Genus invariants of closed 3-manifolds associated to homology cobordisms of surfaces

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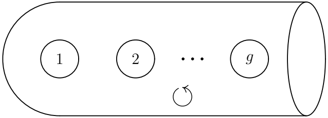
All 3-manifolds are assumed to be oriented and connected.

Introduction (1)

§1. Introduction

- $\Sigma_{g,1} =$  $(g \geq 0)$

Definition (Goussarov, Habiro, Garoufalidis-Levine, Levine)

M: a cpt 3-manifold, $i_+, i_- : \Sigma_{g,1} \hookrightarrow \partial M$: embeddings

(M, i_+, i_-) is called a *homology cylinder (over $\Sigma_{g,1}$)* if

1. i_+: orientation preserving, i_-: orientation reversing,
2. $\partial M = i_+(\Sigma_{g,1}) \cup i_-(\Sigma_{g,1})$ and
   $i_+(\Sigma_{g,1}) \cap i_-(\Sigma_{g,1}) = i_+(\partial\Sigma_{g,1}) = i_-(\partial\Sigma_{g,1})$,
3. $H_*(M, i_+(\Sigma_{g,1})) = H_*(M, i_-(\Sigma_{g,1})) = 0$,
4. $i_+\big|_{\partial\Sigma_{g,1}} = i_-\big|_{\partial\Sigma_{g,1}}$.

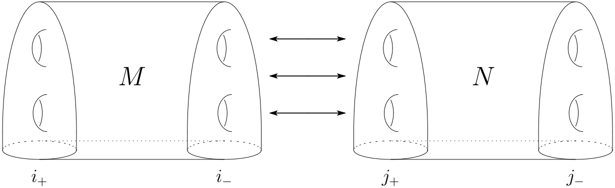
Introduction (2)

Definition

$\mathcal{C}_{g,1} := \{\text{homology cylinders}\} \,/\, (\text{diffeo. preserving } i_+, i_-)$.

- $M = (M, i_+, i_-),\ N = (N, j_+, j_-) \in \mathcal{C}_{g,1}$

  $\Longrightarrow \quad M \cdot N := (M \cup_{j_+ \circ i_-^{-1}} N, i_+, j_-) \in \mathcal{C}_{g,1}$.



$\rightsquigarrow$ $\mathcal{C}_{g,1}$ becomes a monoid.

Introduction (3)

Definition (Closing, cl_g)

$$\mathrm{cl}_g(M, i_+, i_-) := M/(i_+(x) = i_-(x)), \quad x \in \Sigma_{g,1}$$

closed 3-manifold

cf. $\{$braids (or string links)$\}$ $\overset{\text{closing}}{\rightsquigarrow}$ $\{$links$\}$

group (or monoid) set

Remark. For $M = (M, i_+, i_-)$, $N = (N, j_+, j_-) \in \mathcal{C}_{g,1}$,

$$\begin{cases} \mathrm{cl}_g(M \cdot N) = \mathrm{cl}_g(N \cdot M), \\ \mathrm{cl}_g(M) = -\mathrm{cl}_g(-M), \end{cases}$$

where $-M := (-M, i_-, i_+) \in \mathcal{C}_{g,1}$.

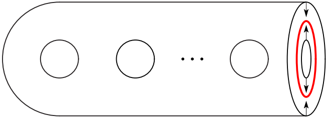
Introduction (4)

Example $[\varphi] \in \mathcal{M}_{g,1}$: the mapping class group of $\Sigma_{g,1}$

$\varphi : \Sigma_{g,1} \xrightarrow{\sim} \Sigma_{g,1}$: a diffeo. s.t. $\varphi\big|_{\partial\Sigma_{g,1}} = \mathrm{id}$.

$\Longrightarrow (\Sigma_{g,1} \times I, \mathrm{id} \times 1, \varphi \times 0) \in \mathcal{C}_{g,1}$,

where corners of $\Sigma_{g,1} \times I$ are rounded, and



- When $M = [\varphi] \in \mathcal{M}_{g,1} \hookrightarrow \mathcal{C}_{g,1}$,

$$\mathrm{cl}_g(M) = (\Sigma_{g,1} \times I) \Big/ \left(\begin{array}{ll} (x,1) = (\varphi(x),0), & x \in \Sigma_{g,1} \\ (y,0) = (y,t)) & y \in \partial\Sigma_{g,1}, t \in I \end{array} \right)$$

: an *open book decomposition* with *monodromy φ.*

$i_+(\partial\Sigma_{g,1}) = i_-(\partial\Sigma_{g,1})$ is a *fibered knot* in $\mathrm{cl}_g(M)$.

Introduction (5)

$$\begin{tikzcd}
\bigsqcup_{g\geq 0} \mathcal{C}_{g,1} \arrow[r, "\sqcup\, \mathrm{cl}_g"] & \{\text{closed 3-manifolds}\} \\
\bigsqcup_{g\geq 0} \mathcal{M}_{g,1} \arrow[u, hook] \arrow[ur, "\sqcup\, \mathrm{cl}_g = \text{open book decomp.}"'] &
\end{tikzcd}$$

Theorem (Alexander, Myers, González-Acunã)

$\bigsqcup_{g\geq 0} \mathrm{cl}_g \big|_{\sqcup \mathcal{M}_{g,1}}$ *is onto.*

So we can define the following:

Definition (Genus invariants)

For a closed 3-manifold X,

$$\mathrm{hc}(X) := \min\{g \mid \mathrm{cl}_g(M) = X \text{ for some } M \in \mathcal{C}_{g,1}\},$$

$$\mathrm{op}(X) := \min\{g \mid \mathrm{cl}_g(M) = X \text{ for some } M \in \mathcal{M}_{g,1}\}.$$

Introduction (6)

Some invariants similar to op

- Rubinstein (2003): *Open book genus*
  (by using fibered links and associated Heegaard splitting),
- Etnyre-Ozbagci (2006): *Supporting genus*
  (with contact structures).

Remark. Heegaard genus $\leq 2\,\mathrm{op}$.

Fundamental properties of hc and op (1)

§2. Fundamental properties of hc and op

Note that

$$\mathrm{cl}_g(M, i_+, i_-) = M/(i_+(x) = i_-(x)) = M \cup_\partial (\Sigma_{g,1} \times I),$$

where $\Sigma_{g,1} \times I \cong H(2g)$, the handlebody of genus $2g$.

Proposition

$$\begin{aligned} \mathrm{hc}(X) = 0 \quad &\Longleftrightarrow \quad X \text{ is a homology 3-sphere}, \\ \mathrm{op}(X) = 0 \quad &\Longleftrightarrow \quad X = S^3. \end{aligned}$$

For a homology 3-sphere X,

$$((\Sigma_{g,1} \times I)\# X, \mathrm{id} \times 1, \mathrm{id} \times 0) \in \mathcal{C}_{g,1}.$$

When $g = 0$, we have $\mathcal{C}_{0,1} \cong (\text{Homology 3-spheres}, \#)$.

Fundamental properties of hc and op (2)

Applying the Mayer-Vietoris sequence to $X = M \cup_\partial (\Sigma_{g,1} \times I)$, we have

Lemma (Primary estimate)

For a closed 3-manifold X,

$$\left\lceil \frac{1}{2} \min \left\{ n \;\middle|\; \begin{array}{l} H_1(X) \textit{ is generated} \\ \textit{by } n \textit{ elements} \end{array} \right\} \right\rceil \leq \mathrm{hc}(X) \leq \mathrm{op}(X).$$

<u>Question</u> Is $\leq$ of the left always $=$? $\cdots\cdots$ No!

Fundamental properties of hc and op (3)

More on op

Genus 1 fibered knots in lens spaces $L(p, q)$

$\Downarrow$

Many lens spaces have $\mathrm{op} = 1$. However,

- $\mathrm{op}(L(19, i)) \geq 2$ if $i = 2, 4, 7$ (Morimoto),
- $\mathrm{op}(L(p, 2)) \geq 2$ if $p \geq 11$, prime (Baldwin).

Moreover, Baker gave a general criterion in terms of 2-bridge links and their 3-braid representations.

We want to find more such examples by using hc ($\leq \mathrm{op}$).

$\Longleftarrow$ hc is easier to compute than op.

Some computations of hc (1)

§3. Some computations of hc

Proposition

For a closed 3-manifold X, if $H_1(X) \cong \mathbb{Z}^{2g}$ or $\mathbb{Z}^{2g-1}$, then

$$\mathrm{hc}(X) = g.$$

(Sketch of Proof)

1. Remove a handlebody $H(2g)$ of genus $2g$ appropriately from X:

   When $H_1(X) = \mathbb{Z}^{2g}$, take an embedding

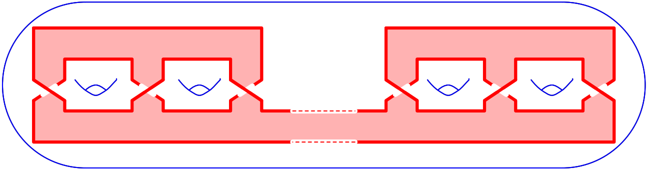
   $$\bigvee_{2g} S^1 \hookrightarrow X \quad s.t.\ H_1\left(\bigvee_{2g} S^1\right) \xrightarrow{\cong} H_1(X),$$

   where $N\left(\bigvee_{2g} S^1\right) = H(2g)$.

Some computations of hc (2)

2. Take an identification between the removed $H(2g)$ and $\Sigma_{g,1} \times I$ so that it satisfies the conditions for homology cylinders:

   We take a ribbon ($\cong \Sigma_{g,1}$) in $H(2g)$ as follows.

   Identify $H(2g) = N(\text{ribbon}) = \Sigma_{g,1} \times I$, and divide $\partial H(2g) = \partial(\Sigma_{g,1} \times I)$ into two $\Sigma_{g,1}$.
   Then $\overline{X - H(2g)}$ becomes a homology cylinder. □

Some computations of hc (3)

Similarly, we can show the following.

Lemma

(1) *For odd q, if* $p(p+4)$ *or* $p(p-4)$ *is a square residue in* $\mathbb{Z}/q\mathbb{Z}$, *then* $\mathrm{hc}(L(p,q)) = 1$.

In particular, if $q = \pm 1, \pm 3, \pm 5$, *then* $\mathrm{hc}(L(p,q)) = 1$.

(2) $\mathrm{hc}\left((S^1 \times S^2)\#L(p,\pm 1)\right) = 1$.

(3) $\mathrm{hc}\left(L(p,\pm 1)\#L(p',\pm 1)\right) = 1$ *for any* p, p'.

Torsion linking form and Borromean surgery (1)

§4. Torsion linking form and Borromean surgery

Torsion linking form X : an oriented closed 3-manifold,
$\mathrm{t}H_1(X)$: torsion part of $H_1(X)$.

$$\langle \cdot, \cdot \rangle_X : \mathrm{t}H_1(X) \otimes \mathrm{t}H_1(X) \longrightarrow \mathbb{Q}/\mathbb{Z}$$

(non-degenerate
symmetric
bilinear form)

is defined by $\displaystyle \langle x, y \rangle_X := \frac{A \cdot C}{n} \in \mathbb{Q}/\mathbb{Z}$

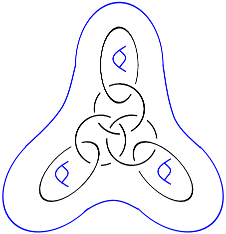
where A, B : simple closed curves representing x and y,
C : a 2-chain with $\partial C = nB$.

Torsion linking form and Borromean surgery (2)

Borromean surgery due to Matveev

($\rightsquigarrow$ Goussarov-Habiro finite type invariant theory)

6-component 0-framed link in the standard genus 3 handle-body:



We embed them in a 3-manifold and do surgery along the framed links. $\cdots$ Borromean surgery

Important property : Borromean surgeries do not change H_1 and its torsion linking form.

Torsion linking form and Borromean surgery (3)

Theorem (Matveev)

Two closed 3-manifolds X_1, X_2 are Borromean equivalent

$$\iff {}^{\exists}\varphi : H_1(X_1) \xrightarrow{\sim} H_1(X_2) \ \textit{ isom.}$$
$$\textit{s.t. } \langle x, y\rangle_{X_1} = \langle \varphi(x), \varphi(y)\rangle_{X_2} \textit{ for } {}^{\forall}x, y \in \mathrm{t}H_1(X).$$

- Borromean surgeries preserve the conditions for homology cylinders.

Theorem (Habiro, Massuyeau-Meilhan)

Two homology cylinders (M, i_+, i_-), (N, j_+, j_-) are Borromean equivalent

$$\iff i_+^{-1} \circ i_- = j_+^{-1} \circ j_- \in \mathrm{Aut}(H_1(\Sigma_{g,1})).$$

Torsion linking form and Borromean surgery (4)

Lemma

If two closed 3-manifolds X_1, X_2 are Borromean equivalent, then

$$\mathrm{hc}(X_1) = \mathrm{hc}(X_2).$$

(Proof)

- Removing $H(2g)$ from a closed 3-manifold.
- Doing Borromean surgeries.

Both operations have their supports in neighborhoods of graphs, which are generically disjoint.

$\rightsquigarrow$ The above two operations are commutative. $\square$

Torsion linking form and Borromean surgery (5)

Theorem

For a closed 3-manifold X, $\mathrm{hc}(X)$ depends only on the isomorphism class of $H_1(X)$ and its torsion linking form.

In particular, we have the following commutative diagram:

$$\begin{tikzcd}
\bigsqcup\limits_{g\geq 0} \mathcal{M}_{g,1} \arrow[r, hook] \arrow[ddr, two heads] \arrow[dddr, bend right, two heads] & \bigsqcup\limits_{g\geq 0} \mathcal{C}_{g,1} \arrow[r, two heads, "\sqcup \mathrm{cl}_g"] \arrow[d, two heads] & \{\text{closed 3-manifolds}\} \arrow[r, "\mathrm{hc}"] \arrow[d, two heads] & \mathbb{Z}_{\geq 0} \\
 & \bigsqcup\limits_{g\geq 0} \mathcal{C}_{g,1}/\mathrm{Bo.} \arrow[r, two heads, "\sqcup \mathrm{cl}_g"] \arrow[d, equal, "\mathrm{Habiro}"] & \{\text{closed 3-manifolds}\}/\mathrm{Bo.} \arrow[ur, "\mathrm{hc}"] \arrow[d, equal, "\mathrm{Matveev}"] & \\
 & \bigsqcup\limits_{g\geq 0} \mathrm{Sp}(2g,\mathbb{Z}) \arrow[r, two heads, "\sqcup \mathrm{cl}_g"] \arrow[d, two heads] & \{H_1 \text{ and tor. link. form}\}/\mathrm{isom} & \\
 & \bigsqcup\limits_{g\geq 0} \mathrm{Sp}(2g,\mathbb{Z})/\mathrm{conj} \arrow[ur, two heads, "\sqcup \mathrm{cl}_g"'] & &
\end{tikzcd}$$

Bo. $=$ Borromean equivalent

3-manifolds with $\mathrm{hc}(X) = 1$ (1)

§5. 3-manifolds with $\mathrm{hc}(X) = 1$

Theorem (In progress)

(1) $\mathrm{hc}\left((S^1 \times S^2)\#L(p,q)\right) = 1$

$\iff$ *q or $-q$ is a quadratic residue in $\mathbb{Z}/p\mathbb{Z}$.*

(2) $\mathrm{hc}\left(L(p,\pm 1)\#L(p',\pm 1)\right) = 1$ *for any p, p'.*

On the other hand, $\mathrm{hc}\left(L(5,1)\#L(5,2)\right) = 2$.

(3) *For odd q, if $p(p+4)$ or $p(p-4)$ is a square residue in $\mathbb{Z}/q\mathbb{Z}$, then $\mathrm{hc}(L(p,q)) = 1$. In particular, if one of the following holds then $\mathrm{hc}(L(p,q)) = 1$:*

- $q = \pm 1, \pm 3, \pm 5$,
- *p is a prime with* $p \not\equiv 1, 49 \pmod{60}$,
- $p \leq 20000$ *(by using a computer).*

3-manifolds with $\mathrm{hc}(X) = 1$ (2)

Remark. While we only mentioned about $L(p,q)$, the same statement holds for 3-manifolds having the same H_1 and torsion linking forms as $L(p,q)$.

When $g = 1$, we need to consider the map

$$\mathrm{cl}_1 \ : \ SL(2,\mathbb{Z})/\mathrm{conj} \longrightarrow \left\{ \begin{array}{l} \text{closed 3-manifolds whose } H_1 \\ \text{can be generated by 2-elements} \end{array} \right\} /\mathrm{Bo.},$$

where

$$\left\{ \begin{array}{l} \#\{\text{LHS with fixed trace}\} < \infty \quad \text{(folklore)} \\ \#\{\text{RHS with fixed isomorphism class of } H_1\} < \infty \\ \qquad\qquad \text{(cf. Wall, Kawauchi-Kojima's classification)} \end{array} \right.$$

3-manifolds with $\mathrm{hc}(X) = 1$ (3)

Problems.

- Does $\mathrm{hc}(L(p,q)) = 1$ hold for any p, q?
- Let $p, p' \geq 2$ and $\gcd(p, p') \geq 2$. Does $\mathrm{hc}\left(L(p,q)\#L(p',q')\right) \geq 2$ hold unless $\pm q$ and $\pm q'$ are both quadratic residues in $\mathbb{Z}/p\mathbb{Z}$ and $\mathbb{Z}/p'\mathbb{Z}$, respectively?