} \to j_* j^* \to 0$$

Remark 19. Sometimes one defines $j^* := j^!$ and $i_! := i_*$ so that the short exact sequences can be written as

$$0 \to j_! j^! \to \mathrm{id} \to i_* i^* \to 0$$

$$0 \to i_! i^! \to \mathrm{id} \to j_* j^* \to 0$$

5 Curves

Example 20 (Milne, Exam.II.3.12)**.** Let A be a discrete valuation ring (e.g., $\mathbb{C}[[z]], \mathbb{F}_p[[z]], \mathbb{Z}_p, \dots$). Let

1. $K = \mathrm{Frac}(A)$,
2. $k = A/\mathfrak{m}$,
3. $G_K = Gal(K^{sep}/K)$,
4. $G_k = Gal(k^{sep}/k)$,

Since A is a discrete valuation ring, $X = \mathrm{Spec}(A)$ has one open point, and one closed point. Let $U = \mathrm{Spec}(K), Z = \mathrm{Spec}(k)$ be the corresponding open and closed subschemes. Recall that the category of étale sheaves over a field is equivalent to the category of discrete Galois modules. That is, $\mathsf{Shv}_{et}(Z) \cong G_k$-mod and $\mathsf{Shv}_{et}(U) \cong G_K$-mod. We can give an analogous description of $\mathsf{Shv}_{et}(X)$ using a similar construction to $T(X)$. It suffices to work out what functor G_K-mod $\to G_k$-mod corresponds to $i^* j_* : \mathsf{Shv}_{et}(U) \to \mathsf{Shv}_{et}(Z)$.

Let A^h be the henselisation of A, and A^{sh} a strict henselisation. Since K^{sep} is separable closed, there are factorisations $A \to A^h \to A^{sh} \to K^{sep}$ which are actually inclusions. The choice of A^{sh} and the inclusion define subgroups $I = \mathrm{Gal}(K^{sep}/\mathrm{Frac}(A^{sh}))$ and $D = \mathrm{Gal}(K^{sep}/\mathrm{Frac}(A^h))$, with $I \subseteq D \subseteq G_K$, and it turns out that D/I is canonically isomorphic to $Gal(k^{sep}/k)$ where we identify $k^{sep} = A^{sh}/\mathfrak{m}_{A^{sh}}$. Then we claim that the functor $i^* j_* : \mathsf{Shv}_{et}(U) \to \mathsf{Shv}_{et}(Z)$ corresponds to the functor of I-invariants.

$$(-)^I : G_K\text{-mod} \to G_k\text{-mod}.$$

Hence, the category $\mathsf{Shv}_{\mathsf{et}}$ is equivalent to the category of triples (M_1, M_2, ϕ) where $M_1 \in G_k$-mod, $M_2 \in G_K$-mod, and $\phi : M_1 \to M_2$ is compatible with the actions of $G_k \cong D/I$ and G_K.

Example 21. Example 20 can be generalised to any normal curve, see Milne Exer.II.3.16 for details.