Computations of noncommutative Alexander invariants for string links

Takuya SAKASAI

The University of Tokyo

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Contents

§1. Introduction

How can we distinguish two (compact) manifolds?

One of the most important way to distinguish two manifolds is to use invariants.

The fundamental group is the most important group invariant.

How can we distinguish two (finitely presentable) groups?

$\rightsquigarrow$ We needs invariants of groups.

Today we discuss the Alexander polynomial and its noncommutative generalizations.

The (classical) Alexander polynomial: $\Delta(G)$

$G = \langle x_1, \ldots, x_l \mid r_1, \ldots, r_m \rangle$: a finitely presentable group
$H := G^{abel}/torsion$ (free abelian)
$\rightsquigarrow \ \Delta(G) \in \mathbb{Z}H/\pm H$

Computation of $\Delta(G)$

1. Calculate the "Jacobi matrix" $\left(\frac{\partial r_j}{\partial x_i}\right)_{i,j} \in M(l, m; \mathbb{Z}G)$.

   $\frac{\partial}{\partial x_i} : \langle x_1, \ldots, x_l \rangle \to \mathbb{Z}G$ is the Fox differential:
   $\frac{\partial 1}{\partial x_i} = 0, \quad \frac{\partial x_j}{\partial x_i} = \delta_{i,j}, \quad \frac{\partial (uv)}{\partial x_i} = \frac{\partial u}{\partial x_i} + u\frac{\partial v}{\partial x_i}.$

2. Apply $\mathfrak{a} : \mathbb{Z}G \to \mathbb{Z}H$ to each entry.
   $\rightsquigarrow$ ${}^{\mathfrak{a}}\left(\frac{\partial r_j}{\partial x_i}\right)_{i,j} \in M(l, m; \mathbb{Z}H)$ $\cdots$ the Alexander matrix
3. Take the ideal generated by all $(l-1)$-minors.
   $\rightsquigarrow$ $E_1(G) \subset \mathbb{Z}H$ $\cdots$ the 1st elementary ideal

Definition (The Alexander polynomial [Alexander '28])

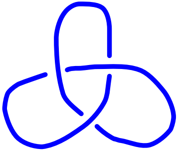
$\Delta(G) := \gcd(E_1(G)) \in \mathbb{Z}H$ well-defined up to $\pm H$.

Remark The computation of $\Delta(G)$ needs

- The Fox differential,
- Computations in $\mathbb{Z}H$ (Laurent polynomial ring).

$\rightsquigarrow$ The calculation by computers is easily implemented.

Example

$K =$  $\subset S^3,$

$$G := \pi_1(S^3 - K)$$
$$= \{x_1, x_2, x_3 \mid x_1x_3x_2^{-1}x_3^{-1},\ \ x_2x_1x_3^{-1}x_1^{-1},\ \ x_3x_2x_1^{-1}x_2^{-1}\}.$$
$$H = G^{abel} = \langle t \rangle. \quad (t = [x_1] = [x_2] = [x_3]).$$

$$\Longrightarrow \quad {}^{\mathfrak{a}}\left(\frac{\partial r_j}{\partial x_i}\right)_{i,j} = \begin{pmatrix} 1 & t-1 & -t \\ -t & 1 & t-1 \\ t-1 & -t & 1 \end{pmatrix},$$

$$\Longrightarrow \quad \underline{\Delta(G) = t^2 - t + 1.}$$

Definition (The Alexander norm: $\|\cdot\|_A$)

G, H as before

$\Delta(G) = \sum_{i=1}^{m} a_i h_i \quad (a_i \in \mathbb{Z} - \{0\}, h_i \in H)$

$\psi \in H^1(G;\mathbb{Z}) = \mathrm{Hom}(G,\mathbb{Z}) = \mathrm{Hom}(H,\mathbb{Z})$

$$\Longrightarrow \quad \|\psi\|_A := \begin{cases} \sup_{i,j} \left(\psi(h_i) - \psi(h_j)\right) \in \mathbb{Z}_{\geq 0} & (\Delta(G) \neq 0) \\ \infty & (\Delta(G) = 0) \end{cases}$$

Application $\qquad M$: a cpt oriented 3-manifold

- Lower estimate of the Thurston norm of M.
- Obstruction to $S^1 \times M$ admitting a symplectic structure.

etc...

§2. Generalizations of the Alexander polynomial

Observation

In the computation of $\Delta(G)$, the operation $\mathfrak{a} : \mathbb{Z}G \to \mathbb{Z}H$ makes the situation very simpler.

It may lose some information on G.

$\Downarrow$ noncommutative generalizations

- *Twisted Alexander polynomials*
  (Wada, Lin)
- *Noncommutative* (*Higher-order*) *Alexander "polynomials"*
  (Cochran, Harvey, Turaev)

Today, we mainly consider the latter.

Generalizations of the Alexander polynomial (2)

1. Twisted Alexander invariants: $\Delta_\rho(G)$ [Wada, Lin '90s]

- Take a matrix representation,

$$\rho : G \longrightarrow GL(n,R) \qquad \left(\begin{array}{l} R : \text{commutative UFD} \\ R = \mathbb{Z}, \mathbb{F}_p \ (p\text{: prime}), \ldots \end{array} \right)$$

of G, and "fatten" each entry of the Jacobi matrix by ρ.

$\rightsquigarrow$ an invariant $\Delta_\rho(G) \in \mathrm{Frac}(RH)$ for each (G, ρ).

- To obtain a canonical invariant for G, we need to "integrate" (gather) all such rational functions.

2. Noncommutative Alexander invariants
[Cochran, Harvey, Turaev '04]

- Take a group Γ having good properties ($\rightsquigarrow$ PTFA) between G and $H = G^{abel}/torsion$:

$$G \longrightarrow \Gamma \longrightarrow H.$$

- Using a localization ($\rightsquigarrow$Ore localization) of $\mathbb{Z}\Gamma$, we can define a noncommutative Alexander polynomial with high indeterminacy.

However, its degree $\delta_\Gamma^\psi(G)$ is well-defined.

$\downarrow$

Noncommutative Alexander invariants

Definition (PTFA group)

Γ is PTFA (Poly-{Torsion Free Abelian})

$\iff$ $^{\exists}$seq. $\Gamma = \Gamma_1 \rhd \Gamma_2 \rhd \cdots \rhd \Gamma_n = \{1\}$
s.t. Γ_i/Γ_{i+1} torsion free abelian.

Remark. $\begin{pmatrix} \text{torsion free} \\ \text{nilpotent} \end{pmatrix} \subset \text{(PTFA)} \subset \begin{pmatrix} \text{torsion free} \\ \text{solvable} \end{pmatrix}$

Theorem

Γ : *PTFA* $\Rightarrow$ $\mathbb{Z}\Gamma, \mathbb{Q}\Gamma$ *are Ore domains.*

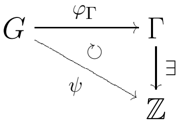
$\rightsquigarrow$ $\mathcal{K}_\Gamma := \mathbb{Z}\Gamma(\mathbb{Z}\Gamma - \{0\})^{-1} = \mathbb{Q}\Gamma(\mathbb{Q}\Gamma - \{0\})^{-1}$ $\cdots$ skew field
is defined. (Ore localization of $\mathbb{Z}\Gamma, \mathbb{Q}\Gamma$)

Computation of $\delta_\Gamma^\psi(G)$

G : a finitely presentable group

(φ_Γ, ψ) : an admissible pair for G:

(Γ : PTFA)



$$\left(\frac{\partial r_j}{\partial x_i}\right)_{i,j} \in M(l, m; \mathbb{Z}G)$$

$\Downarrow$ Apply $\varphi_\Gamma : \mathbb{Z}G \to \mathbb{Z}\Gamma$

$${}^{\varphi_\Gamma}\!\left(\frac{\partial r_j}{\partial x_i}\right)_{i,j} \in M(l, m; \mathbb{Z}\Gamma) \subset M(l, m; \mathcal{K}_\Gamma)$$

$\Downarrow$ Noncommutative linear algebra

$$\delta_\Gamma^\psi(G) \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$$

Remark. $\delta_\Gamma^\psi(G) = \|\psi\|_A$ when $\Gamma = G^{abel}/torsion$.

Using these generalizations, we can obtain various information which cannot be captured by the original Alexander polynomial.

- Twisted Alexander invariants have been studied both theoretically and experimentally.
- On the other hand, noncommutative Alexander invariants have been studied only theoretically (heuristically), since the explicit computations are considered to be difficult.

Problem

Find ways to compute noncommutative Alexander invariants explicitly (by using computers if necessary).

§3. Noncommutative linear algebra

For the computation of $\delta_\Gamma^\psi(G)$, we use

1. *Dieudonné determinant*,
2. *Degree functions*,

defined on matrices with entries in $\mathcal{K}_\Gamma = \mathbb{Z}\Gamma(\mathbb{Z}\Gamma - \{0\})^{-1}$.

In particular, we will construct a map

$$\deg^\psi : M(\mathcal{K}_\Gamma) \to \mathbb{Z}$$

for each $\psi \in H^1(\Gamma; \mathbb{Z})$.

Tool (I): *Dieudonné determinant*

Theorem (Dieudonné determinant)

For a skew field $\mathcal{K}$, there exists a unique homomorphism

$$\det : GL(\mathcal{K}) \longrightarrow (\mathcal{K}^\times)^{abel}$$

characterized by

1. $\det I = 1$,
2. *If A' is obtained by multiplying a row of a matrix $A \in GL(\mathcal{K})$ by $a \in \mathcal{K}^\times$ from the left, then $\det A' = a \cdot \det A$.*
3. *If A' is obtained by adding to a row of a matrix A a left $\mathcal{K}$-linear combination of other rows, then $\det A' = \det A$.*

Example

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a \cdot \det\begin{pmatrix} 1 & a^{-1}b \\ c & d \end{pmatrix}$$
$$= a \cdot \det\begin{pmatrix} 1 & a^{-1}b \\ 0 & d - ca^{-1}b \end{pmatrix}$$
$$= a(d - ca^{-1}b) \ \in (\mathcal{K}^\times)^{abel}$$

As an element in $(\mathcal{K}^\times)^{abel}$, we also have

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = -c(b - ac^{-1}d) = (a - bd^{-1}c)d = \cdots .$$

Tool (II): *Degree functions*

Definition (Degree functions)

Γ : PTFA

$\psi \in \mathrm{Hom}(\Gamma, \mathbb{Z}) = H^1(\Gamma; \mathbb{Z})$

$f = \sum_{i=1}^{m} a_i f_i \in \mathbb{Z}\Gamma \quad (a_i \in \mathbb{Z} - \{0\}, f_i \in N_k)$

$$\deg^\psi(f) := \begin{cases} \sup_{i,j} \left(\psi(f_i) - \psi(f_j)\right) \in \mathbb{Z}_{\geq 0} & (f \neq 0) \\ \infty & (f = 0) \end{cases}.$$

Furthermore, for $f g^{-1} \in \mathcal{K}_\Gamma$ with $g \neq 0$,

$$\deg^\psi(f g^{-1}) := \deg^\psi(f) - \deg^\psi(g) \in \mathbb{Z} \cup \{\infty\}.$$

This function induces a homomorphism

$$\deg^\psi : (\mathcal{K}_\Gamma^\times)^{abel} \to \mathbb{Z}.$$

Combining the above tools,

$$d_\Gamma^\psi(A) := \deg^\psi(\det A) \in \mathbb{Z}.$$

for $\psi \in H^1(\Gamma;\mathbb{Z}),\ A \in GL(\mathcal{K}_\Gamma)$

Remark. $d_\Gamma^\psi(A) = 0$ for $A \in GL(\mathbb{Z}\Gamma) \subset GL(\mathcal{K}_\Gamma)$.

Moreover, we can naturally extend the above function to

$$d_\Gamma^\psi(\,\cdot\,):\ M(\mathcal{K}_\Gamma) \longrightarrow \mathbb{Z} \cup \{\infty\}$$

so that $\delta_\Gamma^\psi(G) = d_\Gamma^\psi\left({}^{\varphi_\Gamma}\!\left(\frac{\partial r_j}{\partial x_i}\right)_{i,j} \right)$ for an admissible pair (φ_Γ, ψ) of a finitely presentable group $G = \langle x_1, \ldots, x_l \mid r_1, \ldots, r_m \rangle$.

§4. Applications to string links

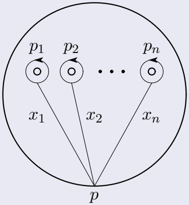
Definition (String links [Habegger-Lin])

D^2 : a 2-dimensional disk

$p_1, \ldots, p_n \in \mathrm{Int} D^2$: fixed $n \geq 2$ points

$D_n := D^2 - \{p_1, \ldots, p_n\}$

$\pi_1 D_n = \langle x_1, \ldots, x_n \rangle$



An *n-string link* L is

$$L : \coprod_{i=1}^{n} I_{(i)} \longrightarrow D^2 \times I \qquad \text{smooth embedding}$$

s.t. $0 \in I_{(i)} \mapsto (p_i, 0)$ and $1 \in I_{(i)} \mapsto (p_{\sigma_L(i)}, 1)$ for $^\exists \sigma_L \in \mathfrak{S}_n$.

Definition

$$\begin{array}{rcl} \mathcal{SL}_n & := & \text{isotopy classes of } n\text{-string links} \\ \cup & & \\ \mathcal{PSL}_n & := & \text{isotopy classes of } \textit{pure} \text{ (i.e. } \sigma_L = 1) \ n\text{-string links} \end{array}$$

- $\mathcal{SL}_n$ becomes a monoid by stacking.
- We have

$$\begin{array}{ccc} \text{(Braid group)} & \subset & \mathcal{SL}_n \\ \cup & & \cup \\ \text{(Pure braid group)} & \subset & \mathcal{PSL}_n \end{array}.$$

$L \in \mathcal{SL}_n$

$i_0, i_1 : D_n \hookrightarrow D^2 \times I - L$ embeddings
with $i_0(D_n) \subset D^2 \times \{0\}$, $i_1(D_n) \subset D^2 \times \{1\}$.

$i_0, i_1 : H_*(D_n) \xrightarrow{\simeq} H_*(D^2 \times I - L)$ isom

$\Downarrow$ Stallings' theorem

Lemma

$i_0, i_1 : N_k(\pi_1 D_n) \xrightarrow{\simeq} N_k(\pi_1(D^2 \times I - L))$ isom for $^\forall k \geq 2$,

where $N_k(G) := G/G_{(k)}$, nilpotent quotients of a group G,

$$G_{(1)} := G, \quad G_{(l)} := [G_{(l-1)}, G] \text{ for } l \geq 2.$$

We put $N_k := N_k(\pi_1 D_n)$.

Applications to string links (4)

Invariant of $\mathcal{SL}_n$ $\quad L \in \mathcal{SL}_n$

Lemma

$H_*(D^2 \times I - L,\ i_j(D_n); \mathcal{K}_{N_k}) = 0$ for $j = 0, 1$.

$$\Downarrow$$

$$\tau_k(L) := \tau\left(C_*(D^2 \times I - L,\ i_0(D_n); \mathcal{K}_{N_k})\right) \in GL(\mathcal{K}_{N_k})/\sim$$

Reidemeister torsion

For each $\psi \neq 0 \in H^1(N_k; \mathbb{Z}) = H^1(N_2; \mathbb{Z})$, we have a map

$$d^{\psi}_{N_k}(\tau_k(\cdot)) : \mathcal{SL}_n \longrightarrow \mathbb{Z}.$$

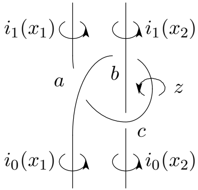
Remark. If L is a braid, then $\tau_k(L) = 1$,

and $d^{\psi}_{N_k}(\tau_k(L)) = 0$ for $\forall \psi \in H^1(N_2; \mathbb{Z})$.

Applications to string links (5)

Example.

$L =$



$$J := \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{pmatrix} i_1(x_1) & i_1(x_2) & z \\ -i_0(x_1)i_1(x_1)^{-1} & 0 & i_0(x_1)i_1(x_1)^{-1}i_0(x_1)^{-1} \\ 0 & -i_0(x_1)i_1(x_2)^{-1} + i_0(x_1)i_1(x_2)^{-1}z^{-1} & -i_0(x_1)i_1(x_2)^{-1}z^{-1} \\ 0 & -i_0(x_2)z^{-1}i_1(x_2)^{-1} & -i_0(x_2)z^{-1} + i_0(x_2)z^{-1}i_1(x_2)^{-1} \end{pmatrix}$$

$\Longrightarrow \quad \tau_2(L) = \det(J) = -1 + x_1^{-1} + x_2^{-1}. \quad (x_i := i_0(x_i))$

For example, $d_{N_2}^{x_1^*}(\tau_2(L)) = d_{N_2}^{x_2^*}(\tau_2(L)) = 1$.

$\tau_3(L) = -[x_1, x_2] + x_1^{-1}[x_1, x_2]^2 + x_2^{-1}$.

So $d_{N_3}^{\psi}(\tau_3(L)) = d_{N_2}^{\psi}(\tau_2(L))$ for $\forall \psi \in H^1(N_2; \mathbb{Z})$.

Generally,

Theorem

For $L_1, L_2 \in \mathcal{SL}_n$ *and* $\psi \in H^1(N_2;\mathbb{Z})$,

$$d^{\psi}_{N_k}(\tau_k(L_1 L_2)) = d^{\psi}_{N_k}(\tau_k(L_1)) + d^{\psi\cdot\sigma_{L_1}}_{N_k}(\tau_k(L_2)).$$

In particular,

$$d^{\psi}_{N_k}(\tau_k(\cdot)) \ : \ \mathcal{PSL}_n \longrightarrow \mathbb{Z}.$$

is a monoid homomorphism. (It vanishes on pure braids.)

Theorem

For each $\psi \in H^1(N_2;\mathbb{Z}) - \{0\}$, *the homomorphisms* $\left\{d^{\psi}_{N_k}(\tau_k(\cdot))\right\}_{k=2}^{\infty}$ *are all non-trivial and linearly independent.*

§5. Toward explicit computations

Problem

Find ways to compute noncommutative Alexander invariants *explicitly*.

$\Uparrow$

Find ways to compute Dieudonné determinants explicitly.

For simplicity, we only consider

$$N_k \ \cdots \ \text{the free nilpotent quotients}$$

$$\text{and } \ \mathbb{Z}N_k \hookrightarrow \mathcal{K}_{N_k} = \mathbb{Z}N_k(\mathbb{Z}N_k - \{0\})^{-1}.$$

Computations in $\mathbb{Z}N_k$ are not so difficult.

$\Uparrow$

We have a *normal form* for each element of N_k,

($\rightsquigarrow$ monomial in $\mathbb{Z}N_k$)

and the *product formula* is easily written down.

Example. In $N_3(\langle x_1, x_2, x_3 \rangle)$,

Normal form : $x_1^a x_2^b x_3^c [x_1, x_2]^d [x_1, x_3]^e [x_2, x_3]^f$

$$=: [a, b, c, d, e, f]$$

Product formula : $[a_1, b_1, c_1, d_1, e_1, f_1] \cdot [a_2, b_2, c_2, d_2, e_2, f_2]$

$$= [a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2 - b_1 a_2, e_1 + e_2 - c_1 a_2, f_1 + f_2 - c_1 b_2].$$

Difficulty appears when we compute in $\mathcal{K}_{N_k}$.

i.e. $(f_1\, g_1^{-1}) \cdot (f_2\, g_2^{-1}) = ?$

$(f_1\, g_1^{-1}) + (f_2\, g_2^{-1}) = ?$

The Ore property for the pair (f_2, g_1) says that

$$\exists f, g \in \mathbb{Z}N_k - \{0\} \quad s.t. \quad f_2\, g = g_1\, f.$$

$$\Longrightarrow \quad (f_1\, g_1^{-1}) \cdot (f_2\, g_2^{-1}) = f_1\, (g_1^{-1}\, f_2)\, g_2^{-1} = f_1\, (f\, g^{-1})\, g_2^{-1} = (f_1\, f)\, (g_2\, g)^{-1}.$$

Similarly we can compute $(f_1\, g_1^{-1}) + (f_2\, g_2^{-1})$.

Definition (Ore pair)

(f, g) is called an Ore pair for (f_2, g_1).

For a general PTFA group, it is not known how to find Ore pairs (while they exist).

$\updownarrow$ However

For finitely generated torsion free nilpotent group, $^{\exists}$algorithm to find Ore pairs.

More specifically, by using

1. Noncommutative Gröbner basis ('97) ,
2. Syzygy module theory ('00)

due to Madlener-Reinert, we can find a non-trivial solution (x, y) of the linear equation

$$fx - gy = 0 \qquad (f, g \in \mathbb{Z}N_k).$$

- We are now trying to implement the computations of Ore pairs (in general, very hard computations) by using *Mathematica* (with Y.Kiriu).
- One goal is to derive a universal formula for Ore pairs. At present, we have:

Proposition

In $\mathbb{Z}N_3$ (the first noncommutative case),

$$(1+x)\sum_{i=1}^{n}\left(y_i \prod_{j=1, j\neq i}^{n} y_j^{-1}(1+x)y_j\right)$$
$$= (y_1+y_2+\cdots+y_n)\left(\prod_{i=1}^{n} y_i^{-1}(1+x)y_i\right).$$