Twisted cohomology pairings of knots

2016. July 21 in Hellas, Greece

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Contents

§0 Introduction; two motivations and results

§1 Computation of twisted pairings

§2 Classical Blanchfield pairings

§3 Relation to infinite cyclic covers.

What is the twisted cohomology pairing?

For a 2-cocycle $\phi \in H^2(S^3 \setminus K, \partial(S^3 \setminus K); M)$ **in local coeff.,**

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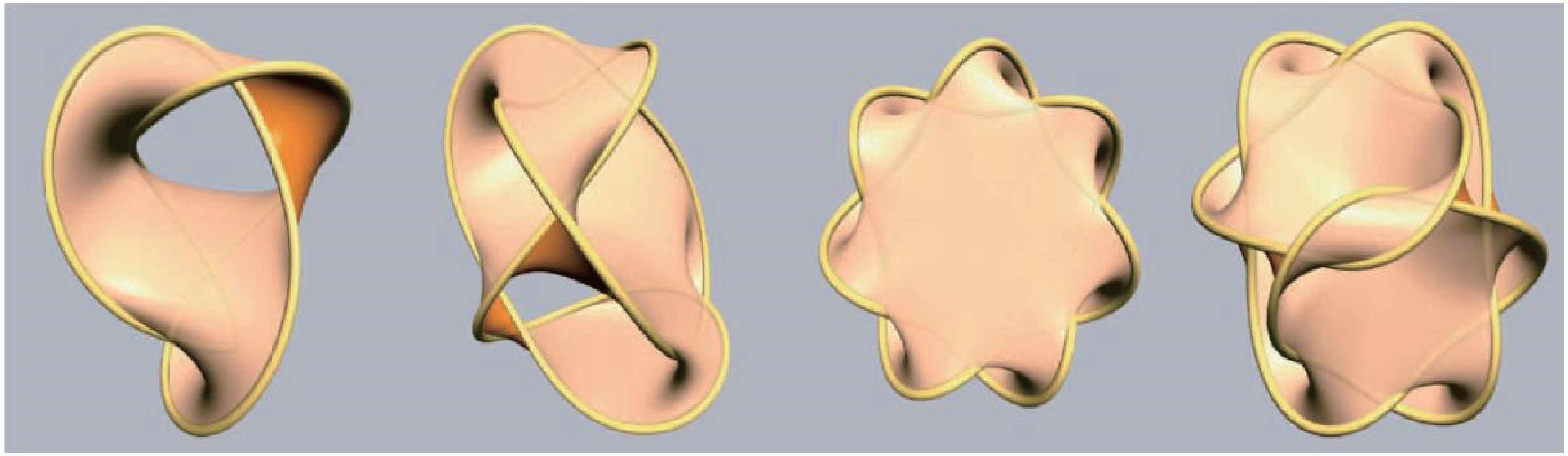

Seifert surfaces

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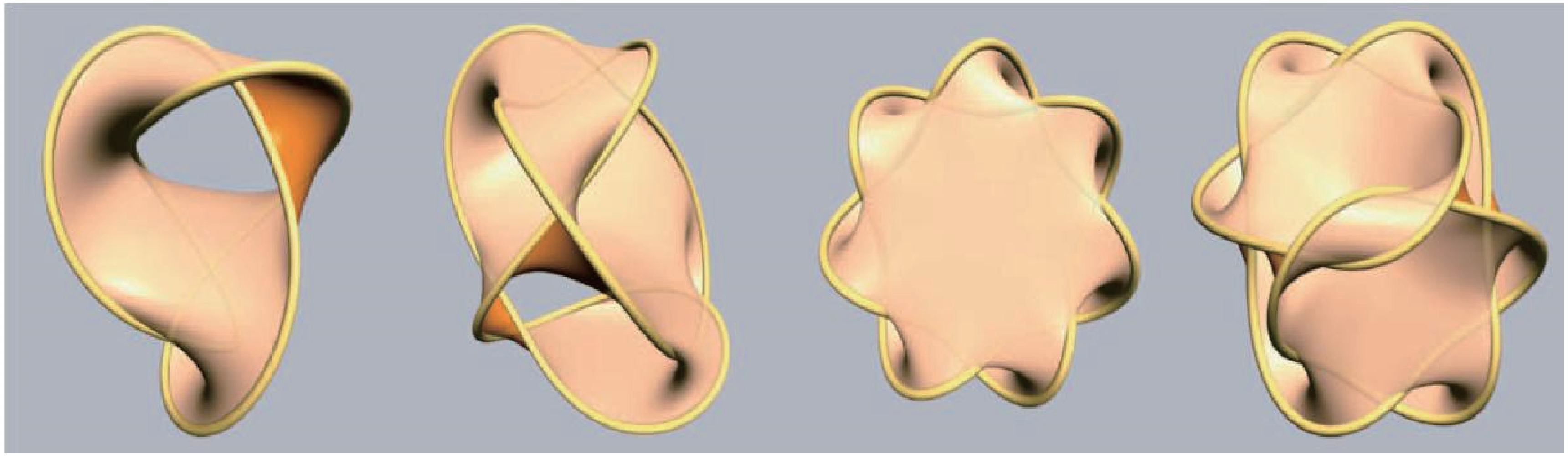
<u>TODAY</u> **The case** $\phi = \theta \smile \theta'$. **(**$\theta$ **are 1-cocycles)**

$\Longrightarrow$ **The pairing is a bilinear form.**



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used by Ordinary homology & Bockstein maps.

Ex.2 Signature. e.g, Casson-Gordon invariant, L^2-signature.

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Q'dl cocycle invariant [Carter-Jelsovsky-Kamada-Langford-Saito' 99]

$$\text{“}\mathrm{Col}_X(D) \longrightarrow M\text{”}.$$

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Q'dl cocycle invariant [Carter-Jelsovsky-Kamada-Langford-Saito' 99]

Today: "$\mathrm{Col}_X(D) \times \mathrm{Col}_X(D) \longrightarrow M$".

diagrammatically constructed $\rightsquigarrow$ Prob. top. meaning.

Results (Outlined)

GOAL: show twisted pairings of $S^3 \setminus L$ are pretty useful.

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Every twisted pairing can be diagrammatically described in terms of quandle theory.

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Recover → **Classical objects**

- Blanchfield paring of knots.
- Casson-Gordon's related signatures

Def. → **∞-ld coverings**

Bilinear forms on twisted Alex. module

w/ non-degeneracy (cf. Milnor duality)

Application. → **4-dim objects**

Def. invariants of 4-dim Lefschetz fibrations over S^2.

(Not told today)

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Rem. **This is a survey, without proofs (13 sheets).**
Several statements today hold for links in S^3.
However, we focus on only knots.

Main Result on the twisted pairings (Keep this in mind)

(Invariant of hom's $f : \pi_1(Y) \longrightarrow G$. Here $Y := S^3 \setminus K$)

Input M : right G-module/ a comm. ring A

$\psi : M \otimes M \longrightarrow A$: bilinear s.t. $\psi(a \cdot g, b \cdot g) = \psi(a, b)$.

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$$\smile_\psi : H^1(Y, \partial Y; M)^{\otimes 2} \xrightarrow{\ \smile\ } H^2(Y, \partial Y; M^{\otimes 2}) \longrightarrow$$

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can be described as a "quandle cocycle inv." diagrammatically.

Rem.

$\smile_\psi$ seems *un*computable (cf. Σ & longitude $\in \pi_1(\partial Y)$).

Description of the q'dl cocycle inv. in 1-page

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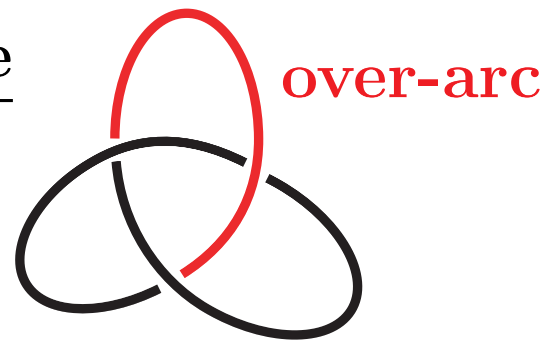
Given $f : \pi_1(S^3 \setminus K) \longrightarrow G$,

Wirtinger present f as $\{$ over-arcs $\} \xrightarrow{\text{map}} G$.



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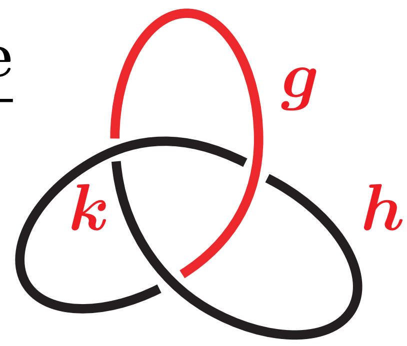

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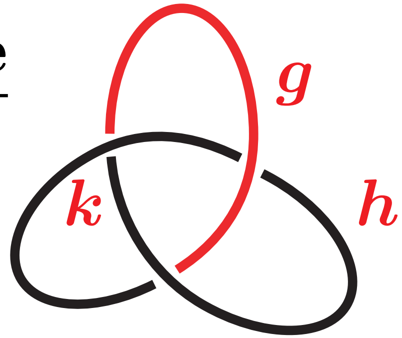

Given $f : \pi_1(S^3 \setminus K) \longrightarrow G$,

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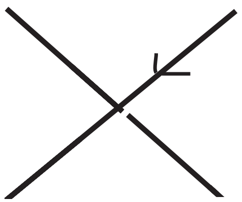
Def.[IIJO] A coloring is $\mathcal{C} : \{ \text{ over-arcs } \} \to M \times G$ over f

s.t. (x, g)  $(y, h) \in M \times G$

$$\Downarrow$$

$$(y + (x - y) \cdot h,\ h^{-1} g h\ )$$

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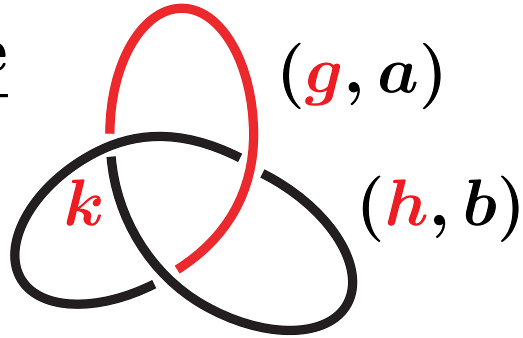

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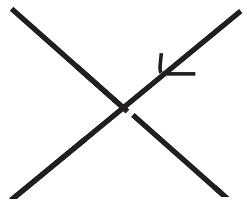

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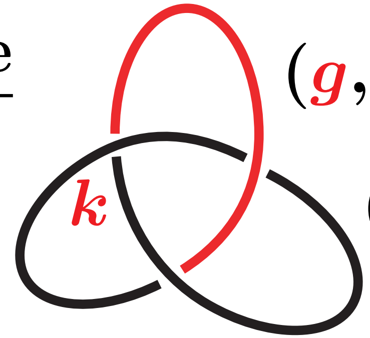

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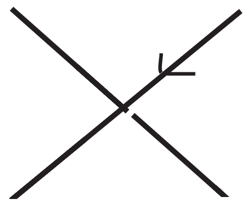

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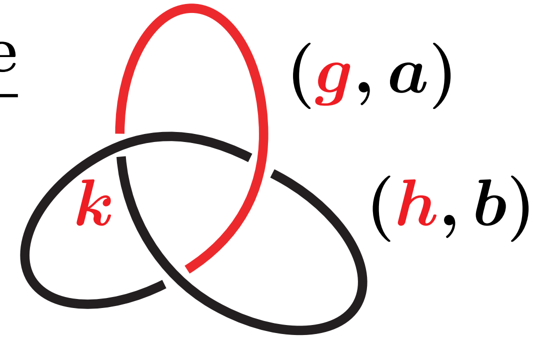

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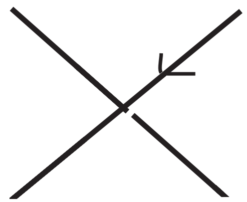
s.t. (x, g)  $(y, h) \in M \times G$

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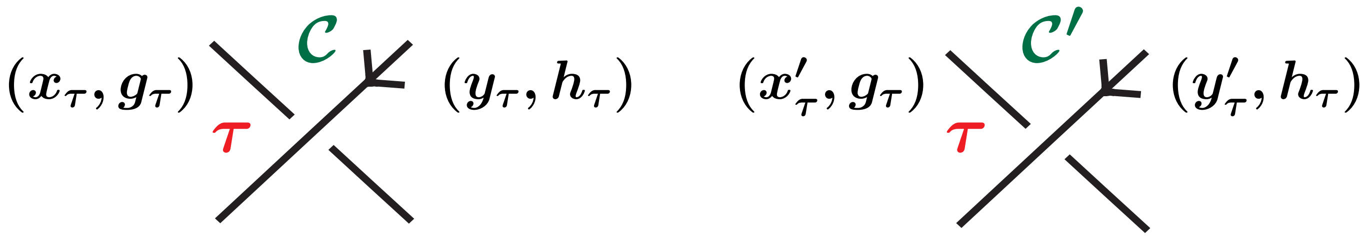
Def.[N.] cf. diagonally [CJKLS].)

Define the quandle cocycle inv. of f by the binary map

$$\mathcal{Q}_\psi : (\mathrm{Col}(D_f))^2 \longrightarrow A$$

$$\begin{array}{ccc} \Psi & & \Psi \\ (\mathcal{C},\ \mathcal{C}') & \longmapsto & \displaystyle\sum_{\tau:\ \text{crossing}} \epsilon_\tau \psi(x_\tau - y_\tau,\ y'_\tau(1 - h_\tau^{-1})). \end{array}$$



Main Result; q'dl inv $\mathcal{Q}_\psi$ VS the twisted pairings $\smile_\psi$

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Thm.[N.] $\mathrm{Col}(D_f)^{\exists} \cong H^1(Y, \partial Y; M) \oplus M$

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Advantages

- $\smile_\psi$ become computable $\Longrightarrow$ We may study only $\smile_\psi$.
- We can compute $\mathcal{Q}_\psi$, w/ describing no Seifert surface.

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Q. Does the above setting include classical objects?

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§2 Classical Blanchfield pairings

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§2 Classical Blanchfield pairings

The pairing of a knot K was roughly

$$\underbrace{H_1(S^3 \setminus K; \mathbb{Z}[t^{\pm 1}])}_{\uparrow\ \text{Alexander module}}{}^{\otimes 2} \xrightarrow{\text{bilinear}} \mathbb{Z}[t^{\pm 1}]/(\Delta_K).$$

§3 Relation to infinite cyclic covers.

Apply $M = \Lambda/(\Delta_K)$ to the Main thm. $(\Lambda := \mathbb{Z}[t^{\pm 1}])$:

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Quite analogous to the Bl_K pairing!!

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Key (Poincaré duality) [Trotter, Levine]

$H^1(Y, \partial Y; M) \overset{\exists}{\cong} H_1(S^3 \setminus K; \mathbb{Z}[t^{\pm 1}])$ explicitly.

Recovery of the classical Blanchfield pairing

Recall the pairing of a knot K is

$$\mathrm{Bl}_K : H_1(S^3 \setminus K; \mathbb{Z}[t^{\pm 1}])^{\otimes 2} \xrightarrow{\text{bilinear}} \mathbb{Z}[t^{\pm 1}]/(\Delta_K).$$

Thm(N.) $x_1, x_2 \in \ H_1(S^3 \setminus K; \mathbb{Z}[t^{\pm 1}]) \overset{\exists}{\cong} H^1(Y, \partial Y; \Lambda/\Delta_K)$.

s.t. $\smile_\psi (x_1,\ x_2) = \dfrac{1+t}{1-t} \cdot \mathrm{Bl}_K(x_1 \otimes x_2).$

<u>p.f.</u> Describe $\smile_\psi$ by a Seifert surface. $\square$

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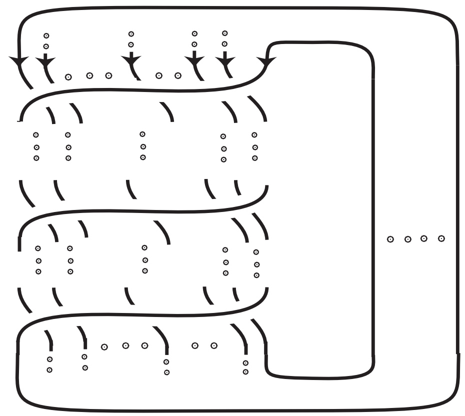
$\Downarrow$ with some discussion in quandle theory.

Cor. (topological meaning)

${}^{\forall}$the 2-cocycle inv. w.r.t ${}^{\forall}$Alexander q'dl recover from Bl_K

i.e., the inv. is a complete inv. of the "S-equivalence".

Corollary: Bl_K of the torus knot $K = T_{m,n}$.



Fact: Alexander module $\mathbb{Z}[t^{\pm 1}]/\Delta_K$,

where $\Delta_K = \frac{(t^{nm}-1)(t-1)}{(t^n-1)(t^m-1)}$.

Thm. (N.) Let $(n, m, a, b) \in \mathbb{Z}^4$ be $an + bm = 1$.

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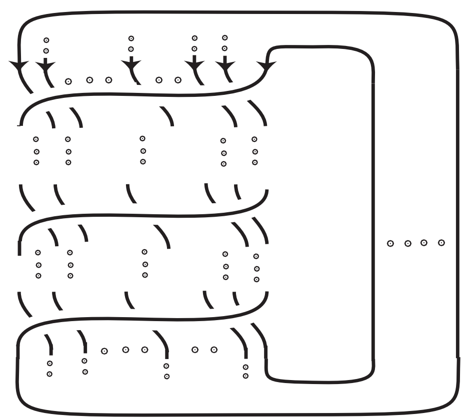
Rem.

Corollary: Bl_K of the torus knot $K = T_{m,n}$.



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Rem. (I) The proof is done within One-page.

(II) Seifert martix is of rank $g_* = (n-1)(m-1)/2$.

(cf. signature [T. Matsumoto, '77])

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- **Infinite cyclic covering & Non-degeneracy**
- **Casson-Gordon "related" signatures.**

Bilinear form on the twisted Alexander (dual) module

Given $f : \pi_K \longrightarrow GL_n(\mathbb{F}), \qquad \rho : \pi_K \xrightarrow{\text{Ab.}} \mathbb{Z} = \langle t^{\pm 1} \rangle.$

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$\smile_\psi H^1(S^3 \setminus K; (\mathbb{F}[t^{\pm 1}]/\Delta_f)^n)^{\otimes 2} \longrightarrow \mathbb{F}[t^{\pm 1}]/\Delta_f$. bilinear.

Assumption (technical, but a week condi. You may forget)

1 The poly $\Delta_f \neq 0$. that is, H_1 is of fin. dim over $\mathbb{F}$.

2 Δ_f is relatively prime to $\det(\mathrm{id} - f \otimes \mathrm{Ab}(\mathrm{meridian}))$.

Topological meaning & degeneracy of $\smile_\psi$ under the assumption.

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Observation (Bilinear form considered [Kirk-Livingston])

$$H^1(Y_\infty, \partial Y_\infty; \mathbb{F}^n)^2 \xrightarrow{\smile} H^2(Y_\infty, \partial Y_\infty; \mathbb{F}^n) \xrightarrow{\psi(\bullet\cap\Sigma)} \mathbb{F}.$$

where Y_∞ is the ∞-cyclic cover of $S^3 \setminus K$.

Thm.[N.]

$$H^1(Y, \partial Y; (\mathbb{F}[t]/\Delta_f)^n)^{\otimes 2} \xrightarrow{\smile_\psi} \mathbb{F}[t]/\Delta_f \quad \text{in } (-1)\text{-page}$$

$$H^1(Y_\infty, \partial Y_\infty; \mathbb{F}^n)^{\otimes 2} \xrightarrow{\smile_{KL}} \mathbb{F} \quad \text{as above}$$

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$$\begin{tikzcd} H^1(Y, \partial Y; (\mathbb{F}[t]/\Delta_f)^n)^{\otimes 2} \arrow[r, "\smile_\psi"] \arrow[d, "\exists\text{Isom.}"', "\wr"] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{F}[t]/\Delta_f \arrow[d, "\text{“Trace map”[Neumann]}"] \\ H^1(Y_\infty, \partial Y_\infty; \mathbb{F}^n)^{\otimes 2} \arrow[r, "\smile_{KL}"] & \mathbb{F} \end{tikzcd}$$

in (-1)-page

as above

Rem. Conversely, $\smile_\psi$ can recover from $\smile_{KL}$ as

$$\smile_\psi (x, y)/\Delta_f = \sum_{j\geq 0} (\tau_*^{-j} x) \smile_{KL} y \cdot t^j \in \mathbb{F}((t))/\mathbb{F}[t].$$

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Cor. [N.] ("Twisted" Milnor duality of Y_∞).

If the $f : \pi_K \longrightarrow GL_n(\mathbb{F})$ is unitary or symplectic,

then the pairing $\smile_\psi$ is non-degeneracy.

Result 1.5 Casson-Gordan signature $\boldsymbol{\sigma}_{\omega}^{\mathrm{CG}}(K, \chi)$

Announce. (N.) Let $\omega \in \mathbb{C}$ be "generic" and $|\omega| = 1$.

Using no 4-mfds, we get a diagrammatic computation for

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Casson-Gordan (meta-abelian) signature (Rough review)

Input $(q,d) \in \mathbb{Z}^2$: prime powers, $\quad w \in \mathbb{C}$

$\chi : H_1(S^3 \setminus K;\Lambda) \longrightarrow \mathbb{Z}/d$ with "Gilmer's condi.".

Output

$\mathrm{Sign}_w(\mathrm{Int}(W_{\mathrm{ab}\otimes\hat{\chi}}) - \mathrm{Int}(W_{\mathrm{tri}})) \in \mathbb{Q}$. concordance inv.

Rem.
- W: 4-mfd s.t. $\partial W = \sharp^q$ (q-fold cover bra. over K)
- [Kirk-Livingston] Cup product without p.f. & Exa.

Four topological Applications from this work (Brief)

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Thm. [N. '14] (in 1-knots cases)

Top. meaning of 2-cocycle inv. w.r.t. Alexander q'dls.

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Thm. [Ishii-Iwakiri-Jang-Oshiro. '12]

They found new handlebody knots which are achiral.

(Here they used q'dl cocycles in [N.'12])

Thm. [N. '14]

References

[1] **T. Nosaka,** *Twisted cohomology pairings of knots I; diagrammatic computation*, **arXiv:1602.01129**

[2] **T. Nosaka,** *Twisted cohomology pairings of knots II; to classical invariants*, **arXiv:1602.01131**

[3] **T. Nosaka,** *Bilinear-form invariants of Lefschetz fibrations over the 2-sphere*, **preprint.**

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GOAL: show twisted pairings of $S^3 \setminus K$ are pretty useful.

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Every twisted pairing can be diagrammatically described in terms of quandle theory.

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(cf. Milnor duality)

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• Casson-Gordon's related signatures	w/ non-degeneracy (cf. Milnor duality)	(Not told today)
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Thank You