

In this talk we define the pro-étale site of a scheme, giving a number of examples of pro-étale schemes. We discuss the case of a field in detail, and in particular, mention the equivalence of categories

$$G\text{-ProFinSet} \cong \text{Spec}(k)_{\text{proét}}^{\text{aff}}.$$

We see that in general the pro-étale topos is locally weakly contractible, and therefore is replete, and left complete. Finally, we observe that the pro-étale topos gives a good setting to study the cohomology of compactly generated topological abelian groups. In particular, for a large class of “nice” topological groups, continuous cohomology agrees with the pro-étale cohomology

$$H_{\text{cont}}^n(G, M) \cong H_{\text{proét}}^n(G\text{-ProFinSet}, F_M).$$

1 The pro-étale site

Recall that a morphism of schemes $f : Y \rightarrow X$ of finite presentation is étale, if it is flat and the diagonal $Y \rightarrow Y \times_X Y$ is flat.¹

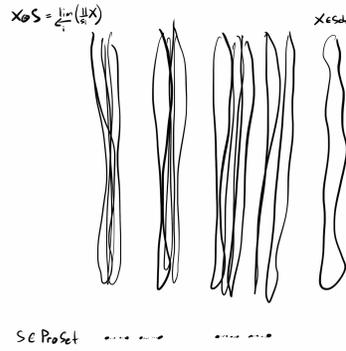
Definition 1 (Def.4.1.1). *A map $f : Y \rightarrow X$ of schemes is weakly étale if it is flat, and the diagonal $\Delta : Y \rightarrow Y \times_X Y$ is flat. The category of weakly étale X -schemes is denoted $X_{\text{proét}}$.*

Example 2.

1. Suppose $A \rightarrow B$ is an ind-étale morphism of rings (so $\text{Spec}(B) \rightarrow \text{Spec}(A)$ is pro-étale). Then we saw previously that $A \rightarrow B$ is a weakly étale morphism of rings [Prop.2.3.3], so $\text{Spec}(B) \rightarrow \text{Spec}(A)$ is a weakly étale morphism of schemes.
2. Bhatt, Scholze choose to work with weakly étale maps instead of pro-étale morphisms of schemes in general because pro-étale morphisms of schemes are not so well-behaved, cf. [Exa.4.1.12] reproduced below.
3. [Exa.4.1.9] Given a scheme X , and a profinite set $S = \varprojlim S_i$, the morphism $X \otimes S := \varprojlim_{i \in I} (\sqcup_{s \in S_i} X) \rightarrow X$ is pro-étale. This defines a functor

$$\text{ProFinSet} \times X_{\text{proét}} \rightarrow X_{\text{proét}}; \quad (S, Y) \mapsto S \otimes Y$$

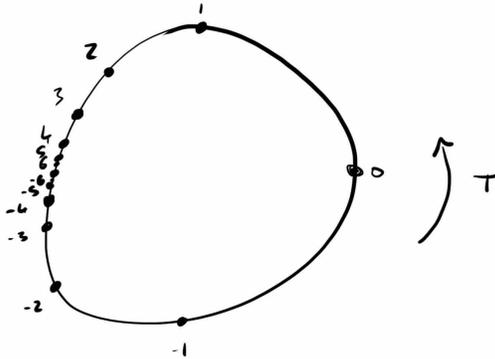
¹The diagonal being flat is one of a number of equivalent definitions for a finite presentation morphism to be unramified.



4. [Exa.4.1.12] Consider the set²

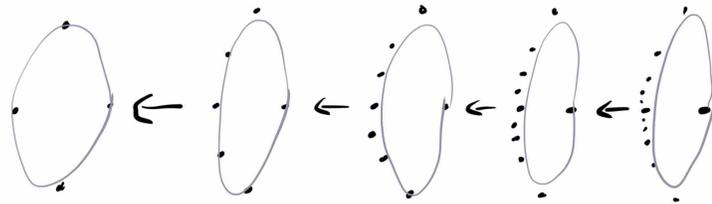
$$S = \{e^{\pi i(1-\frac{1}{2^n})} : n \in \mathbb{Z}, n \geq 0\} \cup \{e^{\pi i(2^n-1)} : n \in \mathbb{Z}, n \leq 0\} \cup \{-1\} \subseteq \mathbb{C}$$

equipped with the translation function T induced by $n \mapsto n + 1$ on the image of \mathbb{Z} , and sending $-1 \in \mathbb{C}$ to -1 .

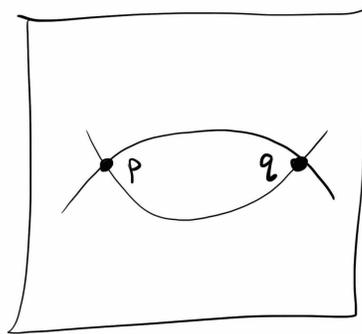


Note that this is a profinite set

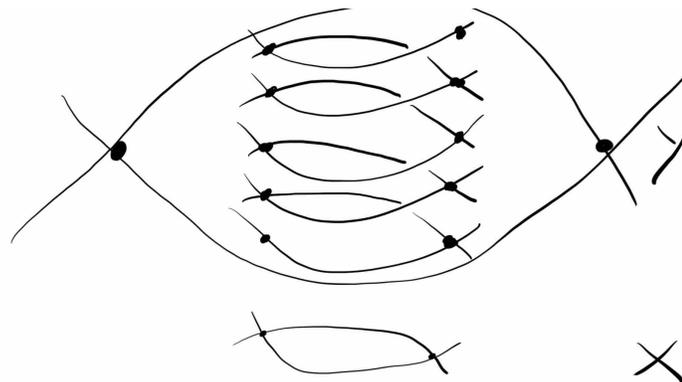
²I.e., the one point compactification of \mathbb{Z} considered as a discrete set.



Now let $X_1, X_2 \subseteq \mathbb{A}_{\mathbb{C}}^2$ be two smooth curves meeting transversally at points p and q , and $X = X_1 \cup X_2$.



Then consider the X -scheme Y which is $S \otimes X_1$ glued to $S \otimes X_2$ using the identity at p and the translation function T at q .



Then $Y \rightarrow X$ is locally pro-étale. Indeed, away from p (or q), it is just $S \otimes (X \setminus \{p\}) \rightarrow (X \setminus \{p\})$. However, $Y \rightarrow X$ is not globally pro-étale.

By closely considering the topology on Y , one can see that it cannot be written as $Y = \varprojlim Y_i$ for étale X schemes Y_i .

5. [Exa.4.1.4] If k is a field, then a morphism $\mathrm{Spec}(R) \rightarrow \mathrm{Spec}(k)$ is weakly étale if and only if $k \rightarrow R$ is ind-étale.³
6. [Exa.4.1.5] For any scheme X , point $x \in X$, and geometric point $\bar{x} \rightarrow X$, the morphisms

$$\mathrm{Spec}(\mathcal{O}_{X,x}) \rightarrow X, \quad \mathrm{Spec}(\mathcal{O}_{X,x}^h) \rightarrow X, \quad \mathrm{Spec}(\mathcal{O}_{X,\bar{x}}^h) \rightarrow X$$

are all weakly étale.

Recall that flat morphisms are preserved by base change and composition.

Exercise 1 ([Lem.4.1.6]). *Weakly étale morphisms are preserved by base change.* Show that if $f : Y \rightarrow X$ is weakly étale then $X' \times_X Y \rightarrow X'$ is weakly étale for any morphism $X' \rightarrow X$.

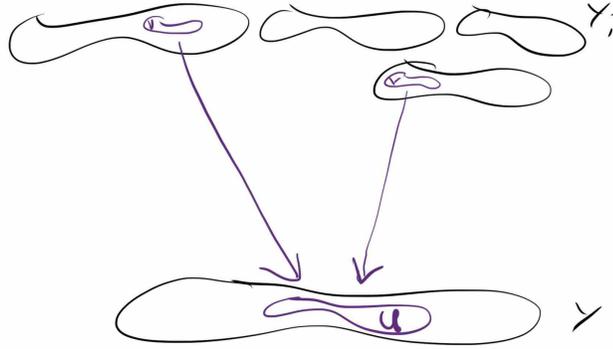
Exercise 2 ([Lem.4.1.6], [Lem.4.1.7]). *Weakly étale morphisms are preserved by composition, and all morphisms in $X_{\mathrm{proét}}$ are weakly étale.* Let $g : W \rightarrow Y$ and $f : Y \rightarrow X$, $f' : Y' \rightarrow X$ be weakly étale morphisms, and $h : Y' \rightarrow Y$ any X -morphism.

1. Use the fact that $W \cong (Y \times_X W) \times_{(Y \times_X Y)} Y$ to show that $W \rightarrow Y \times_X W$ is flat.
2. Use part (1), the fact that $Y \times_X W \times_X W \cong (Y \times_X W) \times_Y (Y \times_X W)$, and a clever factorisation of $W \rightarrow W \times_X W$ to show that $f \circ g : W \rightarrow X$ is weakly étale.
3. As in part (1), show that $Y' \rightarrow Y \times_X Y'$ is flat.
4. Use part (3) and the fact that $Y' \cong (Y' \times_Y Y') \times_{(Y' \times_X Y')} (Y')$ to show that $Y' \rightarrow Y$ is pro-étale.

Exercise 3 ([Lem.4.1.8]). Use Exercise 1 and Exercise 2 to show that $X_{\mathrm{proét}}$ has fibre products. Deduce that $X_{\mathrm{proét}}$ has all finite limits. (Recall, that an exercise earlier in the course was to show that a category has all finite limits if and only if it has fibre products and a terminal object).

Definition 3. *A family $\{Y_i \rightarrow Y\}_{i \in I}$ in $X_{\mathrm{proét}}$ is a covering, if for every open affine $U \subseteq Y$, there is a finite subset $J \subseteq I$ and open affines $V_j \subseteq Y_j$ for each $j \in J$ such that $\coprod_{j \in J} V_j \rightarrow U$ is surjective.*

³By [BS, Thm.2.3.4], for every weakly étale morphism $A \rightarrow B$ there is a faithfully flat ind-étale morphism $B \rightarrow C$ such that $A \rightarrow C$ is ind-étale. In particular, B is a sub- A -algebra of an ind-étale A -algebra. But for fields k , every sub- k -algebra B of an ind-étale k -algebra C is again an ind-étale k -algebra: Indeed, write $C = \varinjlim_{i \in I} C_i$ where the C_i are étale k -algebras, and recall that this means that each C_i is a finite product of finite separable k -field extensions. Replacing each C_i with its image in C , we can assume all morphisms $C_i \rightarrow C$ are injective. Then taking $B_i = C_i \cap B$, we produce a system $(B_i)_{i \in I}$ such that each B_i is an étale k -algebra and $B = \varinjlim_{i \in I} B_i$.



The finiteness in the above definition is important, and affects the topology:

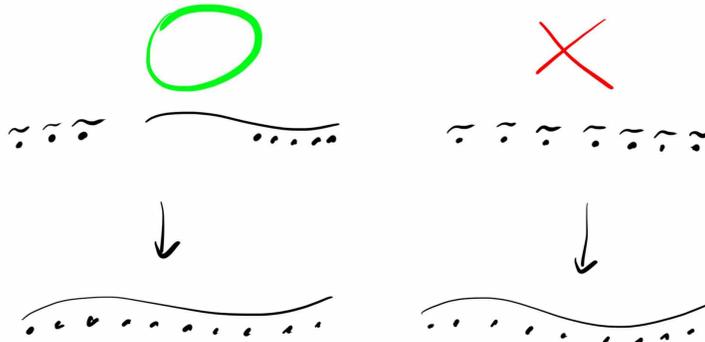
Example 4 ([Exa.4.1.13]). Consider $\text{Spec}(\mathbb{Z})$. If p_1, \dots, p_n are finitely many primes, then

$$\left\{ \text{Spec}(\mathbb{Z}_{(p_1)}^{sh}), \dots, \text{Spec}(\mathbb{Z}_{(p_n)}^{sh}), \text{Spec}(\mathbb{Z}[\frac{1}{p_1}, \dots, \frac{1}{p_n}]) \right\}$$

is a weakly étale cover. However,

$$\left\{ \text{Spec}(\mathbb{Z}_{(p)}^{sh}) : p \text{ is prime} \right\}$$

is *not* a weakly étale cover.



2 The pro-étale site of a field

Example 5. In this example we study affine pro-étale schemes over a separably closed field in great detail, giving a concrete description of them as locally ringed spaces.

Suppose that k is a separably closed field, and R is an ind-étale algebra

$$k \rightarrow R = \varinjlim R_\lambda.$$

So each R_λ is a *finite* product $R_\lambda = \prod_{i \in I_\lambda} k$. Moreover, every morphism $\prod_{i \in I_\lambda} k = R_\lambda \rightarrow R'_\lambda = \prod_{j \in I'_\lambda} k$ is induced by morphisms of sets $\phi_{\lambda, \lambda'} : I'_\lambda \rightarrow I_\lambda$. Since the underlying topological space of Spec of a filtered colimit of rings is the inverse limit of the underlying topological spaces, [EGAIV, §8], we see that the underlying topological space of $\text{Spec}(R)$ is the profinite set $I = \varprojlim I_\lambda$.

$$\text{Spec}(R)_{\text{top}} = I.$$

For any profinite set $\varprojlim S_\lambda$, and any discrete set X , one can see that we have⁴ $\text{hom}_{\text{cont.}}(\varprojlim S_\lambda, X) = \varprojlim \text{hom}(S_\lambda, X)$. Hence,

$$\Gamma(\text{Spec}(R), \mathcal{O}_{\text{Spec}(R)}) = \text{hom}_{\text{cont.}}(I, k)$$

is the set of continuous morphisms where k is given the discrete topology. Moreover, for any open subset of the form $U_{\lambda, i} = \phi_\lambda^{-1}(i)$ where $i \in I_\lambda$ and $\phi_\lambda : I \rightarrow I_\lambda$ is the canonical projection, $U_{\lambda, i}$ is again ind-étale, and so $\Gamma(U_{\lambda, i}, \mathcal{O}_{\text{Spec}(R)}) = \text{hom}_{\text{cont.}}(U_{\lambda, i}, k)$. Finally, if $U \subseteq I$ is any open subset, and $\{U_i \subset U\}$ an open covering, we have

$$\Gamma(U, \mathcal{O}_{\text{Spec}(R)}) = \text{Eq}\left(\prod_{i \in I} \Gamma(U_i, \mathcal{O}_{\text{Spec}(R)}) \rightrightarrows \prod_{i, j \in I} \Gamma(U_i \cap U_j, \mathcal{O}_{\text{Spec}(R)})\right)$$

By definition, every open of I is covered by opens of the form $U_{\lambda, i}$ so we deduce that in general,

$$\Gamma(U, \mathcal{O}_{\text{Spec}(R)}) = \text{hom}_{\text{cont.}}(U, k).$$

Given any point $x \in I$, and any open $x \in U$, there is a smaller open containing x of the form $U_{\lambda, i}$. For any continuous function $f : U \rightarrow k$, there is a refinement $x \in U_{\lambda, i} \subseteq U$. But $U_{\lambda, i}$ is profinite, and therefore quasicompact, so the image $f(U_{\lambda, i})$ is finite, so there is a further refinement $x \in V \subseteq U_{\lambda, i}$ such that $f : V \rightarrow k$ is constant. It follows that all local rings of $\text{Spec}(R)$ are isomorphic to k .

$$\mathcal{O}_{\text{Spec}(R), x} \cong k.$$

Conversely, if (X, \mathcal{O}_X) is a locally ringed space such that $X = \varprojlim X_\lambda$ is profinite, and $\mathcal{O}_X(U) = \text{hom}_{\text{cont.}}(U, k)$ for some separable closed field k , then $(X, \mathcal{O}_X) \cong \text{Spec}(\varprojlim \prod_{X_\lambda} k)$, i.e., (X, \mathcal{O}_X) is the affine scheme associated to an ind-étale k -algebra.

Proposition 6. *Suppose k is a separable closed field and $X \in \text{Spec}(k)_{\text{proét}}$. The following are equivalent.*

1. X is affine.

⁴ If $f : \varprojlim S_\lambda \rightarrow X$ is a continuous morphism, then for every point $x \in X$, $f^{-1}x$ is open, and by definition of the limit topology, admits a covering of the form $\mathcal{U}_x = \{U_{\lambda, s} : \lambda \in \Lambda_x, s \in S_\lambda\}$ where $U_{\lambda, s} = \phi^{-1}(s)$ is the preimage of s under the canonical projection $\phi : S \rightarrow S_\lambda$. Since S is profinite, it is quasicompact, so the covering family $\cup_{x \in X} \mathcal{U}_x$ admits a finite subcovering $\{V_{\lambda_i, s_i} : 1 \leq i \leq n, s_i \in S_{\lambda_i}\}$. Then by construction any λ with $\lambda \leq \lambda_1, \dots, \lambda_n$, has the property that f is constant on the fibres of $S \rightarrow S_\lambda$. Hence, f factors as $S \rightarrow S_\lambda \rightarrow X$.

2. X is the spectrum of an ind-étale algebra.
3. X is qcqs.
4. X is of the form $\text{Spec}(k) \otimes S$ for a profinite set S .

Proof. (1 \iff 2) We have seen, Exa.2(5) that the affine schemes in $\text{Spec}(k)_{\text{proét}}$ are precisely the spectra of ind-étale k -algebras.

(2 \iff 3) All affine schemes are qcqs, so consider the other direction. Suppose that X is qcqs. A scheme is qcqs if and only if it admits a finite open affine cover $\{U_i \rightarrow X\}_{i=1}^n$ such that each $U_i \cap U_j$ for $1 \leq i, j \leq n$ is also affine. Since affines in $\text{Spec}(k)_{\text{proét}}$ have profinite underlying topological space (i.e., compact, Hausdorff, totally disconnected topological space), it follows that any qcqs X also has profinite underlying topological space (see the lemma below). Moreover, the structure sheaf of X has the form $V \mapsto \text{hom}_{\text{cont.}}(V, k)$ since those of the U_i and $U_i \cap U_j$ have this form. Hence, it follows from Example 5 that if X is qcqs, it is the spectrum of an ind-étale algebra.

(2 \iff 4) This follows from the definition of $- \otimes S$ and the equivalence between the category of finite sets and the category of étale k -algebras. \square

Lemma 7. *Suppose that X is a topological space admitting a finite open cover $\{U_i \rightarrow X\}_{i=1}^n$ such that all U_i and $U_i \cap U_j$ are compact, Hausdorff, totally disconnected topological spaces. Then show that X is also compact, Hausdorff, and totally disconnected.*

Proof. X is compact: Suppose that $\{V_j \rightarrow X\}_{j \in J}$ is an open covering. then each $\{V_j \cap U_i\}$ is an open covering. But each U_i is compact, so for each $i = 1, \dots, n$, there is a finite subset $J_i \subseteq J$ such that $U_i = \cup_{j \in J_i} U_i \cap V_j$. It follows that $X = \cup_{i=1}^n \cup_{j \in J_i} V_j$.

X is Hausdorff: Suppose that $x \neq y \in X$ are two points. Choose i_x, i_y such that $x \in U_{i_x}$ and $y \in U_{i_y}$ and set $U_x = U_{i_x}, U_y = U_{i_y}, U_{xy} = U_{i_x} \cap U_{i_y}$. If, say, $y \in U_{xy} \subseteq U_x$, then since U_x is Hausdorff, we can find opens $x \in V, y \in W$ such that $V \cap W = \emptyset$. So suppose that $x, y \notin U_{xy}$. Since U_{xy} is compact, and U_x, U_y are both profinite, U_{xy} is both closed and open in both U_x and U_y . In particular, $V = (U_x \setminus U_{xy}) \subseteq U_x$ and $W = (U_y \setminus U_{xy}) \subseteq U_y$ are also both closed and open in U_x, U_y respectively. This means that V and W are both open in X . By construction, $x \in V$ and $y \in W$ and $V \cap W = \emptyset$, so we are done.

X is totally disconnected: First recall that a subset $W \subseteq X$ is open (resp. closed) if and only if $W \cap U_i$ is open (resp. closed) for all i . Let us write $Y \subseteq W$ to indicate that Y is both open and closed in W . Suppose that $W \subseteq X$ is a subset containing more than one point. We want to find a proper nonempty $Y \subseteq W$. If $W \cap U_i$ has a single point, say w , for some i , then $\{w\}$ is open in W . But all U_i are totally disconnected, so $\{w\}$ is closed in all U_i , and therefore closed in X , and therefore closed in W . Hence, $Y = \{w\} \subseteq W$ works.

So suppose each $W \cap U_i$ has more than one point. Since the U_i are totally disconnected, for each i there is some proper nonempty $Y_i \subseteq W \cap U_i$. For any other j , we then have that $Y_i \cap U_j \subseteq (W \cap U_i) \cap U_j$. Now as above, since $U_i \cap U_j$ is quasicompact, $U_i \cap U_j \subseteq U_j$, so, $W \cap U_i \cap U_j \subseteq W \cap U_j$, and we find that in

fact, $Y_i \cap U_j \in W \cap U_i \cap U_j \in W \cap U_j$. Now define T_i inductively by setting $T_0 = W$. If one of $T_{i-1} \cap Y_i$ or $T_{i-1} \cap (W \cap U_i \setminus Y_i)$ are nonempty then choose one and set T_i to be this nonempty intersection. If both are empty, then define $T_i^a = T_{i-1}^a$. Now note that since $Y_i \cap U_j \in W \cap U_j$ for every i, j , it follows that each $T_i \in W \cap U_j$ for every $1 \leq j \leq i$. In particular, $T_n \in W \cap U_j$ for all j , and therefore $T_n \in W$. It is nonempty and proper by construction. \square

Write $\text{Spec}(k)_{\text{proét}}^{\text{aff}}$ for the fullsubcategory of $\text{Spec}(k)_{\text{proét}}$ of those objects satisfying the equivalent conditions of the previous lemma.

Corollary 8. *If k is a separably closed field, there is an equivalence of categories*

$$\begin{aligned} \text{ProFinSet} &\cong \text{Spec}(k)_{\text{proét}}^{\text{aff}} \\ S &\mapsto \text{Spec}(k) \otimes S \\ X(k) &\leftarrow X \end{aligned}$$

Under this identification, coverings of $\text{Spec}(k) \otimes S$ are precisely the jointly surjective families of profinite sets $\{S_i \rightarrow S\}_{i \in I}$ that admit a jointly surjective finite subfamily $\{S_{i_j} \rightarrow S\}_{j=1}^n$.

Example 9. If S is any nonfinite profinite set then the family $\{s \rightarrow S\}_{s \in S}$ of inclusions of its points is *not* a covering family.

The following is basically a version of the equivalence we saw in Galois theory between étale k -algebras and finite G -sets.

Proposition 10. *Let k be any field, choose a separable closure k^{sep}/k , and let $G = \text{Gal}(k^{sep}/k)$. There is an equivalence of categories between profinite sets equipped with a continuous G -action and the affine objects in $\text{Spec}(k)_{\text{proét}}$.*

$$G\text{-ProFinSet} \cong \text{Spec}(k)_{\text{proét}}^{\text{aff}}$$

Under this identification, coverings are precisely the jointly surjective families $\{S_i \rightarrow S\}_{i \in I}$ that admit a jointly surjective finite subfamily $\{S_{i_j} \rightarrow S\}_{j=1}^n$.

Sketch of proof. In one direction, we use the functor

$$\text{Spec}(k)_{\text{proét}}^{\text{aff}} \xrightarrow{k^{sep} \otimes_k -} \text{Spec}(k^{sep})_{\text{proét}}^{\text{aff}}$$

and the equivalence

$$\text{Spec}(k^{sep})_{\text{proét}}^{\text{aff}} \cong \text{ProFinSet}$$

The G -action is induced by the canonical G -action on $\text{Spec}(k^{sep})$. In the other direction, given a pro-finite set S equipped with a continuous G -action, we take $\text{Spec}(\text{hom}_{\text{cont}}(S, k^{sep})^G)$, i.e., the spectrum of the ring of those continuous functions which are invariant for the action of G acting via its action on S . \square

3 The pro-étale topos

Definition 11 ([Def.4.2.1]). *Let X be a scheme. An object $U \in X_{\text{proét}}$ is called a pro-étale affine if it is of the form $U = \varprojlim U_i$ for some small filtered diagram $(U_i)_{i \in I}$ of (absolutely) affine schemes $U_i = \text{Spec}(A_i)$ in $X_{\text{ét}}$. The expression $U = \varprojlim U_i$ is called a presentation of U . The full subcategory of $X_{\text{proét}}$ spanned by pro-étale affines is denoted $X_{\text{proét}}^{\text{aff}}$. We make it a site by saying a family in $X_{\text{proét}}^{\text{aff}}$ is a covering in $X_{\text{proét}}^{\text{aff}}$ if it is a covering in $X_{\text{proét}}$.*

Lemma 12. *For X a scheme, every scheme $Y \in X_{\text{proét}}$ admits a pro-étale covering $\{Y_i \rightarrow Y\}$ such that each Y_i is in $X_{\text{proét}}^{\text{aff}}$.*

Proof. Choose an open affine covering $\{\text{Spec}(A_i) \rightarrow X\}_{i \in I}$ of X , and for each i , choose an open affine covering $\{\text{Spec}(B_{ij}) \rightarrow \text{Spec}(A_i) \times_X Y\}_{j \in J_i}$ of the preimage of $\text{Spec}(A_i)$ in Y . Now by [Thm.2.3.4], since the morphisms $A_i \rightarrow B_{ij}$ are weakly étale for each i, j , there is a faithfully flat ind-étale morphism $B_{ij} \rightarrow C_{ij}$ such that $A_i \rightarrow C_{ij}$ is ind-étale. Consequently, $\{\text{Spec}(C_{ij}) \rightarrow Y\}_{i \in I, j \in J_i}$ is a covering of the desired form. \square

Corollary 13 ([Lem.4.2.4, Rem.4.2.5]). *For any scheme X , the canonical restriction functor induces an equivalence of categories of sheaves*

$$\text{Shv}(X_{\text{proét}}) \xrightarrow{\sim} \text{Shv}(X_{\text{proét}}^{\text{aff}}).$$

Proof. This is a general fact about Grothendieck sites. Consider any site (C, τ) and full subcategory $D \subseteq C$ equipped with the induced topology. If every object of C has a covering by objects of D , then there is an equivalence $\text{Shv}(C) \cong \text{Shv}(D)$. \square

Proposition 14 ([Prop.4.2.8]). *For any scheme X , the topos $\text{Shv}(X_{\text{proét}})$ is locally weakly contractible [Def.3.2.1]. In particular, it is replete [Def.3.1.1], and so $D(X_{\text{proét}})$ is left-complete [Def.3.3.1].*

Proof. [Prop.3.2.3] says that a locally weakly contractible topos is replete. [Prop.3.3.3] says that the derived category of a replete topos is left-complete. It suffices to show that for every scheme $Y \in X_{\text{proét}}$ there is a covering $\{Y_i \rightarrow Y\}_{i \in I}$ with $Y_i \in X_{\text{proét}}^{\text{aff}}$ locally weakly contractible. Lemma 12 says that every scheme admits a pro-étale affine covering. So it remains only to see that affine schemes have locally weakly contractible coverings. This was the main result of the Algebra II lecture. \square

On the pro-étale site, one can define interesting “constant” sheaves associated to topological spaces.

Lemma 15 ([Lem.4.2.12]). *Suppose X is a scheme and T is a topological space. Then the presheaf*

$$F_T : X_{\text{proét}}^{\text{op}} \rightarrow \text{Set}; \quad U \mapsto \text{Map}_{\text{cont}}(U, T)$$

which sends a scheme U to the set of continuous maps from the underlying topological space of U to T is a sheaf.

Sketch of proof. This uses [Lem.4.2.6] which we did not do. It says that a presheaf F on $X_{\text{proét}}$ is a sheaf if and only if it satisfies the sheaf condition for Zariski covers, and surjective maps in $X_{\text{proét}}^{\text{aff}}$. In the category of topological spaces, any representable presheaf is a sheaf for the topology generated by usual open coverings of topological spaces, and surjective morphisms $Y \rightarrow X$ such that X has the quotient topology induced from Y . Hence, in our setting, it suffices to check that for any surjective morphism $f : \text{Spec}(B) \rightarrow \text{Spec}(A)$ in $X_{\text{proét}}^{\text{aff}}$ a subset $U \subseteq \text{Spec}(A)$ is open if and only if f^{-1} is open. This is proved in a really neat way using the constructible topology, and the fact that a subset of a scheme is open if and only if it is constructible and closed under generisation. \square

4 Addendum

We did not have time for the following comments. There are of course many more details in Bhatt, Scholze.

Let k be a field, k^{sep} a separable closure, and $G = \text{Gal}(k^{\text{sep}}/k)$. Recall that we had an equivalence of categories

$$\text{Shv}(k_{\text{ét}}, \text{Ab}) \cong G\text{-mod}$$

between the category of étale sheaves on k , and discrete G -modules. A consequence of this was that for any discrete G -module M with associated sheaf F_M , the group cohomology of M is isomorphic to the étale sheaf cohomology of F_M ,

$$H_{\text{ét}}^n(k, F_M) \cong H^n(G, M).$$

The pro-étale site allows us to upgrade this, although things become more technical and complicated.

Recall that we have already seen an equivalence of categories

$$k_{\text{proét}}^{\text{aff}} \cong G\text{-ProFinSet}$$

between the subcategory of affine objects in $k_{\text{proét}}$ and the category of profinite sets equipped with a continuous action. The covering families in the left side are just surjective families.

Definition 16. *Given an arbitrary profinite group G , we define a topology on the category $G\text{-ProFinSet}$ whose covering families are surjective families.*

Definition 17. *Let $G\text{-Spc}$ be the category of topological spaces equipped with a continuous G -action. Let $G\text{-Spc}_{\text{cg}} \subseteq G\text{-Spc}$ be the full subcategory of $X \in G\text{-Spc}$ whose underlying topological space can be written as a quotient of a disjoint union of compact Hausdorff spaces. These spaces are called compactly generated.*

Lemma 18 ([Lem.4.3.2]). *The association $T \mapsto \text{hom}_{\text{cont}, G}(-, T)$ produces a functor $G\text{-Spc} \rightarrow \text{Shv}(G\text{-ProFinSet})$. This functor is fully faithful on $G\text{-Spc}_{\text{cg}}$, admits a left adjoint (everywhere), and its essential image generates $\text{Shv}(G\text{-ProFinSet})$ under colimits.*

Definition 19. *We write $G\text{-Mod}$ for the category of topological abelian groups equipped with a continuous G -action. We write $G\text{-Mod}_{cg}$ for the full subcategory whose underlying space is compactly generated (i.e., lies in $G\text{-Spc}_{cg}$).*

As above, given $M \in G\text{-Mod}$, we get an abelian sheaf $F_M : X \mapsto \text{hom}_{cont,G}(-, M)$ on $G\text{-ProFinSet}$.

We did not define continuous cohomology, but the main result about it is the following.

Lemma 20 ([Lem.4.3.9]). *For a large class of “nice” $M \in G\text{-Mod}$ we have*

$$H_{cont}^n(G, M) \cong H_{\text{proét}}^n(G\text{-ProFinSet}, F_M).$$