# Factorization formulas of higher-order Alexander invariants for homological fibered knots (joint work with Hiroshi Goda)

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## §1. Introduction

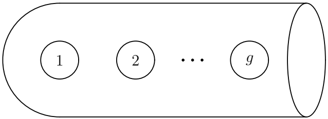
- $K \subset S^3$ : a knot,
  $S$ : a Seifert surface of $K$.

The **Alexander polynomial** of $K$ is

$$\Delta_K(t) = \det(A^T - tA),$$

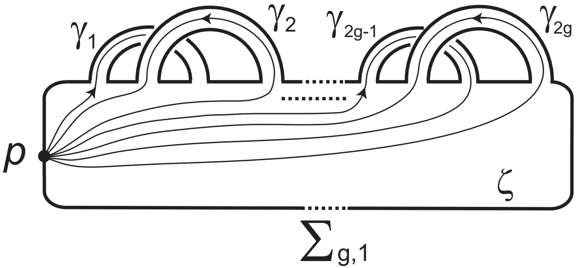
where $A$ is a **Seifert matrix** of $S$.

## Seifert matrix

- $\Sigma_{g,1} =$  ($g \geq 0$, oriented)



with a standard cell decomposition:



- Fix an identification

$$i : \Sigma_{g(S),1} \xrightarrow{\cong} S.$$

The Seifert matrix (for the identification $i$ ) is

$$A = \big(link(i(\gamma_k), i(\gamma_\ell)^+)\big)_{1\le k,\ell,\le 2g(S)}$$

Easy observation

$$\deg \Delta_K(t) = \deg(\det(A^T - tA)) \le 2g(K)$$

holds, where $g(K) = \min\{g(S) \mid S : \text{a Seifert surface of } K\}$.

Now assume that

$A$ is invertible over $\mathbb{Q}$.

Then

$$\begin{aligned}\Delta_K(t) &= \det(A^T - tA)\\ &= \det(A^T)\det(I_{2g(S)} - t(A^T)^{-1}A)\end{aligned}$$

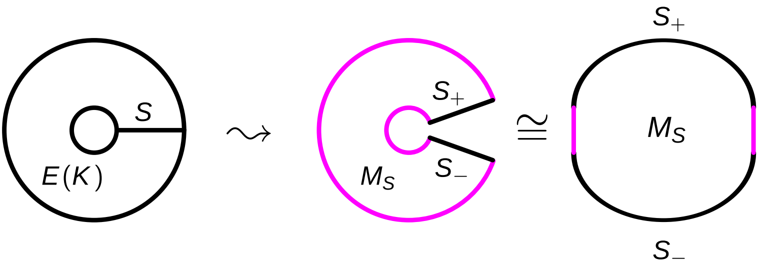
What does this *factorization* mean?

Remark In this case, $g(S) = g(K) = 2\deg \Delta_K(t)$ holds.

## §2. Sutured manifold

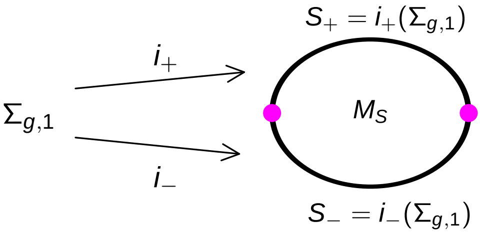
- $K \subset S^3$ : a knot,
  $S$ : a Seifert surface of $K$ of genus $g$.

  $M_S$ : the cobordism obtained from $E(K)$ by cutting along $S$
  $=$ the (complementary) sutured manifold for $S$.



# Sutured manifold (2)

• By using the identification $i : \Sigma_{g,1} \xrightarrow{\cong} S$, we obtain a *marked* sutured manifold $(M_S, i_+, i_-)$:



Return to our factorization:

$$\Delta_K(t) = \det(A^T)\det(I_{2g(S)} - t(A^T)^{-1}A) \quad \text{if } A \in GL(2g(S),\mathbb{Q})$$

We can check:

- $A^T$ and $A$ are representation matrices of the maps

$$i_+, i_- : \mathbb{Z}^{2g} \cong H_1(\Sigma_{g,1}) \longrightarrow H_1(M_S) \cong \mathbb{Z}^{2g}$$

under certain bases of $H_1(\Sigma_{g,1})$ and $H_1(M_S)$. In fact,

$$\begin{aligned} \det(A) &= \text{The top (bottom) coefficient of } \Delta_K(t) \\ &= \pm|H_1(M, i_+(\Sigma_{g,1})| \\ &= \tau(C_*(M_S, i_+(\Sigma_{g,1}); \mathbb{Q})) \quad \text{torsion.} \end{aligned}$$

# Sutured manifold (4)

- $\sigma(M_S) := (A^T)^{-1}A$ belongs to $Sp(2g, \mathbb{Q})$.
  (Regard $\sigma(M_S)$ as an $H_1$-monodromy of $M_S$.)

So, roughly speaking, our factorization formula says

$$\Delta_K(t) = (\text{torsion of } M_S) \cdot (\textit{effect} \text{ of } H_1\text{-monodromy of } M_S).$$

Remark Milnor showed that

$$\frac{\Delta_K(t)}{1-t} = \tau_{\mathbb{Z}}(K),$$

where $\tau_{\mathbb{Z}}(K)$ is the Reidemeister torsion associated with the $\mathbb{Z}$-cover of $E(K)$.

**Definition**

A knot $K$ is (rational) homologically fibered if

$\exists S$ : a (minimal genus) Seifert surface of $K$
s.t. $M_S$ is a (rational) homology product.
(i.e. Seifert matrix for $S$ is invertible over $\mathbb{Z}$ (or $\mathbb{Q}$))

Notation

$$\begin{aligned} \text{HFknot} &:= \text{homological fibered knot,} \\ \mathbb{Q}\text{-HFknot} &:= \text{rational homological fibered knot.} \end{aligned}$$

- We have seen that $\Delta_K(t)$ of a $\mathbb{Q}$-HFknot $K$ is factorized by invariants of $M_S$.

## §3. Homological fibered knots

**Proposition (Crowell-Trotter, . . . , Goda-S)**

- A knot $K$ is a $\mathbb{Q}$-HFknot if and only if $\deg \Delta_K(t) = 2g(K)$.
- A $\mathbb{Q}$-HFknot $K$ is an HFknot if and only if $\Delta_K(t)$ is monic.

Remark

(Fibered knots) $\subset$ (HFknots) $\subset$ ($\mathbb{Q}$-HFknots) $\subset$ (all knots).

most interesting

# Homological fibered knots (2)

- For an HFknot $K$,

$$\begin{aligned}\Delta_K(t) &= \det(A^T)\det(I_{2g(S)} - t(A^T)^{-1}A)\\ &= \pm\det(I_{2g(S)} - t(A^T)^{-1}A).\end{aligned}$$

$\rightsquigarrow$ The factorization formula is useless for HFknots!

$\rightsquigarrow$ We will give a generalization by using twisted homology.

Twisted coefficients

- $K$: an HFknot,
- $M_S = (M_S, i_+, i_-)$ : a sutured manifold associated with a minimal genus Seifert surface $S$,
- $\mathcal{K} := \mathrm{Frac}(\mathbb{Z}H_1(M_S)) \cong \mathbb{Q}(t_1, \ldots, t_{2g})$.

**Lemma**

For $\pm \in \{+, -\}$, $H_*(M_S, i_\pm(\Sigma_{g,1}); \mathcal{K}) = 0$.

cf. classical case: $H_*(M_S, i_\pm(\Sigma_{g,1}); \mathbb{Z}) = 0$.

### Definition

- The Magnus matrix $r_{\mathcal{K}}(M_S) \in GL(2g, \mathcal{K})$ is the representation matrix of the right $\mathcal{K}$-isom.:

$$\begin{array}{ccccc} H_1(\Sigma_{g,1}, p; \mathcal{K}) & \xrightarrow[i_-]{\cong} & H_1(M_S, p; \mathcal{K}) & \xrightarrow[i_+^{-1}]{\cong} & H_1(\Sigma_{g,1}, p; \mathcal{K}) \\ \| & & & & \| \\ \mathcal{K}^{2g} & & \xrightarrow[r_{\mathcal{K}}(M_S)\,\cdot]{\cong} & & \mathcal{K}^{2g} \end{array}$$

- The $\mathcal{K}$-torsion $\tau_{\mathcal{K}}(M_S)$ is

$$\tau_{\mathcal{K}}(M_S) := \tau(C_*(M_S, i_+(\Sigma_{g,1}); \mathcal{K})) \in GL(\mathcal{K})/\sim .$$

### Remarks

- If we substitute $t_i \mapsto 1$, we have

$$r_{\mathcal{K}}(M_S) \mapsto \sigma(M_S), \qquad \tau_{\mathcal{K}}(M_S) \mapsto \det A = \pm 1.$$

# Homological fibered knots (5)

- If $K$ is fibered, we have $\varphi \in MCG(\Sigma_{g,1})$ as the monodromy. Then

$$r_{\mathcal{K}}(M_S) = \overline{\left(\frac{\partial \varphi(\gamma_j)}{\partial \gamma_i}\right)}_{1 \le i,j \le 2g}.$$

- We can use $r_{\mathcal{K}}(M_S)$ and $\tau_{\mathcal{K}}(M_S)$ as fibering obstructions of HFknots:

**Theorem (Fibering obstructions)**

$K$, $M_S$: as before.

If $K$ is fibered, then

1. all the entries of the Magnus matrix $r_{\mathcal{K}}(M_S)$ are Laurent polynomials in $\mathbb{Q}[t_1^{\pm}, \ldots, t_{2g}^{\pm}] \subset \mathcal{K} = \mathbb{Q}(t_1, \ldots, t_{2g})$,
2. the $\mathcal{K}$-torsion $\tau_{\mathcal{K}}(M_S)$ is trivial.

# Homological fibered knots (5)

## Theorem (Factorization formula)

Let

$$\rho : \pi_1(E(K)) \longrightarrow \frac{\pi_1(E(K))}{\pi_1(E(K))''} \cong H_1(M_S) \rtimes H_1(E(K))$$

be the natural projection, and let $t \in H_1(E(K))$ be a meridian loop. Then we have

$$\begin{aligned} \tau_{\mathcal{K}(t^{\pm};\sigma)}(E(K)) &= \frac{\tau_{\mathcal{K}}(M_S) \cdot (I_{2g} - t \cdot r_{\mathcal{K}}(M_S))}{1-t} \\ &\in K_1(\mathcal{K}(t^{\pm};\sigma))/ \pm \rho(\pi_1(E(K))), \end{aligned}$$

where LHS is the noncommutative higher-order torsion associated with $\rho$ (defined by Cochran, Harvey and Friedl).

## §4. 12-crossings non-fibered homological fibered knots

### Facts on fibered knots vs. HFknots

- HFknots with at most 11-crossings are all fibered.
- There are 13 non-fibered HFknots with 12-crossings. In particular, Friedl-Kim showed that these 13 knots are not fibered by using twisted Alexander polynomial associated with finite representations.

We computed $r_{\mathcal{K}}(M_S)$ and $\tau_{\mathcal{K}}(M_S)$ for these 13 knots and checked that each of them also detects the non-fiberedness of all 13 HFknots.

Recipe

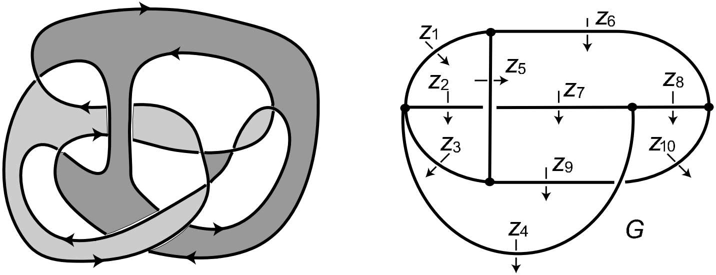
1. Get all the pictures of those 13 knots.
   [By Computer (Database (KnotInfo) on Internet)]
2. For each of them,
   1. Find a minimal genus Seifert surface $S$.
      [By hand]
   2. Calculate an admissible presentation of $\pi_1(M_S)$.
      [By hand]
   3. Compute $r_{\mathcal{K}}(M_S)$ and $\tau_{\mathcal{K}}(M_S)$.
      [By hand and also by computer program]

# 12-crossings non-fibered homological fibered knots (3)

1. List of non-fibered HFknots:

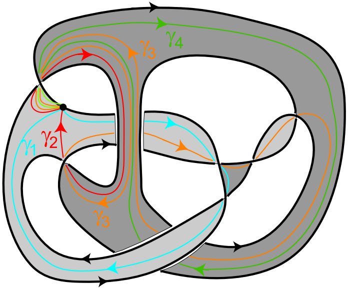
| Knot 12n– | Genus | Alexander polynomial |
|---|---|---|
| 0057 | 2 | $1 - 2t + 3t^2 - 2t^3 + t^4$ |
| 0210 | 3 | $1 - t - t^2 + 3t^3 - t^4 - t^5 + t^6$ |
| 0214 | 3 | $1 - t - t^2 + 3t^3 - t^4 - t^5 + t^6$ |
| 0258 | 2 | $1 - 4t + 5t^2 - 4t^3 + t^4$ |
| 0279 | 2 | $1 - 6t + 11t^2 - 6t^3 + t^4$ |
| 0382 | 2 | $1 - 5t + 7t^2 - 5t^3 + t^4$ |
| 0394 | 2 | $1 - 6t + 11t^2 - 6t^3 + t^4$ |
| 0464 | 2 | $1 - 4t + 5t^2 - 4t^3 + t^4$ |
| 0483 | 2 | $1 - 4t + 5t^2 - 4t^3 + t^4$ |
| 0535 | 2 | $1 - 7t + 11t^2 - 7t^3 + t^4$ |
| 0650 | 2 | $1 - 4t + 7t^2 - 4t^4 + t^4$ |
| 0801 | 2 | $1 - 5t + 7t^2 - 5t^3 + t^4$ |
| 0815 | 2 | $1 - 2t + t^2 - 2t^3 + t^4$ |

2. Example of calculation of admissible presentation




12n0057

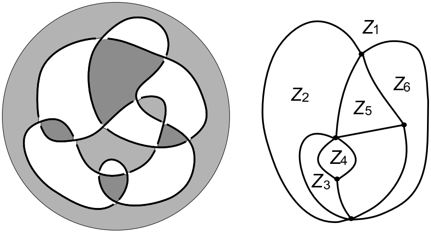
# 12-crossings non-fibered homological fibered knots (5)



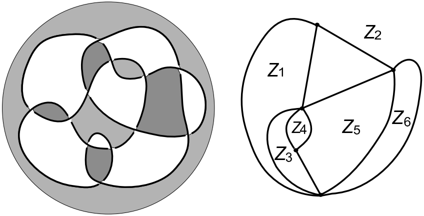

12n0057

Generators $i_-(\gamma_1), \ldots, i_-(\gamma_4),\ z_1, \ldots, z_{10},\ i_+(\gamma_1), \ldots, i_+(\gamma_4)$

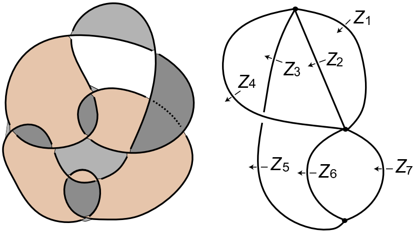
Relations $z_1 z_5 z_6^{-1},\ z_2 z_3 z_4 z_1,\ z_3 z_9^{-1} z_5^{-1},\ z_7 z_4 z_8^{-1},\ z_8 z_{10} z_6,$
$z_2 z_5 z_7^{-1} z_5^{-1},\ z_9 z_4 z_{10}^{-1} z_4^{-1},\ i_-(\gamma_1) z_1^{-1} z_5^{-1},\ i_-(\gamma_2) z_2,$
$i_-(\gamma_3) z_4 z_8 z_7 z_5^{-1},\ i_-(\gamma_4) z_4,\ i_+(\gamma_1) z_5^{-1},\ i_+(\gamma_2) z_9^{-1} z_6^{-1},$
$i_+(\gamma_3) z_6 z_4 z_7 z_5^{-1} z_3^{-1} z_5 z_6^{-1},\ i_+(\gamma_4) z_6 z_7^{-1} z_6^{-1}$



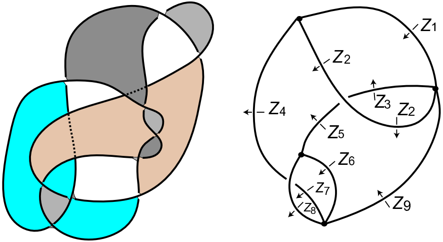

12n0210




12n0214

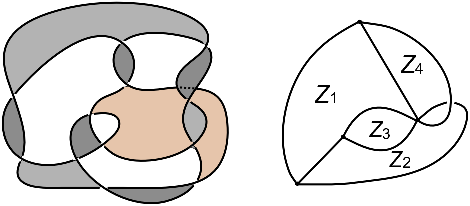



12n0258

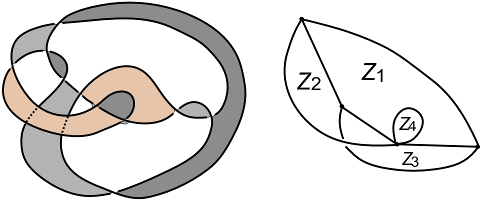



12n0279

# 12-crossings non-fibered homological fibered knots (8)

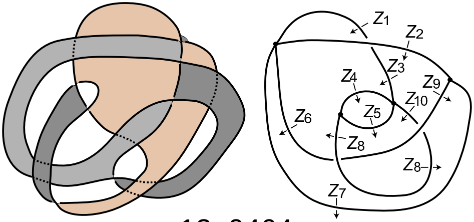



12n0382

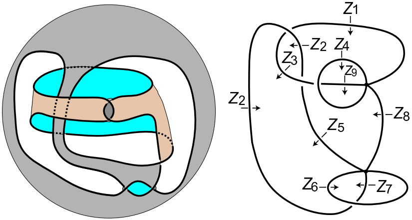



12n0394

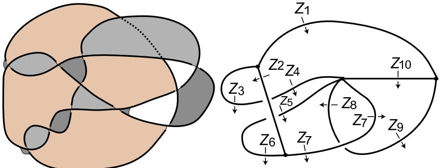
# 12-crossings non-fibered homological fibered knots (9)



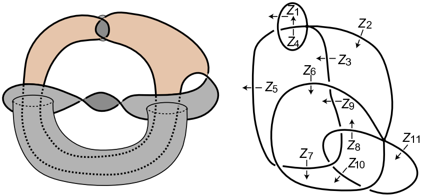

12n0464



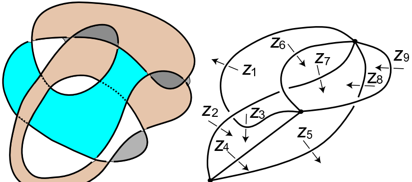

12n0483



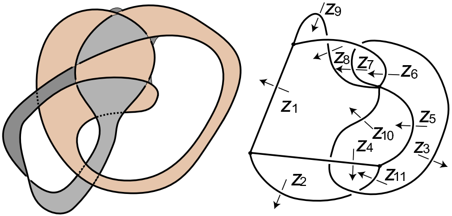

12n0535




12n0650




12n0801




12n0815

3. Computations of $r_{\mathcal{K}}(M_S)$ and $\tau_{\mathcal{K}}(M_S)$:

We wrote a Mathematica program and used it.

Input : admissible presentation of $\pi_1(M_S)$
Output : invariants

## Computational results for 12n0057

$$r_{\mathcal{K}}(M_S) = \begin{pmatrix} \frac{x3+x1\,x2^2\,(-1+x2\,(-1+x4))-x2\,x3\,x4}{x1\,x2^2\,(-1+x2\,(-1+x4))} & -\frac{(-1+x4)\,(-1+x2\,x4)}{-1+x2\,(-1+x4)} & \frac{x4}{1+x2-x2\,x4} & 0 \\ -\frac{(1+x1\,x2)\,x3}{x1^2\,x2\,(-1+x2\,(-1+x4))} & -\frac{x2\,(1+x1\,x2)\,(-1+x4)}{x1\,(-1+x2\,(-1+x4))} & -\frac{(1+x2)\,\left(1+x1\,x2^2\,(-1+x4)\right)}{x1\,x2\,(-1+x2\,(-1+x4))} & \frac{1}{x4} \\ \frac{x3}{x1\,(-1+x2\,(-1+x4))} & \frac{x2^2\,(-1+x4)}{-1+x2\,(-1+x4)} & \frac{x2\,(1+x2)\,(-1+x4)}{-1+x2\,(-1+x4)} & 0 \\ \frac{\left(x1\,x2^2-x3\right)\,x4}{x1^2\,x2\,(-1+x2\,(-1+x4))} & \frac{x2\,x4\,(x1\,x2+x3-x3\,x4)}{x1\,x3\,(-1+x2\,(-1+x4))} & \frac{(1+x2)\,\left(x1\,x2^2-x3\right)\,x4}{x1\,x2\,x3\,(-1+x2\,(-1+x4))} & 1 \end{pmatrix},$$

$$\tau_{\mathcal{K}}(M_S) = x1\,x2^4 + x1\,x2^5 - x1\,x2^5\,x4,$$

where $\mathtt{xj} = i_+(\gamma_j)$.

Each of $r_{\mathcal{K}}(M_S)$ and $\tau_{\mathcal{K}}(M_S)$ shows that 12n0057 is not fibered!

## §5. Further projects

- Detection of non-fiberedness by Johnson-Morita homomorphism:

  (2nd Johnson homomorphism + *Yokomizo cokernel*)

- *Categorification* of factorization formulas:

$$\begin{array}{ccc} \widehat{HFK}(K) & \rightsquigarrow & \Delta_K(t), \\ SFH(M_S, K) & \rightsquigarrow & \tau_{\mathcal{K}}(M_S), \\ ??? & \rightsquigarrow & r_{\mathcal{K}}(M_S). \end{array}$$

*decategorification*