

Reference: [BS] Bhatt, Scholze, “The pro-étale topology for schemes”.

In this lecture we consider replete topoi. This is a nice class of topoi that include the pro-étale topoi, in which inverse limits work well. In particular, for any object $K \in D(\mathcal{X})$ in the derived category of a replete topos \mathcal{X} , the chain complex K is the derived inverse limit of its truncations

$$K \cong R\varprojlim \tau^{-n} K,$$

where $\tau^{-n} K = (\cdots \rightarrow 0 \rightarrow 0 \rightarrow K^n/d(K^{n-1}) \rightarrow K^{n+1} \rightarrow K^{n+2} \rightarrow \cdots)$. This allows us to prove things about unbounded chain complexes using bounded below chain complexes (e.g., we will do this in the proof of Proposition 30).

In Section 3 we will see that, for any topos \mathcal{X} , there is a canonical way to complete its derived category with respect to inverse limits $D(\mathcal{X}) \rightarrow \widehat{D}(\mathcal{X})$. In later lectures we will see that in the case $\mathcal{X} = \mathbf{Shv}_{et}$, we have

$$\widehat{D}(\mathbf{Shv}_{et}) = D(\mathbf{Shv}_{proét}).$$

One of the motivations we gave at the beginning of the course for the pro-étale topology was that it gives a better way of constructing the derived categories of l -adic sheaves. Instead of an ad-hoc 2-limit $D(\mathbf{Shv}_{et}(X, \mathbb{Z}_l)) := 2\text{-}\varprojlim_n D(\mathbf{Shv}_{et}(X, \mathbb{Z}/l^n))$, we would like the derived category of l -adic sheaves to be just that: the category of sheaves of \mathbb{Z}_l -modules on a site. If we consider sheaves of \mathbb{Z}_l -modules on the pro-étale site, we get a bigger category than $D(\mathbf{Shv}_{et}(X, \mathbb{Z}_l))$. In Section 4 we discuss the notion of *derived complete* objects in $D(\mathcal{X})$. It is the subcategory $D_{\text{comp}}(\mathbf{Shv}_{proét}(X, \mathbb{Z}_l))$ of derived complete objects which will be equivalent to $D(\mathbf{Shv}_{et}(X, \mathbb{Z}_l))$.

$$2\text{-}\varprojlim_n D(\mathbf{Shv}_{et}(X, \mathbb{Z}/l^n)) \cong D_{\text{comp}}(\mathbf{Shv}_{proét}(X, \mathbb{Z}_l)) \subseteq D(\mathbf{Shv}_{proét}(X, \mathbb{Z}_l)).$$

1 Replete topoi

Definition 1. A topos is a category equivalent to a category of the form $\mathbf{Shv}_\tau(\mathcal{C})$ for some category \mathcal{C} and some Grothendieck topology τ on \mathcal{C} . Given an object $X \in \mathcal{C}$, we write h_X for the sheaf represented by X . I.e., the sheafification of the presheaf $\text{hom}_{\mathcal{C}}(-, X)$.

Example 2. Given a topos \mathcal{X} , the category $\prod_{\mathbb{N}} \mathcal{X}$ of sequences (\dots, F_2, F_1, F_0) of objects in \mathcal{X} is also a topos.¹ The category $\mathcal{X}^{\mathbb{N}}$ of sequences $(\dots \rightarrow F_2 \rightarrow F_1 \rightarrow F_0)$ of morphisms in \mathcal{X} is also a topos.²

Definition 3. Let $\mathcal{X} = \mathbf{Shv}_\tau(\mathcal{C})$ be a topos. A morphism $F \rightarrow G$ of objects of \mathcal{X} is surjective if for every object $X \in \mathcal{C}$ and $s \in G(X)$, there exists a covering $\{U_i \rightarrow X\}_{i \in I}$ such that $s|_{U_i}$ is in the image of $F(U_i) \rightarrow G(U_i)$ for each $i \in I$.

¹It is the category of sheaves on the disjoint union $\coprod_{n \in \mathbb{N}} \mathcal{C}$ equipped with the coarsest topology such that the inclusions $\mathcal{C} \rightarrow \coprod_{n \in \mathbb{N}} \mathcal{C}$ send covers to covers.

²It is the category of sheaves on the category $\mathbb{N} \times \mathcal{C}$ whose objects are pairs (n, X) consisting of an $n \in \mathbb{N}$ and an object $X \in \mathcal{C}$. Morphisms are $\text{hom}((n, X), (m, Y)) = \emptyset$ if $n > m$ and $\text{hom}(X, Y)$ otherwise. Again, the topology is the coarsest topology such that the inclusions $\mathcal{C} \rightarrow \mathbb{N} \times \mathcal{C}$ send covers to covers.

Remark 4. It can be shown that a morphism $F \rightarrow G$ of sheaves is surjective if and only if for every sheaf H the induced map $\text{hom}(G, H) \rightarrow \text{hom}(F, H)$ is injective. I.e., if and only if $F \rightarrow G$ is surjective in the categorical sense. Another equivalent condition for $F \rightarrow G$ to be surjective is asking that $\text{im}(F \rightarrow G) \rightarrow G$ become an isomorphism (of presheaves) after sheafification. Here, by $\text{im}(F \rightarrow G)$ we mean the *presheaf* image, i.e., $\text{im}(F \rightarrow G)(U) = \text{im}(F(U) \rightarrow G(U))$ (this is not necessarily a sheaf).

exer:epi

Exercise 1. Note that:

- (*) For any presheaf F with sheafification aF , object $X \in \mathcal{C}$, and section $s \in aF(X)$, there exists a covering $\{U_i \rightarrow X\}_{i \in I}$ such that each $s|_{U_i}$ is in the image of $F(U_i) \rightarrow aF(U_i)$.

Let $\{V_i \rightarrow Y\}_{i \in I}$ be a covering family in a site \mathcal{C} . Using the axioms of a Grothendieck topology, and (*) show that $\coprod_{i \in I} h_{V_i} \rightarrow h_Y$ is a surjective morphism of sheaves.

Remark 5. In the SGA definition of a covering family, the converse is also true: a family $\{V_i \rightarrow Y\}_{i \in I}$ is a covering family if and only if the induced morphism of sheaves $\coprod_{i \in I} h_{V_i} \rightarrow h_Y$ is a surjective.

Definition 6 ([BS, Def.3.1.1]). A topos is replete (充実した) if for every sequence of surjective morphisms $\cdots \rightarrow F_2 \rightarrow F_1 \rightarrow F_0$ the induced morphisms

$$\begin{array}{ccccccc} \varprojlim_{i \geq 0} F_i & & & & & & \\ & \searrow & & \searrow & & \searrow & \\ & & \cdots & \longrightarrow & F_2 & \longrightarrow & F_1 & \longrightarrow & F_0 \end{array}$$

are surjective for all n .

rema:limits

Remark 7. Note: the inclusion $\text{Shv}_\tau(\mathcal{C}) \subseteq \text{PreShv}(\mathcal{C})$ preserves limits (but not all colimits). That is, $(\varprojlim_{i \in I} F_i)(X) = \varprojlim_{i \in I} (F_i(X))$ for any diagram of sheaves $I \rightarrow \text{Shv}_\tau(\mathcal{C})$ and $X \in \mathcal{C}$ (to calculate colimits of sheaves, one takes the colimit in the category of presheaves and then sheafifies).

Exercise 2.

1. Show that the category of sets is replete. (Note, this is a topos: Set is the category of sheaves on the category $\overset{\text{id}}{*}$ with only one object equipped with the trivial Grothendieck topology).
2. Let \mathcal{C} be a category equipped with the trivial Grothendieck topology,³ so every presheaf is a sheaf. Show that $\text{PreShv}(\mathcal{C})$ is replete.

³I.e., the only covering families are families of the form $\{X \xrightarrow{\text{id}} X\}$.

3. Let G be a (discrete) group. Deduce that the category of G -sets is replete. Note: G -sets is the category of presheaves on the category BG which has one object, one morphism for every element of G , and composition is defined by the multiplication in G .

exam:etNotReplete

Example 8. Let k be a field such that k^{sep}/k is not finite. Then the category $\text{Shv}_{\text{et}}(k)$ of étale sheaves on k is not replete: Since k^{sep}/k is not finite there exists a tower $\dots/L_2/L_1/L_0 = k$ of nontrivial finite separable field extensions. Since each $\text{Spec}(L_n) \rightarrow \text{Spec}(L_{n-1})$ is a covering, each morphism in the tower induces a surjective morphism of sheaves. However,

$$\varprojlim_i h_{\text{Spec}(L_i)} \rightarrow h_{\text{Spec}(k)} \quad (1)$$

equa:idSurjEq

cannot be surjective.

Exercise 3. By evaluating on $X = \text{Spec}(k)$ and considering $s = \text{id}_k$ prove the claim that Example 8(1) is not surjective. Hint: recall that the coverings of $\text{Spec}(k)$ are of the form $\{\text{Spec}(K_j) \rightarrow \text{Spec}(k)\}_{j \in J}$ with K_j/k finite separable field extensions.

Example 9 ([BS, Example 3.1.7]). The category of affine schemes with the fpqc topology⁴ is replete. Suppose $\dots \rightarrow F_2 \rightarrow F_1 \rightarrow F_0$ is a tower of surjective morphisms, and consider some affine scheme $X = \text{Spec}(A)$ and some $s \in F_0(X)$. Since $F_1 \rightarrow F_0$ is surjective, there is a faithfully flat morphism $A \rightarrow B_0$ such that $s|_{B_0}$ is in the image of $F_1(B_0) \rightarrow F_0(B_0)$. That is, there is some $s_1 \in F_1(B_0)$ mapping to $s|_{B_0}$. Repeating the argument, we find a sequence of faithfully flat morphisms $A \rightarrow B_0 \rightarrow B_1 \rightarrow B_2 \rightarrow \dots$ and elements $s_i \in F_i(B_{i-1})$ such that s_i maps to $s_{i-1}|_{B_{i-1}}$. Set $B = \varinjlim B_i$. Now $A \rightarrow B$ is again a faithfully flat morphism, and the sequence $(s_n \in F_n(B_{n-1}))$ induces a sequence $(t_n \in F_n(B))$ such that $t_n \mapsto t_{n-1}$ for all n . In other words, it induces an element $t \in (\varinjlim F_i)(B)$. By construction, $s|_B = t$, and so we deduce that $\varinjlim F_i \rightarrow F_0$ is surjective. The same argument shows each $\varinjlim F_i \rightarrow F_n$ is surjective.

Remark 10. Note that the reason the fpqc site is replete and the étale site is not replete is precisely because limits of coverings exist in the category, and are still coverings.

Our first goal is to show that countable products are exact in replete topoi. This is Proposition 14. Knowing that products are exact, makes derived limits easy to calculate, cf. Remark 16. The first step is the following lemma.

lem:3.1.8

Lemma 11 ([BS, Lem.3.1.8]). *Let*

$$\begin{array}{ccccccc} \dots & \longrightarrow & F_2 & \longrightarrow & F_1 & \longrightarrow & F_0 \\ & & \downarrow & & \downarrow & & \downarrow \\ \dots & \longrightarrow & G_2 & \longrightarrow & G_1 & \longrightarrow & G_0 \end{array}$$

⁴I.e., the topology whose coverings are families $\{\text{Spec}(B_i) \rightarrow \text{Spec}(A)\}_{i \in I}$ such that each $A \rightarrow B_i$ is flat, and $\coprod \text{Spec}(B_i) \rightarrow \text{Spec}(A)$ is surjective.

be morphisms in a replete topos, and suppose $F_i \rightarrow G_i$ and $F_{i+1} \rightarrow F_i \times_{G_i} G_{i+1}$ are surjective for all i . Then $\varprojlim F_i \rightarrow \varprojlim G_i$ is surjective.

Exercise 4. Prove Lemma 11 when the topos is the category of sets.

Exercise 5. This exercise shows that limits do not preserve surjections in general. So the hypotheses of Lemma 11 are really necessary.

1. Show that $(\cdots \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}) \rightarrow (\cdots \rightarrow \mathbb{Z}/l^3 \rightarrow \mathbb{Z}/l^2 \rightarrow \mathbb{Z}/l)$ does not satisfy the hypotheses of Lemma 11.
2. Show that the limit of the above sequence of morphisms is $\mathbb{Z} \rightarrow \mathbb{Z}_l$. Show that this is not surjective.

exer:surjective

Exercise 6. Suppose that $(f_i : B_i \rightarrow C_i)_{i \in \mathbb{N}}$ is a sequence of surjections. Show that the conditions of Lemma 11 are satisfied for $F_n = \prod_{0 \leq i \leq n} B_i$, and $G_n = \prod_{0 \leq i \leq n} C_i$. Note: $X \times_Y (Y \times Y') \cong X \times Y'$ so

$$(B_0 \times \cdots \times B_n) \times_{(C_0 \times \cdots \times C_n)} (C_0 \times \cdots \times C_{n+1}) \cong (B_0 \times \cdots \times B_n \times C_{n+1})$$

Deduce that $\prod_{i \in \mathbb{N}} f_i : \prod_{i \in \mathbb{N}} B_i \rightarrow \prod_{i \in \mathbb{N}} C_i$ is surjective.

exer:EpiLemma

Exercise 7. Suppose that $\cdots \rightarrow F_2 \xrightarrow{t_2} F_1 \xrightarrow{t_1} F_0$ is a sequence of surjections in a replete topos \mathcal{X} .

1. Show that each map

$$\prod_{i=0}^{n+1} F_i \xrightarrow{t-\text{id}} \prod_{i=0}^n F_i;$$

$$(s_{n+1}, \dots, s_2, s_1, s_0) \mapsto (t_{n+1}s_{n+1} - s_n, \dots, t_2s_2 - s_1, t_1s_1 - s_0)$$

is surjective.

2. Using $B_n = \prod_{i=0}^{n+1} F_i$ and $C_n = \prod_{i=0}^n F_i$ and Exercise 6, show that

$$\prod_{\mathbb{N}} F_i \xrightarrow{t-\text{id}} \prod_{\mathbb{N}} F_i$$

is surjective where $t-\text{id}$ is the morphism $(\dots, c_2, c_1, c_0) \mapsto (\dots, tc_3 - c_2, tc_2 - c_1, tc_1 - c_0)$.

Definition 12. If $\mathcal{X} = \text{Shv}_\tau(\mathcal{C})$ is a topos we write $D(\mathcal{X}) = D(\text{Shv}_\tau(\mathcal{C}, \text{Ab}))$ for its derived category.

Recall that if \mathcal{X} is a topos, then the category $\prod_{\mathbb{N}} \mathcal{X}$ of sequences of objects and the category $\mathcal{X}^{\mathbb{N}}$ of sequences of morphisms are also topoi. We can thus consider the right derived functors associated to product and limit

$$R\Pi : D\left(\prod_{\mathbb{N}} \mathcal{X}\right) \rightarrow D(\mathcal{X}),$$

$$R\varprojlim : D(\mathcal{X}^{\mathbb{N}}) \rightarrow D(\mathcal{X}).$$

We prove the following proposition in an appendix to this lecture.

prop:rlimDef

Proposition 13 (See Proposition 39 below). *Let \mathcal{A} be a Grothendieck abelian category with products and $(\dots \rightarrow C_2 \xrightarrow{t} C_1 \xrightarrow{t} C_0)$ a sequence of chain complexes (the t 's are different, but we omit the indices). Then there is an isomorphism in $D(\mathcal{A})$*

$$R\varprojlim C_n \cong \text{Cone} \left(R\Pi C_n \xrightarrow{t-\text{id}} R\Pi C_n \right) [-1]$$

where $t - \text{id}$ is the morphism $(\dots, c_2, c_1, c_0) \mapsto (\dots, tc_3 - c_2, tc_2 - c_1, tc_1 - c_0)$.

One of the reasons we are interested in replete topoi is that limits work very well.

prop:derivedProd

Proposition 14. *Let \mathcal{X} be a replete topos. Then the functor $\Pi : \prod_{\mathbb{N}} \mathcal{X} \rightarrow \mathcal{X}$ preserves injections and surjections. In particular, Π preserves quasi-isomorphism of chain complexes and so induces a well-defined functor*

$$\Pi : D\left(\prod_{\mathbb{N}} \mathcal{X}\right) \rightarrow D(\mathcal{X})$$

which is just Π on each object.

Proof. We want to show that if $(f_i : F_i \rightarrow G_i)_{i \in \mathbb{N}}$ is a sequence of morphisms in \mathcal{X} which is injective (resp. surjective) then $\prod f_i$ is injective (resp. surjective). It is automatically injective because limits always preserve monomorphisms. The surjective case is exactly Exercise 6. \square

Proposition 15 ([BS, 3.1.10]). *Let $\mathcal{X} = \text{Shv}_{\tau}(\mathcal{C})$ be a replete topos and suppose $\dots \rightarrow F_2 \xrightarrow{t} F_1 \xrightarrow{t} F_0$ is a sequence of surjective morphisms in $\text{Shv}_{\tau}(\mathcal{C}, \text{Ab})$. Then we have $\varprojlim F_i \cong R\varprojlim F_i$ in $D(\mathcal{X})$.*

Proof. Since each $F_{i+1} \rightarrow F_i$ is surjective, the morphism $t - \text{id}$ is surjective by Exercise 7. So, we have a short exact sequence

$$0 \rightarrow \varprojlim F_i \rightarrow \prod F_i \xrightarrow{t-\text{id}} \prod F_i \rightarrow 0.$$

Since products are automatically derived by Proposition 13 and Lemma 14, we have

$$\varprojlim F_i \cong \text{Cone} \left(\prod F_i \xrightarrow{\text{shift} - \text{id}} \prod F_i \right) [-1].$$

\square

rema:derivedLimCalc

Remark 16. Proposition 14 shows that in a replete topos, we could define $R\varprojlim F_i$ as $\text{Cone}(\prod F_i \rightarrow \prod F_i)[-1]$. We will work with this definition from now on.

2 Locally weakly contractible topoi

coherentSite

Definition 17 (Cf.[Johnstone, Topos theory, Thm.7.35]). *A topos is said to be coherent if there is a site (C, τ) such that C has finite limits and for every covering $\{U_i \rightarrow X\}_{i \in I}$ there is a finite set $\{i_1, \dots, i_n\} \subseteq I$ such that $\{U_{i_j} \rightarrow X\}_{j=1}^n$ is also a covering. We will call (C, τ) a coherent site of definition.*

Exercise 8. Let SCH be the category of all schemes and AFF the category of all affine schemes (both equipped with the Zariski topology).

1. Show that the canonical restriction functor $\text{Shv}_{\text{Zar}}(SCH) \rightarrow \text{Shv}_{\text{Zar}}(AFF)$ induces an equivalence of categories.
2. Show that $(C, \tau) = (AFF, \text{Zar})$ satisfies the conditions of Definition 17.
3. Show that $(C, \tau) = (SCH, \text{Zar})$ does not satisfy the conditions of Definition 17, but that none-the-less, $\text{Shv}_{\text{Zar}}(SCH)$ is a coherent topos.

Definition 18 ([Bs, Def.3.2.1]). *An object F of a topos \mathcal{X} is weakly contractible if every surjection $G \rightarrow F$ has a section. Suppose \mathcal{X} is a coherent topos with coherent site of definition (C, τ) . We say that \mathcal{X} is locally weakly contractible if every object $X \in \mathcal{X}$ admits a surjection $\coprod_{i \in I} Y_i \rightarrow X$ with Y_i weakly contractible objects, which are representable by objects of C .*

Exercise 9. 1. Suppose that $\mathcal{X} = \text{Shv}_{\tau}(C)$ is a topos such that C is small. Show that for any sheaf F , the canonical morphism

$$\coprod_{X \in C, s \in F(X)} \text{hom}_C(-, X) \rightarrow F$$

is surjective.

2. Suppose that $\mathcal{X} = \text{Shv}_{\tau}(C)$ is a topos such that C is small, and suppose that $C' \subseteq C$ is a full subcategory such that C' is a coherent site of definition for \mathcal{X} (e.g., $C = SCH, C' = AFF$ with the Zariski topology⁵). Show that \mathcal{X} is locally weakly contractible if and only if for each $X \in C$, there is a covering family $\{U_i \rightarrow X\}_{i \in I}$ such that each U_i is in C' , and is weakly contractible.
3. Let Sch_R be the category of schemes of finite presentation over a ring R equipped with the Zariski topology. Using the results about w -local spaces from the lecture on commutative algebra, show that $\text{Shv}_{\text{Zar}}(Sch_R)$ is locally weakly contractible.

Example 19. The pro-étale site that we define in the next lecture is locally weakly contractible.

⁵The categories SCH and AFF are not small, but we can instead consider a variant such as: choose an uncountable strong limit cardinal κ and only consider those schemes that can be build using sets of size $< \kappa$. Another alternative is to choose some base ring R , and consider the categories Sch_R and Aff_R of R -schemes (resp. affine R -schemes) locally of presentation.

Proposition 20 ([BS, Prop.3.2.3].) *] Let \mathcal{X} be a locally weakly contractible topos. Then \mathcal{X} is replete, and for any object $K \in D(\mathcal{X})$ we have $R\lim_n \tau^{\geq n} K \cong K$ where*

$$\tau^{\geq n} K = (\dots \rightarrow 0 \rightarrow (K^n/dK^{n-1}) \rightarrow K^{n+1} \rightarrow K^{n+2} \rightarrow \dots).$$

Remark 21. The property $R\lim_n \tau^{\geq n} K \cong K$ means that all the information of K is contained in its truncations. This lets us deduce properties of unbounded complexes from bounded below complexes.

Sketch of proof. Since \mathcal{X} is locally weakly contractible, a morphism f in \mathcal{X} is an isomorphism (resp. surjection) if and only if evaluating on each weakly contractible object of \mathcal{C} is an isomorphism (resp. surjection). It follows that \mathcal{X} is replete.

Similarly, for any complex of sheaves K and weakly contractible object U we have $(H^i K)(U) = H^i(K(U))$. It follows that $R\lim_n \tau^{\geq n} K \cong K$. \square

3 Truncation completing derived categories

sec:TruncationCompletion

Recall that if \mathcal{X} is a category, then $\mathcal{X}^{\mathbb{N}}$ is the category of sequences $(\dots \rightarrow F_2 \rightarrow F_1 \rightarrow F_0)$ of morphisms in \mathcal{X} . If $\mathcal{X} = \text{Shv}_\tau(\mathcal{C})$ is a topos, then $D(\mathcal{X}^{\mathbb{N}})$ is the derived category of the abelian category $\text{Shv}_\tau(\mathcal{C}, \text{Ab})^{\mathbb{N}}$.

Now that we are working with sequences of chain complexes, we will have two indices: an upper index for the terms in the chain complex, and a lower index for the terms in the sequence.

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots \\
 & & \uparrow d & & \uparrow d & & \uparrow d \\
 \dots & \rightarrow & K_2^{i+1} & \xrightarrow{t} & K_1^{i+1} & \xrightarrow{t} & K_0^{i+1} \\
 & & \uparrow d & & \uparrow d & & \uparrow d \\
 \dots & \rightarrow & K_2^i & \xrightarrow{t} & K_1^i & \xrightarrow{t} & K_0^i \\
 & & \uparrow d & & \uparrow d & & \uparrow d \\
 \dots & \rightarrow & K_2^{i-1} & \xrightarrow{t} & K_1^{i-1} & \xrightarrow{t} & K_0^{i-1} \\
 & & \uparrow & & \uparrow & & \uparrow \\
 & & \vdots & & \vdots & & \vdots
 \end{array}$$

Here, the d 's and t 's should have indices too, but we did not write them. Note that $Ch(\mathcal{A})^{\mathbb{N}} = Ch(\mathcal{A}^{\mathbb{N}})$. That is, we can think about objects in this category as sequences of chain complexes $\xrightarrow{t_2} \begin{pmatrix} \vdots \\ \vdots \\ \vdots \end{pmatrix} \xrightarrow{t_1} \begin{pmatrix} \vdots \\ \vdots \\ \vdots \end{pmatrix} \xrightarrow{t_0} \begin{pmatrix} \vdots \\ \vdots \\ \vdots \end{pmatrix}$ or chain complexes of

sequences

$$\begin{array}{c}
 \uparrow d^{i+1} \\
 (\dots) \\
 \uparrow d^i \\
 (\dots) \\
 \uparrow
 \end{array}$$

Definition 22 ([BS, 3.3.1]). *Let $\mathcal{X} = \text{Shv}_\tau(\mathcal{C})$ be a topos. We define the left-completion $\widehat{D}(\mathcal{X})$ of $D(\mathcal{X})$ as the full subcategory of $D(\mathcal{X}^{\mathbb{N}})$ spanned by the projection systems $(\dots \rightarrow K_2 \rightarrow K_1 \rightarrow K_0)$ in $\text{Ch}(\text{Shv}_\tau(\mathcal{C}, \text{Ab})^{\mathbb{N}})$ such that*

1. $K_n \in D^{\geq -n}(\mathcal{X})$. That is, $H^i K_n = 0$ for $i < -n$.
2. The canonical map $\tau^{\geq -n} K_{n+1} \rightarrow K_n$ is an equivalence. In other words, the map $H^i K_{n+1} \rightarrow H^i K_n$ is an isomorphism for all $i \geq -n$.

We say that $D(\mathcal{X})$ is left-complete if the map

$$\tau : D(\mathcal{X}) \rightarrow \widehat{D}(\mathcal{X}); \quad K \mapsto \{\tau^{\geq -n} K\}$$

is an equivalence.

Remark 23. The definition is equivalent to asking that when we take cohomology, we get the following picture:

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots \\
 \dots & \xrightarrow{\mathbb{R}} & H^2 K_2 & \xrightarrow{\mathbb{R}} & H^2 K_1 & \xrightarrow{\mathbb{R}} & H^2 K_0 \\
 \dots & \xrightarrow{\mathbb{R}} & H^1 K_2 & \xrightarrow{\mathbb{R}} & H^1 K_1 & \xrightarrow{\mathbb{R}} & H^1 K_0 \\
 \dots & \xrightarrow{\mathbb{R}} & H^0 K_2 & \xrightarrow{\mathbb{R}} & H^0 K_1 & \xrightarrow{\mathbb{R}} & H^0 K_0 \\
 \dots & \xrightarrow{\mathbb{R}} & H^{-1} K_2 & \xrightarrow{\mathbb{R}} & H^{-1} K_1 & \longrightarrow & 0 \\
 \dots & \xrightarrow{\mathbb{R}} & H^2 K_2 & \longrightarrow & 0 & \longrightarrow & 0 \\
 \dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \vdots & & \vdots & & \vdots
 \end{array}$$

Remark 24. The inclusion $\widehat{D}(\mathcal{X}) \subseteq D(\mathcal{X})$ is not an inclusion of triangulated categories (because $\widehat{D}(\mathcal{X})$ is not preserve by the deshift $[-1]$ from $D(\mathcal{X})$).

We just state the main facts about completions without giving too many details.

Theorem 25. *Let $\mathcal{X} = \text{Shv}_\tau(\mathcal{C})$ be a topos.*

1. [BS, Lem.3.3.2] *The functor $R\varprojlim : \widehat{D}(\mathcal{X}) \rightarrow D(\mathcal{X}^{\mathbb{N}}) \rightarrow D(\mathcal{X})$ is the right adjoint of τ . In particular, if $D(\mathcal{X})$ is left-complete, then $K \cong R\varprojlim \tau^{-n}K$ for any $K \in \text{Ch}(\text{Shv}_\tau(\mathcal{C}, \text{Ab}))$.*
2. [BS, Prop.3.3.3] *If \mathcal{X} is a replete topos then $D(\mathcal{X})$ is left-complete.*
3. [BS, Exam.3.3.5] *If $k = \mathbb{C}(x_1, x_2, \dots)$, then $D(\text{Spec}(k)_{\text{et}})$ is not left-complete.*
4. [BS, Prop.3.3.7] *If $U \in \mathcal{C}$ is an object such that $\Gamma(U, -)$ is exact then for any $K \in D(\mathcal{X})$ we have $R\Gamma(U, K) \cong R\varprojlim R\Gamma(U, \tau^{-n}K)$.*
5. [BS, Prop.3.3.7] *If for each $K \in D(\mathcal{X})$ and $U \in \mathcal{C}$ there exists some $d \in \mathbb{N}$ such that $H^p(U, \underline{H}^q K) = 0$ for $p > d$, then $D(\mathcal{X})$ is left-complete.*

Example 26. The finiteness condition of [BS, Prop.3.3.7] above is satisfied for the étale sites of $\text{Spec}(\mathbb{F}_q)$, and X when \bar{X} is a smooth affine variety over an algebraically closed field.

4 ℓ -adically completing objects

sec:l-adic

Through-out this section we work with the discrete valuation ring \mathbb{Z}_ℓ . We also fix a replete topos $\mathcal{X} = \text{Shv}_\tau(\mathcal{C})$, and now our derived category will always be the derived category of sheaves of \mathbb{Z}_ℓ -modules

$$D(\mathcal{X}, \mathbb{Z}_\ell) = D(\text{Shv}_\tau(\mathcal{C}, \mathbb{Z}_\ell)).$$

Definition 27. *We say that $M \in \text{Mod}_{\mathbb{Z}_\ell}$ is classically complete if $M \cong \varprojlim M/\ell^n M$. We write $\text{Mod}_{\mathbb{Z}_\ell, \text{comp}} \subseteq \text{Mod}_{\mathbb{Z}_\ell}$ for the full subcategory of classically complete modules.*

We say that $K \in D(\mathcal{X}, \mathbb{Z}_\ell)$ is derived complete if $T(K) \cong 0$ in $D(\mathcal{X}, \mathbb{Z}_\ell)$ where

$$T(K) := R\varprojlim(\cdots \xrightarrow{\ell} K \xrightarrow{\ell} K \xrightarrow{\ell} K).$$

Here the transition maps are multiplication ℓ . We use the notation $D_{\text{comp}}(\mathcal{X}, \mathbb{Z}_\ell) \subseteq D(\mathcal{X}, \mathbb{Z}_\ell)$ for the full subcategory of derived complete objects.

Remark 28. Since we are assuming that \mathcal{X} is replete, by Proposition 14 we have

$$T(K) = \text{Cone} \left(\prod_{\mathbb{N}} K \xrightarrow{\text{id} - \ell} \prod_{\mathbb{N}} K \right) [-1].$$

Exercise 10. Show that \mathbb{Z}_l and \mathbb{Z}/l^n are (classically) complete \mathbb{Z}_l -modules but \mathbb{Q}_l is not.

exer:derivedCompleteCone

Exercise 11.

1. Show that if K is derived complete then so is $K[n]$ for any n .
2. Suppose that $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is a short exact sequence of chain complexes in $Ch(\mathcal{Shv}_\tau(\mathcal{C}, \mathbb{Z}_\ell))$. Using the fact that products in a replete topos are exact, show that $0 \rightarrow TA \rightarrow TB \rightarrow TC \rightarrow 0$ is also a short exact sequence. Deduce that if two of A, B, C are derived complete, then so is the third.
3. Consider a morphism $K \rightarrow L$ in $Ch(\mathcal{Shv}_\tau(\mathcal{C}, \mathbb{Z}_\ell))$ and define $C = Cone(K \rightarrow L)$. Use the second part above to show that if two of K, L, C are derived complete then the third is also derived complete.

exer:derivedCompleteLim

Exercise 12. Using the fact that for any *double* sequence of chain complexes $(K_{n,m})$ we have $R\varprojlim_n R\varprojlim_m K_{n,m} \cong R\varprojlim_m R\varprojlim_n K_{n,m}$, show that if $(\dots \rightarrow K_2 \rightarrow K_1 \rightarrow K_0)$ is a sequence of derived complete chain complexes then $R\varprojlim K_n$ is derived complete.

The relationship between classical complete and derived complete is the following.

Proposition 29 ([BS, Prop.3.4.2]). *An \mathbb{Z}_ℓ -module $M \in \text{Mod}_{\mathbb{Z}_\ell}$ is classically complete if and only if it is ℓ -adically separated⁶ and derived complete.*

In particular, for classical \mathbb{Z}_ℓ -modules, classical completeness is strictly stronger than derived completeness.

We omit the proof of 3.4.2 as it is not used elsewhere.

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Proposition 30 ([BS, Prop.3.4.4]). *An \mathbb{Z}_ℓ -complex $K \in D(\mathcal{X}, \mathbb{Z}_\ell)$ is derived complete if and only if each $H^i K \in \mathcal{Shv}_\tau(\mathcal{C}, \mathbb{Z}_\ell)$ is derived complete.*

Recall that for a chain complex K we define

$$\begin{aligned} \tau^{\geq n} K &= [\dots \rightarrow 0 \rightarrow 0 \rightarrow (K^n/dK^{n-1}) \rightarrow K^{n+1} \rightarrow K^{n+2} \rightarrow \dots] \\ \tau^{\leq n} K &= [\dots \rightarrow K^{n-2} \rightarrow K^{n-1} \rightarrow (\ker d) \rightarrow 0 \rightarrow 0 \rightarrow \dots] \end{aligned}$$

Exercise 13. Show that $H^i \tau^{\leq n} K = H^i K$ for $i \leq n$ and $H^i \tau^{\leq n} K = 0$ for $i > n$. Similarly, show that $H^i \tau^{\geq n} K = H^i K$ for $i \geq n$ and $H^i \tau^{\geq n} K = 0$ for $i < n$.

Proof. Suppose that each $H^i K$ is derived complete. We will show that K is derived complete. For any $i \in \mathbb{N}, n \in \mathbb{Z}$ we have

$$\text{Cone}(\tau^{\leq n+i} \tau^{\geq n} K \rightarrow \tau^{\leq n+i+1} \tau^{\geq n} K) \xrightarrow{q \cdot i} \underset{\sim}{H^{i+1} K}$$

so by induction on i , and Exercise 11, each $\tau^{\leq n+i} \tau^{\geq n} K$ is derived complete. Now we are assuming that \mathcal{X} is replete, so in particular, we have

$$\tau^{\leq m} K \cong R\varprojlim_{n \in \mathbb{N}} \tau^{-n} \tau^{\leq m} K.$$

⁶ ℓ -adically separated means that $\bigcap_{n \in \mathbb{N}} \ell^n M = 0$.

So by Exercise 12, we find that $\tau^{\leq m}K$ is derived complete. Now consider the short exact sequence of complexes

$$0 \rightarrow K \rightarrow \text{Cone}(\tau^{\leq m}K \rightarrow K) \rightarrow \tau^{\leq m}K[1] \rightarrow 0$$

By Exercise 11 the functor T takes short exact sequence to short exact sequences. Since $\tau^{\leq m}K$ is derived complete, we deduce that

$$TK \xrightarrow{q,i} T \text{Cone}(\tau^{\leq m}K \rightarrow K)$$

But

$$\text{Cone}(\tau^{\leq m}K \rightarrow K) \xrightarrow{q,i} \tau^{\geq m+1}K$$

so

$$TK \xrightarrow{q,i} T\tau^{\geq m+1}K$$

Finally, from the definition we see that $(T\tau^{\geq m+1}K)^i = 0$ for $i < m$. Since this is valid for any m , we deduce that $H^iTK = 0$ for all i . \square

Definition 31. Suppose that $K \in \text{Ch}(\text{Shv}_\tau(\mathcal{C}, \mathbb{Z}_\ell))$ is a chain complex. Then we define

$$K \otimes_{\mathbb{Z}_\ell}^L \mathbb{Z}_\ell/\ell^n := \text{Cone}(K \xrightarrow{\ell^n} K).$$

Remark 32. The functor $-\otimes_{\mathbb{Z}_\ell}^L \mathbb{Z}_\ell/\ell^n$ that we defined above actually calculates the left derived functor of $-\otimes_{\mathbb{Z}_\ell} \mathbb{Z}_\ell/\ell^n$ where here $\otimes_{\mathbb{Z}_\ell}$ is the usual tensor product. Since we only need the derived product in this case, we just take this as the definition.

exer:derCompletion

Exercise 14.

1. Show that there is a canonical morphism of sequences of chain complexes from $(\dots \xrightarrow{\ell} K \xrightarrow{\ell} K \xrightarrow{\ell} K)$ to $(\dots \xrightarrow{\text{id}} K \xrightarrow{\text{id}} K \xrightarrow{\text{id}} K)$
2. Deduce that there is a canonical morphism from $(\dots \xrightarrow{\text{id}} K \xrightarrow{\text{id}} K \xrightarrow{\text{id}} K)$ to $(\dots \rightarrow K \otimes_{\mathbb{Z}_\ell}^L \mathbb{Z}_\ell/\ell^2 \rightarrow K \otimes_{\mathbb{Z}_\ell}^L \mathbb{Z}_\ell/\ell \rightarrow K)$.
3. Show that there is a short exact sequence

$$0 \rightarrow K \rightarrow \widehat{K} \rightarrow TK \rightarrow 0$$

where

$$\widehat{K} := R\varprojlim (K \otimes_{\mathbb{Z}_\ell}^L \mathbb{Z}_\ell/\ell^n).$$

Deduce that K is derived complete if and only if the morphism $K \rightarrow \widehat{K}$ is a quasi-isomorphism.

Proposition 33 ([BS, Lem.3.4.9, Prop.3.5.1]). The functor sending K to \widehat{K} defines a left adjoint to the inclusion $D_{\text{comp}}(\mathcal{X}, \mathbb{Z}_\ell) \subseteq D(\mathcal{X}, \mathbb{Z}_\ell)$.

Sketch of proof. By Exercise 14 we see that \widehat{K} is derived complete. Suppose that $L \in D(\mathcal{X}, \mathbb{Z}_\ell)$ is also derived complete. Then we want to show that

$$\mathrm{hom}_{D(\mathcal{X}, \mathbb{Z}_\ell)}(\widehat{K}, L) \rightarrow \mathrm{hom}_{D(\mathcal{X}, \mathbb{Z}_\ell)}(K, L)$$

is an isomorphism. By the short exact sequence in Exercise 14(3) it suffices to show that

$$\mathrm{hom}_{D(\mathcal{X}, \mathbb{Z}_\ell)}(TK, L) = 0$$

(this uses some homological algebra that we have not covered, but it is not difficult homological algebra). Now we make two claims.

Claim 1. [BS, Lem.3.4.7] We have that $\mathrm{hom}(M, L) = 0$ for all $M \in D(\mathcal{X}, \mathbb{Q}_\ell)$.

Claim 2. [BS, Lem.3.4.8] We have that TK is in the essential image of the canonical functor $D(\mathcal{X}, \mathbb{Q}_\ell) \rightarrow D(\mathcal{X}, \mathbb{Z}_\ell)$.

The proof of these claims is not difficult, but is omitted. \square

Definition 34. We define a tensor product on $D_{\mathrm{comp}}(\mathcal{X}, \mathbb{Z}_\ell)$ using the tensor product on $D(\mathcal{X}, \mathbb{Z}_\ell)$:

$$K \widehat{\otimes}_{\mathbb{Z}_\ell} L := K \widehat{\otimes}_{\mathbb{Z}_\ell}^L L.$$

Here, $\widehat{\otimes}_{\mathbb{Z}_\ell}^L$ is the derived tensor product on $D(\mathcal{X}, \mathbb{Z}_\ell)$.

A Derived limits

In this subsection we consider a Grothendieck abelian category \mathcal{A} that admits products (in other words, satisfies Grothendieck's axiom (AB3^{*})). We are concerned with the derived functors

$$R\Pi : D\left(\prod_{\mathbb{N}} \mathcal{A}\right) \rightarrow D(\mathcal{A})$$

$$R\varprojlim : D(\mathcal{A}^{\mathbb{N}}) \rightarrow D(\mathcal{A})$$

associated to product $\Pi : \prod_{\mathbb{N}} \mathcal{A} \rightarrow \mathcal{A}$ and limit $\varprojlim : \mathcal{A}^{\mathbb{N}} \rightarrow \mathcal{A}$. Note that $\prod_{\mathbb{N}} \mathcal{A}$ and $\mathcal{A}^{\mathbb{N}}$ are again Grothendieck abelian categories (since they are functor categories from a small category to a Grothendieck abelian category).

Recall from the lecture Homological Algebra I that for a general left exact functor between Grothendieck abelian categories $F : \mathcal{B} \rightarrow \mathcal{B}'$, the derived functor $RF : D(\mathcal{B}) \rightarrow D(\mathcal{B}')$ can be calculated as follows. If $C \in \mathrm{Ch}^+(\mathcal{B})$ is a bounded below chain complex, then there exists a quasi-isomorphism $C \rightarrow I$ with I a bounded below chain complex of injective objects,⁷ and $RF(C) \cong F(I)$ in $D(\mathcal{B}')$. More generally, for any chain complex $C \in \mathrm{Ch}(\mathcal{B})$, there exists a quasi-isomorphism $C \rightarrow Q$ to a fibrant chain complex,⁸ and $RF(C) \cong F(Q)$ in $D(\mathcal{B}')$.

⁷Recall that an object $I \in \mathcal{B}$ is injective if for every monomorphism $A \rightarrow B$, every morphism $A \rightarrow I$ factors through $A \rightarrow B$.

⁸Recall that a chain complex $Q \in \mathrm{Ch}(\mathcal{B})$ is fibrant if for every monomorphic quasi-isomorphism $A \rightarrow B$ of chain complexes, every morphism $A \rightarrow Q$ factors through $A \rightarrow B$.

lemm:injProd

Lemma 35. *An object $(I_i)_{i \in \mathbb{N}}$ in $\prod_{\mathbb{N}} \mathcal{A}$ is injective if and only if each I_i is injective in \mathcal{A} .*

Exercise 15. Prove Lemma 35.

lemm:injectiveSequences

Lemma 36. *An object $(\cdots \rightarrow A_2 \rightarrow A_1 \rightarrow A_0)$ in $\mathcal{A}^{\mathbb{N}}$ is injective if and only if each A_i is injective and each $A_{i+1} \rightarrow A_i$ is a split surjection.*

Proof. Suppose $\mathcal{I}_{\bullet} = (\cdots \rightarrow I_2 \rightarrow I_1 \rightarrow I_0)$ is an injective object in $\mathcal{A}^{\mathbb{N}}$. Let $\lambda_n : \mathcal{A} \rightarrow \mathcal{A}^{\mathbb{N}}$ be the functor sending $A \in \mathcal{A}$ to $(\cdots \rightarrow 0 \rightarrow \underbrace{A \xrightarrow{id} \cdots \xrightarrow{id} A}_{n \text{ morphisms}})$. Then

λ_n is exact and a left adjoint to the “evaluation at n ” functor Ev_n (which sends $(\cdots \rightarrow B_2 \rightarrow B_1 \rightarrow B_0)$ to B_n). Since Ev_n has an exact left adjoint it sends injectives to injectives, and hence, each $I_n = Ev_n \mathcal{I}_{\bullet}$ is injective in \mathcal{A} . To see that each $I_{n+1} \rightarrow I_n$ is split surjective, consider the canonical monomorphism $\lambda_n I_n \rightarrow \lambda_{n+1} I_n$. Since \mathcal{I}_{\bullet} is injective, the canonical morphism $\lambda_n I_n \rightarrow \mathcal{I}_{\bullet}$ factors as $\lambda_n I_n \rightarrow \lambda_{n+1} I_n \rightarrow \mathcal{I}_{\bullet}$. The degree $n+1, n, n-1$ piece of this factorisation is

$$\begin{array}{ccccc}
 \vdots & & \vdots & & \vdots \\
 \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & I_n & \longrightarrow & I_{n+1} \\
 \downarrow & & \downarrow id & & \downarrow \\
 I_n & \longrightarrow & I_n & \xrightarrow{id} & I_n \\
 id \downarrow & & id \downarrow & & \downarrow \\
 I_n & \longrightarrow & I_n & \longrightarrow & I_{n-1} \\
 id \downarrow & & id \downarrow & & \downarrow \\
 \vdots & & \vdots & & \vdots
 \end{array}$$

So $I_{n+1} \rightarrow I_n$ is split surjective.

Conversely, suppose that $\mathcal{I}_{\bullet} = (\cdots \rightarrow I_2 \rightarrow I_1 \rightarrow I_0)$ is an object of $\mathcal{A}^{\mathbb{N}}$ such that each I_n is injective in \mathcal{A} , and each $I_{n+1} \rightarrow I_n$ is split surjective. Suppose that $A_{\bullet} = (\cdots \rightarrow A_2 \rightarrow A_1 \rightarrow A_0) \rightarrow (\cdots \rightarrow B_2 \rightarrow B_1 \rightarrow B_0) = B_{\bullet}$ is a monomorphism in $\mathcal{A}^{\mathbb{N}}$, and that $A_{\bullet} \rightarrow \mathcal{I}_{\bullet}$ is some morphism. We will show by induction that it factors through $A_{\bullet} \rightarrow B_{\bullet}$. In degree 0, this follows from the fact that I_0 is injective: $A_0 \rightarrow B_0$ is a monomorphism and I_0 injective so $A_0 \rightarrow I_0$ factors as $A_0 \rightarrow B_0 \rightarrow I_0$. Suppose that we have factorisations $A_i \rightarrow B_i \rightarrow I_i$ for all $0 \leq i < n$ which are compatible with the transition morphisms of $A_{\bullet}, B_{\bullet}, \mathcal{I}_{\bullet}$.

respectively. In particular, we have the following diagram

$$\begin{array}{ccccc}
A_n & \xrightarrow{\quad} & B_n & \dashrightarrow & I_n \\
\downarrow & & \downarrow & & \downarrow \\
A_{n-1} & \xrightarrow{\quad} & B_{n-1} & \xrightarrow{\quad} & I_{n-1}
\end{array}$$

and we are looking for the dashed morphism which makes the diagram commute. By hypothesis, $I_n \rightarrow I_{n-1}$ is split surjective. That is, $I_n \cong I_{n-1} \oplus J$ for some J , which is also injective as it is a direct summand of the injective object I_n . As J is injective, the induced morphism $A_n \rightarrow J$ factors as $A_n \rightarrow B_n \xrightarrow{\alpha} J$. On the other hand, we have the morphism $b : B_n \rightarrow B_{n-1} \rightarrow I_{n-1}$ from the above diagram. Then we define the dashed morphism to be $(b, a) : B_n \rightarrow I_{n-1} \oplus J \cong I_n$. One checks that this makes the diagram commute. \square

Now that we consider chain complexes in $\prod_{\mathbb{N}} \mathcal{A}$ and $\mathcal{A}^{\mathbb{N}}$ we will have two indices, (an upper) one for the chain complex direction, and (a lower) one for the $\prod_{\mathbb{N}} \mathcal{A}$, $\mathcal{A}^{\mathbb{N}}$ direction. We will implicitly use the canonical equivalences of categories $Ch(\prod_{\mathbb{N}} \mathcal{A}) \cong \prod_{\mathbb{N}} Ch(\mathcal{A})$ and $Ch(\mathcal{A}^{\mathbb{N}}) \cong Ch(\mathcal{A})^{\mathbb{N}}$.

注意 Beware, however, that the canonical inclusions $Ch^+(\prod_{\mathbb{N}} \mathcal{A}) \subseteq \prod_{\mathbb{N}} Ch^+(\mathcal{A})$ and $Ch^+(\mathcal{A}^{\mathbb{N}}) \subseteq Ch^+(\mathcal{A})^{\mathbb{N}}$ are *not* essentially surjective.

Lemma 37. *A chain complex $(Q_i^{\bullet})_{i \in \mathbb{N}}$ in $Ch(\prod_{\mathbb{N}} \mathcal{A})$ is fibrant if and only if each Q_i^{\bullet} is fibrant in $Ch(\mathcal{A})$.*

Proof. It suffices to note that a morphism $(A_i^{\bullet})_{i \in \mathbb{N}} \rightarrow (B_i^{\bullet})_{i \in \mathbb{N}}$ is a monomorphic quasi-isomorphism if and only if each $A_i^{\bullet} \rightarrow B_i^{\bullet}$ is a monomorphic quasi-isomorphism. \square

lemm:seqFibrant

Lemma 38. *If a chain complex $(Q_i^{\bullet})_{i \in \mathbb{N}}$ in $Ch(\mathcal{A}^{\mathbb{N}})$ is fibrant then each Q_i^{\bullet} is fibrant in $Ch(\mathcal{A})$.*

Proof. As in the proof of Lemma 36, the “evaluation at n ” functor $Ev_n : Ch(\mathcal{A}^{\mathbb{N}}) \rightarrow Ch(\mathcal{A})$ has a left adjoint $\lambda_n : Ch(\mathcal{A}) \rightarrow Ch(\mathcal{A}^{\mathbb{N}})$ which preserves monomorphisms and quasi-isomorphisms. Consequently, Ev_n sends fibrant objects to fibrant objects. \square

prop:limTriangle

Proposition 39. *Suppose that \mathcal{A} is a Grothendieck abelian category with products. Then for any object $(\dots \rightarrow C_2^{\bullet} \rightarrow C_1^{\bullet} \rightarrow C_0^{\bullet})$ in $Ch(\mathcal{A}^{\mathbb{N}})$, there is an isomorphism*

$$R\varprojlim C_n^{\bullet} \cong \text{Cone} \left(R\text{IIC}_n^{\bullet} \xrightarrow{id\text{-shift}} R\text{IIC}_n^{\bullet} \right)$$

in $D(\mathcal{A})$.

Proof. In order to calculate $R\varprojlim C_n^{\bullet}$, replace $(\dots \rightarrow C_2^{\bullet} \rightarrow C_1^{\bullet} \rightarrow C_0^{\bullet})$ with a quasi-isomorphic fibrant complex $(\dots \rightarrow Q_2^{\bullet} \rightarrow Q_1^{\bullet} \rightarrow Q_0^{\bullet})$ in $Ch(\mathcal{A}^{\mathbb{N}})$. Recall that every fibrant chain complex is a chain complex of injective objects (the converse is

true if the complex is bounded below). In particular, for each i the sequence $(\dots \rightarrow Q_2^i \rightarrow Q_1^i \rightarrow Q_0^i)$ is injective in $\mathcal{A}^{\mathbb{N}}$, and therefore by Lemma 36, the morphisms $Q_{n+1}^i \rightarrow Q_n^i$ are split surjective. We will use this fact later.

Now by Lemma 38 each Q_n^\bullet is fibrant. Hence, (Q_\bullet^\bullet) can also be used to calculate the derived products as well. That is,

$$R\varprojlim C_n^\bullet \cong \varprojlim Q_n^\bullet, \quad R\Pi C_n^\bullet \cong \Pi Q_n^\bullet$$

So it suffices to show that the canonical morphism

$$\varprojlim Q_n^\bullet \rightarrow \text{Cone} \left(\Pi Q_n^\bullet \xrightarrow{id\text{-shift}} \Pi Q_n^\bullet \right) [-1]$$

is a quasi-isomorphism. But since each $Q_{n+1}^i \rightarrow Q_n^i$ is split surjective, it follows that each $\Pi Q_n^i \xrightarrow{id\text{-shift}} \Pi Q_n^i$ is surjective. So the sequence

$$0 \rightarrow \varprojlim Q_n^\bullet \rightarrow \Pi Q_n^\bullet \xrightarrow{id\text{-shift}} \Pi Q_n^\bullet \rightarrow 0$$

is exact, and therefore the left term is quasi-isomorphic to the shifted cone of the right morphism. \square