In this lecture we define étale sheaves using the étale topology.

1 Sheaves

We begin with the general theory of Grothendieck sites. This is a generalisation of the notion of a topological space, which allows us to use more general morphisms in place of open immersions.

Definition 1. *A (Grothendieck) topology on a category $\mathcal{C}$ is the data of: for every object $U \in \mathcal{C}$, a collection of families of morphisms $\{U_i \to U\}_{i \in I}$. The families in these collections are called* coverings *of U. This data is required to satisfy the following axioms:*

1. *$\{U \overset{\mathrm{id}}{\to} U\}$ is a covering, for every object U.*
2. *If $\{U_i \to U\}_{i \in I}$ is a covering of U, and $V \to U$ is a morphism, then each fibre product $U_i \times_U V$ exists, and $\{U_i \times_U V \to V\}_{i \in I}$ is a covering of V.*
3. *If $\{U_i \to U\}_{i \in I}$ is a covering of U, and for each $i \in I$ we have a covering $\{U_{ij} \to U_i\}_{j \in J_i}$ of U_i, then $\{U_{ij} \to U\}_{i \in I, j \in J_i}$ is a covering of U.*

A category equipped with a Grothendieck topology is called a site.

Exercise 1. Suppose that X is a topological space in the conventional sense.[1] Define $Op(X)$ to be the category whose objects are open sets of X, and morphisms are inclusions. For $U \in Op(X)$, define the coverings of U to be the families $\{U_i \to U\}_{i \in I}$ such that $\cup_{i \in I} U_i = U$. Show that this defines a Grothendieck topology on $Op(X)$. (Note that in this category $V \times_U W = V \cap W$.)

Exercise 2. Let X be a topological space, and define $LH(X)$ to be the category whose objects are local homeomorphisms[2] $Y \to X$ and morphisms are commutative triangles $\begin{smallmatrix} Y' \to Y \\ \searrow \circlearrowleft \swarrow \\ X \end{smallmatrix}$. Show that this category has fibre products.

For $Y \in LH(X)$, define the coverings of Y to be the families $\{f_i : Y_i \to Y\}_{i \in I}$ such that $\cup_{i \in I} f_i(Y_i) = Y$. Show that this defines a Grothendieck topology on $LH(X)$.

Exercise 3. Recall that a morphism $f : Y \to X$ of schemes is étale if it is locally of finite presentation, and for every $y \in Y$, the ring morphism $\mathcal{O}_{X,f(x)} \to \mathcal{O}_{Y,y}$ is étale. Let $Et(X)$ denote the category whose objects are étale morphisms $Y \to X$, and morphisms are commutative triangles. Do Exercise 2 with $Et(X)$ instead of $LH(X)$.

Definition 2. *A* presheaf *F on a category $\mathcal{C}$ is just a functor $\mathcal{C}^{op} \to$* Set. *A morphism of presheaves $F \to G$ is just natural transformation of functors $F \to G$.*

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[1] I.e., a set equipped with a collection of subsets of X declared to be *open*, preserved by finite intersection, arbitrary union, and containing X and $\varnothing$.

[2] A morphism $f : Y \to X$ is a local homeomorphism if for every point $y \in Y$, there is an open neighbourhood $V \ni y$ such that $f : V \to f(V)$ is a homeomorphism.

Definition 3. *If $\mathcal{C}$ is equipped with a Grothendieck topology, then a presheaf F is called a* sheaf *if for any object U and any covering $\{U_i \to U\}_{i\in I}$ we have*

$$F(U) = \mathrm{eq}\left(\prod_{i\in I} F(U_i) \rightrightarrows \prod_{i,j\in I} F(U_i \times_U U_j)\right). \tag{III}$$

A morphism of sheaves is just a morphism of presheaves. A sheaf on $Et(X)$ for some scheme X is called an étale sheaf on X.

Remark 4. If A is a ring, we will write $Et(A)$ instead of $Et(\mathrm{Spec}(A))$ and if $A \to B$ is an étale algebra, and F a presheaf on $Et(A)$ we will write $F(B)$ instead of $F(\mathrm{Spec}(B))$.

Example 5. We have the following important examples of étale sheaves on $Et(k)$, cf. Exercise 10.

1. $\mathcal{O} : L \mapsto (L, +)$.
2. $\mathcal{O}^* : L \mapsto (L^*, *)$.
3. $\mu_n : L \mapsto \{a \in L^* : a^n = 1\}$.

Remark 6. If a presheaf takes values in the category of abelian groups, then the sheaf condition (III) is equivalent to asking that the sequence

$$0 \to F(U) \to \prod_{i\in I} F(U_i) \to \prod_{i,j\in I} F(U_i \times_U U_j)$$

be exact, where the last morphism is the difference of the two morphisms induced by the two projections $U_i \times_U U_j \rightrightarrows U_i, U_j$.

Exercise 4. Let X be a topological space in the conventional sense. Consider the Grothendieck topology defined on $Op(X)$ in Exercise 1. Show that a presheaf on X is the same thing as a presheaf on $Op(X)$, and a preseheaf on X is a sheaf if and only if its associated presheaf on $Op(X)$ is a sheaf. That is, Definition 3 is an honest generalisation of the classical notion of a sheaf.

Exercise 5. Let $\mathrm{Spec}(L) \to \mathrm{Spec}(L')$ be a morphism in $Et(k)$ such that L/L' is Galois with Galois group $G = Aut(L/L')$. Recall that there is a canonical isomorphism

$$L \otimes_{L'} L \cong \prod_G L$$

where two morphisms $L \rightrightarrows L \otimes_{L'} L; a \mapsto 1 \otimes a, a \otimes 1$ are identified with $a \mapsto (a, a, \dots, a)$ and $a \mapsto (a^{g_1}, \dots, a^{g_n})$ where g_i are the elements of G. Show that if F is an étale sheaf on $\mathrm{Spec}(k)$, then $F(\prod_G L) \cong \prod_G F(L)$, and

$$F(L') = F(L)^G$$

where $F(L)^G = \{s \in F(L) : g^* s = s \ \forall\ g \in G\}$. Deduce that if $F_1 \to F_2$ is a morphism of étale sheaves such that $F_1(L) \cong F_2(L)$ for every Galois extension L/k, then $F_1 \cong F_2$.

Remark 7. We will be able to show later on that a presheaf F on $Et(k)$ is a sheaf if and only if

1. $F(\amalg_{i\in I}U_i) \cong \prod_{i\in I} F(U_i)$ for any collection $U_i, i \in I$, and
2. $F(L) = F(L')^{Aut(L'/L)}$ for every Galois extension L'/L.

Theorem 8 (cf.Milne, Thm.II.1.9)**.** *Suppose that k is a field, k^{sep}/k is a separable closure, and $G = Gal(k^{sep}/k)$. Then there is a canonical equivalence between the category G-set of discrete[3] G-sets[4] and the category $\mathsf{Shv}(Et(k)$ of étale sheaves on k.*

Remark 9. An easy case of the above theorem is $k = \mathbb{R}$. In this case the equivalence $\mathsf{Shv}(Et(k)) \to G$-set is given by $F \mapsto F(\mathbb{C})$. In general, however, k^{sep}/k will not be finite, and therefore $\operatorname{Spec}(k^{sep})$ is not in $Et(k)$. This "problem" will go away next quarter when we discuss the pro-étale topology.

Proof. For $F \in \mathsf{Shv}(Et(k))$ we define

$$X_F = \varinjlim_{k^{sep}/L/k} F(L) \tag{1}$$

as the colimit over all subfields L of k^{sep} which are finite Galois extensions of k.

X_F is a discrete G-set. For any Galois L/k and any $\sigma \in G$ we have $\sigma(L) = L$ so σ restricts to a (finite) automorphism of L/k (and hence an automorphism of $F(L)$) via the canonical map $G \to Gal(L/k) \cong G/Aut(k^{sep}/L)$ where $Aut(k^{sep}/L) = \{g \in G : g(a) = a \ \forall\ a \in L\}$. These actions are compatible with inclusions $L \subseteq L'$ (and hence, the morphisms $F(L) \to F(L')$), hence we get an action of G on X_F. Moreover, every $x \in X_F$ is the image of some $y \in F(L)$, so X_F is a discrete G-set. The assignment $F \mapsto X_F$ is clearly natural in F, that is, it defines a functor.

For future reference, we note that since F is an étale sheaf, for each extension L'/L, the morphism $F(L) \to F(L')$ is injective, and moreover, for any two Galois extensions $L'/L/k$ of k, by Exercise 5 we have $F(L) = F(L')^{Aut(L'/L)}$. Since the action of G commutes with the colimit (1), we get

$$\begin{aligned} X_F^{Aut(k^{sep}/L)} = \varinjlim_{k^{sep}/L'/L/k} F(L')^{Aut(k^{sep}/L)} &= \varinjlim_{k^{sep}/L'/L/k} F(L')^{Aut(L'/L)} \\ &= \varinjlim_{k^{sep}/L'/L/k} F(L) = F(L). \end{aligned}$$

Now suppose we have a discrete G-set X. Recall that every étale k-algebra is of the form $\prod_{i=1}^n L_i$ for some finite separable field extensions L_i. We define a presheaf on $Et(k)$ as

$$F_X(U) = \hom_G\Big(\hom_{\operatorname{Spec}(k)}\big(\operatorname{Spec}(k^{sep}), U\big), X\Big),$$

[3]Here *discrete* means that for every $x \in X$, there is a finite Galois extension L/k with stabiliser $Stab(L) \subseteq G$ such that $x \in X^{Stab(L)}$.

[4]That is, a set X equipped with an action of G.

where $\hom_G$ means G-equivariant morphisms, and $G = Gal(k^{sep}/k) = \hom_k(k^{sep}, k^{sep})$ acts on $\hom_{\mathrm{Spec}(k)}(\mathrm{Spec}(k^{sep}), U)$ by composition.

F_X is an étale sheaf. Cf. Milne, Lem.I.1.8. By Remark 7, to show F_X is a sheaf, it suffices to check that

$$F_X(L) = F_X(L')^{Aut(L'/L)}$$

for finite Galois extensions L'/L. Note that for any Galois extension L'/k and any subextension $L'/L/k$ we have

$$\hom_k(L', k^{sep})_{Aut(L'/L)} \xrightarrow{\sim} \hom_k(L, k^{sep}).$$

it follows from this that $F_X(L) = F_X(L')^{Aut(L'/L)}$. Note that for any finite Galois subextension $k^{sep}/L/k$ we have $\hom_{\mathrm{Spec}(k)}(\mathrm{Spec}(k^{sep}), \mathrm{Spec}(L)) = \mathrm{Gal}(L/k)$. So

$$F_X(L) = \hom_G(\mathrm{Gal}(L/k), X) = X^{Aut(k^{sep}/L)}. \tag{2}$$

Combining (1) and (2) we get

$$X_{F_X} = \varinjlim_L F_X(L) = \varinjlim_L X^{Aut(k^{sep}/L)} = X.$$

On the other hand, by (2) we get

$$F_{X_F}(L) = X_F^{Aut(k^{sep}/L)} = F(L)$$

for Galois extensions L/k. Then by Exercise 5 we have $F_{X_F} = F$.

So the assignements $X \mapsto F_X$ and $F \to X_F$ are inverse equivalences. □

Exercise 6 (Omitted from lecture)**.** Suppose that F is a presheaf on a category $\mathcal{C}$ equipped with a Grothendieck topology. Suppose that $\{U_i \to U\}_{i \in I}$ and $\{U_{ij} \to U_i\}_{j \in J_i}$ are coverings. Using the diagram

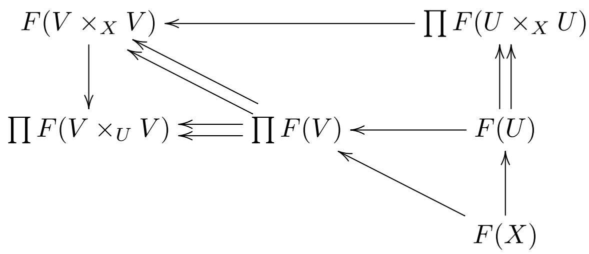
$$\begin{array}{ccccc} F(U) & \longrightarrow & \prod F(U_i) & \rightrightarrows & \prod F(U_i \times_U U_{i'}) \\ \| & & \downarrow & & \downarrow \\ F(U) & \longrightarrow & \prod F(U_{ij}) & \rightrightarrows & \prod F(U_{ij} \times_U U_{i'j'}) \end{array}$$

show that if F satisfies the sheaf condition (III) for $\{U_{ij} \to U\}_{i \in I, j \in J_i}$ and each $F(U_i) \to \prod_{J_i} F(U_{ij})$ is injective, then F satisfies the sheaf condition for $\{U_i \to U\}_{i \in I}$.

Deduce that a presheaf F on $LH(X)$ from Exercise 2 is a presheaf if and only if $F|_{Op(Y)}$ is a sheaf on $Op(Y)$ from Exercise 1 for every $Y \in LH(X)$.

Exercise 7 (Omitted from lecture)**.** Suppose that F is a presheaf on a category $\mathcal{C}$ equipped with a Grothendieck topology. Suppose that $\{V \to U\}$ and $\{U \to$

$X\}$ are coverings consisting of single morphisms. Using the diagram



show that if F satisfies the sheaf condition (Ш) for $\{V \to U\}$ (cf.middle row) and $\{U \to X\}$ (cf.right column), and each $F(U \times_X U) \to F(V \times_X V)$ is injective (cf. top row), then F satisfies the sheaf condition for $\{V \to X\}$ (cf. diagonal).

Exercise 8 (Advanced. Omitted from lecture)**.** Do Exercise 7 for coverings $\{U_i \to X\}_{i \in I}$ and $\{V_{ij} \to U_i\}_{j \in J_i}$ containing more than one element.

Exercise 9 (Advanced)**.** Let X be a scheme. Deduce from Exercises 6 and Exercise 8 that a presheaf F on $Et(X)$ is a sheaf if and only if $F|_{Op(Y)}$ is a sheaf for every $Y \in Et(X)$, and F satisfies the sheaf condition (Ш) for every covering $\{Y_i \to Y\}_{i \in I}$ such that Y and each Y_i are affine schemes.

Exercise 10. Recall that for any faithfully flat ring morphism $A \to B$ the sequence $0 \to A \to B \to B \otimes_A B$ is exact. Deduce from this and Exercise 9 that for any scheme X and any affine scheme T, the presheaf $\hom(-, T)$ is a sheaf on $Et(X)$. (Actually, its also true without the affine hypothesis, and for the category $Fppf(X)$).

Corollary 10. *The following representable presheaves are étale sheaves.*

1. $\hom(-, \mathbb{A}^1)$*;* $X \mapsto \Gamma(X, \mathcal{O}_X)$*,*
2. $\hom(-, \mathbb{G}_m)$*;* $X \mapsto \Gamma(X, \mathcal{O}_X^*)$*,*
3. $\mu_n = \hom(-, \operatorname{Spec}(\frac{\mathbb{Z}[T]}{T^n - 1}))$*;* $X \mapsto \{a \in \Gamma(X, \mathcal{O}_X^*) : a^n = 1\}$*,*
4. $GL_n = \hom\left(-, \operatorname{Spec}\left(\frac{\mathbb{Z}[U, T_{ij} : 1 \leq i,j \leq n]}{U \cdot \det T_{ij} - 1}\right)\right)$*;* $X \mapsto GL_n(\Gamma(X, \mathcal{O}_X))$*,*

2 Sheafification

Definition 11. *A presheaf F on a category equipped with a Grothendieck topology is called* separated *if the morphism $F(U) \to \prod_{i \in I} F(U_i)$ is injective for every covering $\{U_i \to U\}_{i \in I}$.*

Remark 12. Every sheaf is separated.

Exercise 11. Suppose that $\mathcal{C}$ is a category equipped with a Grothendieck topology, and let F be a presheaf. For $U \in \mathcal{C}$ define $F^s(U)$ as the quotient group

$$F^s(U) = F(U) \Big/ \bigcup \ker \left(F(U) \to \prod_{i \in I} F(U_i) \right)$$

where the union is over all covering families $\{U_i \to U\}_{i \in I}$. Show that for any morphism $V \to U$ in $\mathcal{C}$, there is an induced morphism $F^s(U) \to F^s(V)$, that is, F^s is a presheaf. Show that F^s is separated. Show that if $F \to G$ is any morphism from F to a separated presheaf G, there exists a unique factorisation $F \to F^s \to G$. In particular, this is true for every sheaf G.

Proposition 13. *Let $\mathcal{C}$ be a category equipped with a Grothendieck topology. For every presheaf F on $\mathcal{C}$, there exists a universal morphism $F \to F^a$ to a sheaf. That is, a morphism towards a sheaf such that for any other morphism $F \to G$ towards a sheaf, there is a unique factorisation $F \to F^a \to G$.*

In other words, the (fully faithful) inclusion $\mathsf{Shv}(\mathcal{C}) \to \mathsf{PreShv}(\mathcal{C})$ *admits a left adjoint* $(-)^a : \mathsf{PreShv}(\mathcal{C}) \to \mathsf{Shv}(\mathcal{C})$.

Proof. By Exercise 11 it suffices to consider the case that F is separated. For $U \in \mathcal{C}$ define

$$\check{H}^0(U, F) = \varinjlim \operatorname{eq} \left(\prod_{i \in I} F(U_i) \rightrightarrows \prod_{i,j \in I} F(U_i \times_U U_j) \right).$$

Omitted from lecture: Note that this is functorial in F, and if F is a sheaf we have $\check{H}^0(U, F) = F(U)$ by the sheaf condition. It follows from this (with a little bit of work) that we get a unique factorisation $F \to \check{H}^0 F \to G$ for any sheaf G. So it suffices to show that $\check{H}^0 F$ is a sheaf. For simplicity we assume that F is a sheaf of abelian groups, and all covers have a single element. The general case is the same proof, just more confusing chasing indices around.

So suppose that $\{V \to U\}$ is a covering of U. We want to show that

$$0 \to \check{H}^0(U, F) \to \check{H}^0(V, F) \to \check{H}^0(V \times_U V, F)$$

is exact. Let $(U', s \in F(U'))$ represent an element of $\check{H}^0(U, F)$ and suppose that it gets sent to zero in V. Putting in the definitions, we see that this means that there is a refinement $V' \to V \times_U U' \to V$ of the covering $V \times_U U' \to V$ such that $s|_{V'} = 0$. But this is also a refinement of $\{U' \to U\}$, so $(U', s \in F(U'))$ and $(V', 0 \in F(V'))$ represent the same element of $\check{H}^0(U, F)$.

Showing exactness in the middle is fiddly and not very informative, so we omit it. It can be found in [Artin, Grothendieck topologies, 1962, Lemma.2.1.2(ii)]. □

Definition 14. *The sheaf F^a in Proposition 13 is called the* sheafification *or associated sheaf of F.*

Corollary 15. *Let $\mathcal{C}$ be a category equipped with a Grothendieck topology. Then the category $\mathsf{Shv}(\mathcal{C}, Ab)$ of sheaves of abelian groups is an abelian category.*

Sketch of proof. Limits (i.e., products and kernels) can be calculated sectionwise. E.g., $\ker(F \to G)(U) = \ker(F(U) \to G(U))$. Colimits (i.e., sums and cokernels) are calculated sectionwise, and then sheafified. E.g., the sheaf cokernel of $F \to G$ is the sheafification of the presheaf $U \mapsto \operatorname{coker}(F(U) \to G(U))$. $\square$

3 Stalks

Definition 16. *A* geometric point *of a scheme X is a morphism $\overline{x} \to X$ such that $\overline{x} = \operatorname{Spec}(\Omega)$ for some separably closed field Ω.*

Definition 17. *Let F be a presheaf on $Et(X)$. For a geometric point $\overline{x} \to X$ we define the* stalk *at $\overline{x}$ as*

$$F_{\overline{x}} = \varinjlim_{\overline{x} \to Y \to X} F(Y)$$

where the colimit is over factorisations of $\overline{x} \to X$ via some $Y \in Et(X)$.

Remark 18. If X is a topological space, F is a sheaf on X, and $x \in X$ is a point, then classically, the stalk of F at x is defined as the colimit

$$F_x = \varinjlim_{x \in U \subseteq X} F(U)$$

over open subsets of X containing x. The above definition is the étale analogue of this classical definition.

Remark 19. If F is a presheaf defined on all schemes that commutes with filtered colimits, then $F_{\overline{x}} = F(\mathcal{O}^{sh}_{X,x})$ where $x = \operatorname{im}(\overline{x}) \in X$ and $\mathcal{O}^{sh}_{X,x}$ is the strict henselisation of $\mathcal{O}_{X,x}$ defined by the separably closed extension $k(\overline{x})/k(x)$. In particular, if $F = \mathcal{O} : Y \mapsto \Gamma(Y, \mathcal{O}_Y)$, then $F_{\overline{x}} = \mathcal{O}^{sh}_{X,x}$.

Remark 20. If k^{sep}/k is a separable closure, then $\overline{x} = \operatorname{Spec}(k^{sep}) \to \operatorname{Spec}(k)$ is a geometric point, and $F_{\overline{x}}$ is the G-set X_F defined above.

Proposition 21. *Suppose that F is a sheaf of abelian groups on $Et(X)$ and $Y \in Et(X)$. Then a section $s \in F(Y)$ is zero if and only if for any geometric point $\overline{x} \to Y$ its image in each $F_{\overline{x}}$ is zero.*

Proof. Since all sheaves are separated, it suffices to show that for every $s \in F(Y)$, there exists a covering $\{U_i \to Y\}_{i \in I}$ such that $s|_{U_i} = 0$ for all $i \in I$. For every point $x \in Y$, choose a separable closure $k(x)^s/k(x)$, and let $\overline{x} \to X$ be the corresponding geometric point. Since the image of s in $F_{\overline{x}}$ is zero, there is some $\overline{x} \to V \to Y$ such that $s|_V = 0$. Since V is associated to x, let us write $V_x = V$. We do this for every point $x \in Y$, and obtain a family $\{V_x \to Y\}_{x \in Y}$ of étale morphisms indexed by points of Y. Since $x \in \operatorname{im}(V_x \to Y)$ for each $x \in Y$, the family is surjective, and therefore is a covering. By construction $s|_{Y_x} = 0$ for each Y_x, so $s = 0$. $\square$

Corollary 22. *A sheaf of abelian groups F on $Et(X)$ is zero if and only if $F_{\overline{x}} = 0$ for each $x \in X$.*

Proof. (Omitted from lecture). We want to show that $s = 0$ for every $Y \in Et(X)$, $s \in F(Y)$. By Proposition 21, it suffices to show that $F_{\overline{x}} = 0$ for every geometry point $\overline{x} \to Y$. We claim that $F_{\overline{x}\to Y} = F_{\overline{x}\to Y\to X}$. Indeed, there is a canonical morphism

$$F_{\overline{x}\to Y\to X} = \varinjlim_{\overline{x}\to U\to X} F(U) \to \varinjlim_{\overline{x}\to V\to Y} F(V) = F_{\overline{x}\to Y}$$

defined by sending a representative $(U, s \in F(U))$ to $(U\times_Y V, s|_{U\times_Y V})$. Injectivity is straight-forward. For surjectivity, note that any representative $(V, s \in F(V))$ of $F_{\overline{x}\to Y}$ can be considered as a representative of an element $s' \in F_{\overline{x}\to Y\to X}$. Then due to the factorisation $V \to V \times_X Y \to Y$, the image of s' is precisely the element represented by $(V, s \in F(V))$. □

Corollary 23. *A morphism of sheaves of abelian groups $\phi : F \to G$ is a monomorphism, (resp. epimorphism, resp. isomorphism) if and only if $\phi_{\overline{x}} : F_{\overline{x}} \to G_{\overline{x}}$ is for each geometric point $\overline{x} \to X$.*

Proof. (Omitted from lecture). Since the definition of $(-)_{\overline{x}}$ is defined by a filtered colimit, it commutes with kernels and cokernels. Applying Corollary 22 to $\ker\phi$ and $\operatorname{coker}\phi$ gives the result. □