On the fundamental relative 3-classes of knot group representations

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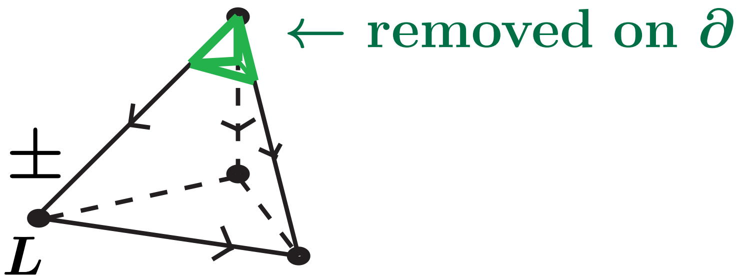
Contents & Plan

§0 Motivations and three results

$E_L := S^3 \setminus \nu L$: link complement with triangulation

$[E_L,\ \partial E_L]$: the fund. 3-class $\in H_3(E_L,\ \partial E_L; \mathbb{Z}) \cong \mathbb{Z}$.

Here, $[E_L,\ \partial E_L] := \sum_{\text{tetrahedra of } L} \pm$ 

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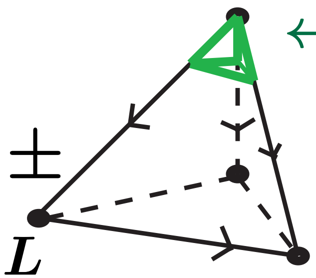
Here, $[E_L,\ \partial E_L] := \sum_{\text{tetrahedra of } L} \pm$ 

Def. Relative fund. 3-class w.r.t. $H \subset G$ grps.

INPUT $f : \pi_1(S^3 \setminus L) \longrightarrow G$ s.t. $f(\partial(S^3 \setminus \nu L)) \subset H$.

$\phi : G^3 \longrightarrow A$ "relative" group 3-cocycle.

OUTPUT: $\langle f^*(\phi),\ [E_L,\ \partial E_L] \rangle \in A$

POINT: The value seems uncomputable, from def.

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Examples

- Complex volume of hyperbolic links   [W.Neumann],

  if f: the holonomy, $G = SL_2(\mathbb{C})$, ϕ: Chern-Simons 3-class.

- It is called Dijkgraaf-Witten invariants, if $|G| < \infty$.
- Milnor link inv., if the settings are "nilpotent" [Turaev, Porter]
- Triple cup products, if ϕ is a trilinear map   (seen later)

$\Longrightarrow$ So, we hope theorems in the above setting.

(Again) **Relative fund. 3-class** $H \subset G$ **grps.**

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L: non-cable knot, or hyperbolic link.

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L: non-cable knot, or hyperbolic link.
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the output is computable from only link diagrams.

Theorem 2 (N.) [depend on only $H \subset G$. Applicable]

If L is a knot and $H \subset G$ is "malnormal",
$\Longrightarrow$
the computation forms a "quandle cocycle inv."

- $H \subset G$ is malnormal, if $gHg^{-1} \cap H = 1$ for $^\forall g \in G \setminus H$.

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Theorem 3 (N.)

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(Trilinear output from hom $f : \pi_1(E) \longrightarrow G$, where $E := S^3 \setminus \nu L$)

Input M : right G-module/ a ring A

$$\phi : M^3 \xrightarrow{\text{trilinear}} A \quad \text{s.t. } \phi(a\cdot g, b\cdot g, c\cdot g) = \phi(a, b, c).$$

Output: $H^1(E, \partial E; M)^3 \xrightarrow{\text{cup prod.}} H^3(E, \partial E; M^{\otimes 3}) \to$

$$\xrightarrow{\bullet\cap \text{rel. fund. 3-class}} M \otimes M \otimes M \xrightarrow{\langle \phi, \bullet \rangle} A.$$

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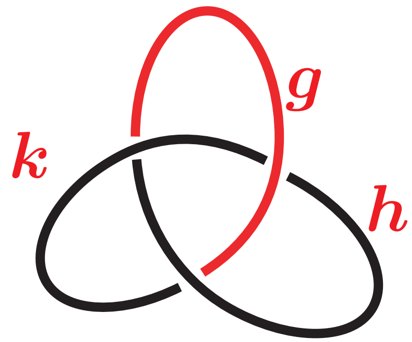
On $H^1(E, \partial E; M)$ w.r.t. $\pi_1(S^3 \setminus L) \xrightarrow{f} G$

$\{$ over-arcs $\} \xrightarrow{\text{map}} G$:   Wirtinger pre. of f.



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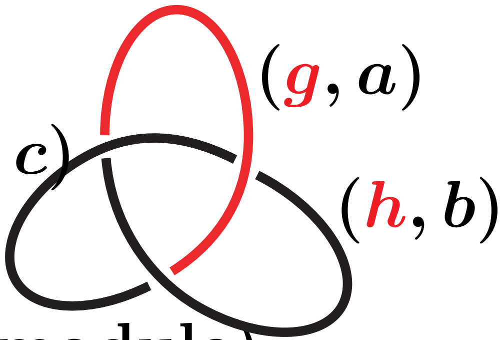
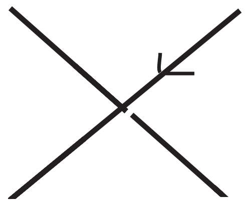


Def. [Ishii-Iwakiri-Jang-Oshiro]. (M: right $\mathbb{Z}[G]$-module)

- **Coloring over f** is $\mathcal{C}$: $\{$ over-arcs $\} \to M \times G$ over f

  s.t. (x, g)  $(y, h) \in M \times G$

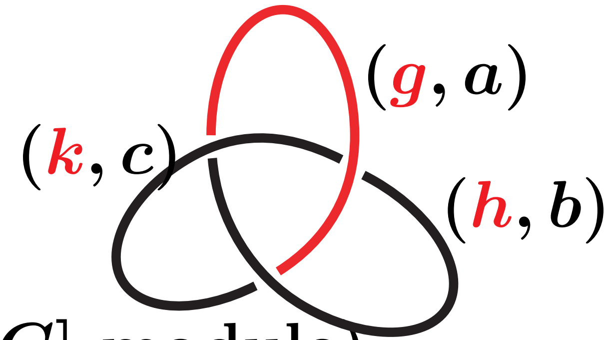
  $\Downarrow$

  $(y + (x - y) \cdot h,\ h^{-1}gh) \in M \times G$

- **A shadow of $\mathcal{C}$** is λ : $\{$ regions of D $\} \to M$ s.t.

  (i) (h, b)  a $\quad b + (a - b) \cdot h$ $\qquad$ (ii) $\lambda(\{\infty\}) = 0 \in M$.

On $H^1(E, \partial E; M)$ w.r.t. $\pi_1(S^3 \setminus L) \xrightarrow{f} G$



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Thm.[N.14]

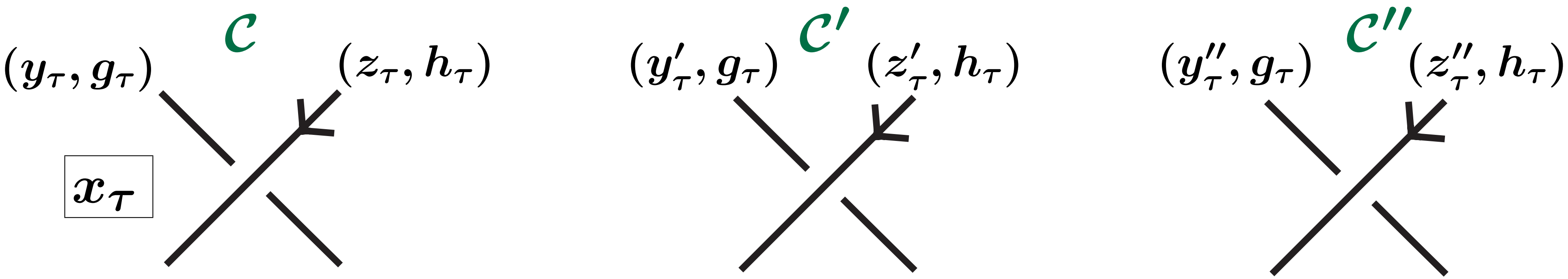
$^{\exists}$isom. : $\mathrm{SCol}(D_f) \cong H^1(E, \partial E; M) \oplus M$.

Def.[N.] $\mathcal{T}_\phi : (\mathrm{SCol}(D_f))^3 \to A.$

For shadow coloring $\mathcal{C},\ \mathcal{C}',\ \mathcal{C}'' \in \mathrm{SCol}(D_f)$, we define

$$\mathcal{T}(\mathcal{C},\ \mathcal{C}',\ \mathcal{C}'') := \sum_{\tau:\ \text{crossing}} \phi\Big((x_\tau - y_\tau)(1-g_\tau^{\epsilon_\tau}),\ y'_\tau - z'_\tau,\ z''_\tau\,(1-h_\tau^{-1})\Big).$$

Here, the three colorings around the τ are given by



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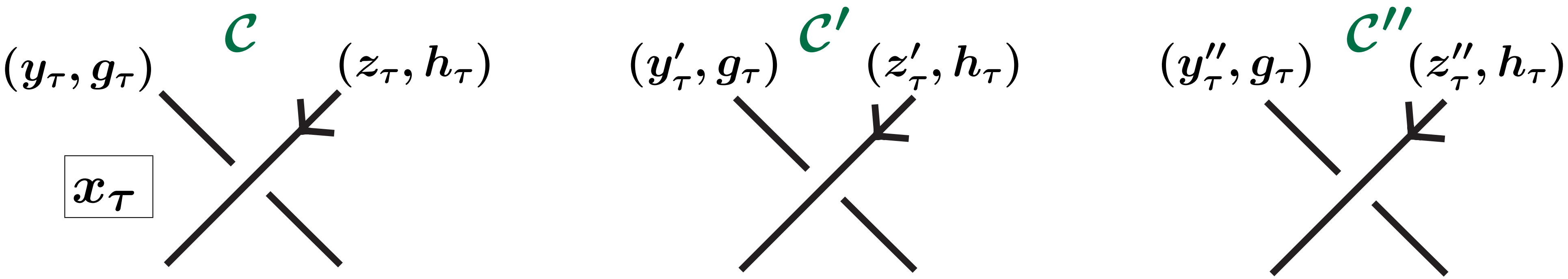

Prop.(Invariance w.r.t. f)[N.]

If $D \overset{R\text{-moves}}{\longleftrightarrow} D'$,

$$\mathrm{SCol}(D_f)^3 \overset{\exists\text{isom.}}{\longleftrightarrow} \mathrm{SCol}(D'_f)^3, \quad \mathcal{T}_\phi : \mathrm{SCol}(D_f)^3 \to A, \quad \mathcal{T}'_\phi : \mathrm{SCol}(D'_f)^3 \to A \quad (\circlearrowleft)$$

Thm.[N.]

$^{\exists}$**Isom** : $\mathrm{SCol}(D_f) \cong H^1(E, \partial E; M) \oplus M$.

Further, if L is a knot or a hyperbolic link,

$\mathcal{T}_\phi : \mathrm{SCol}(D_f)^3 \to A$ **agrees with the triple product.**

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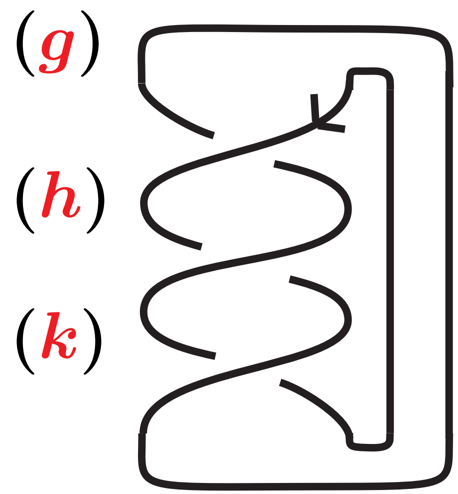

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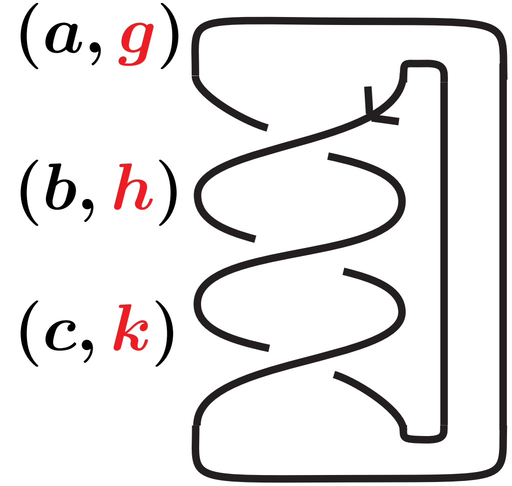

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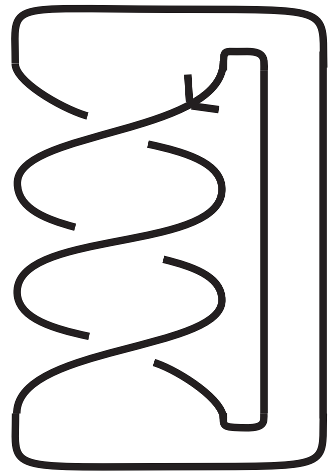

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$$\iff \quad (a-b)\cdot(1-g+hg) = (a-b)\cdot(1-h+gh) = 0. \quad (\clubsuit)$$

Prop. (by the previous thm.)

$$H^1(E, \partial E; M) \oplus M$$
$$\cong \left\{\ (a, b) \in M^2 \ \middle|\ (\clubsuit)\text{ is satisfied.}\ \right\}.$$

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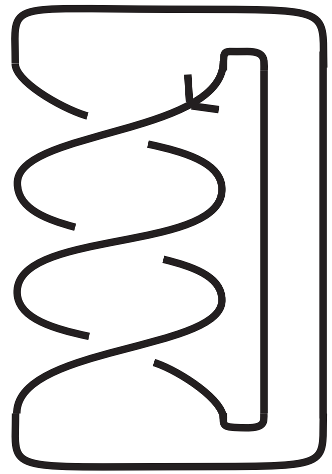

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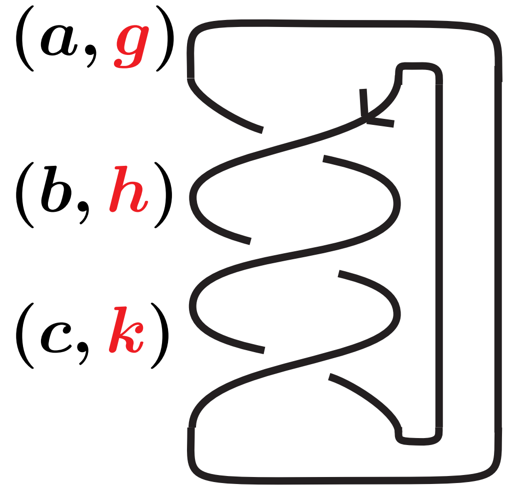

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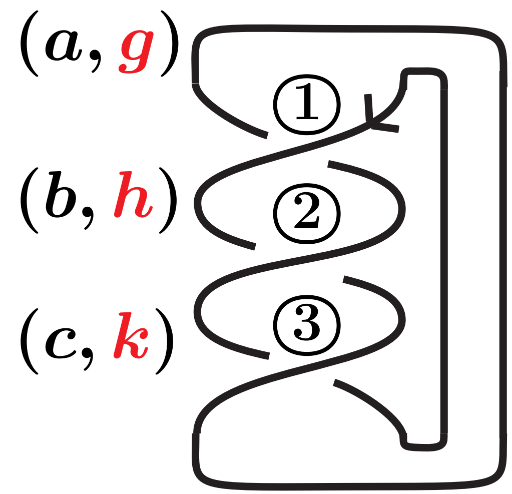

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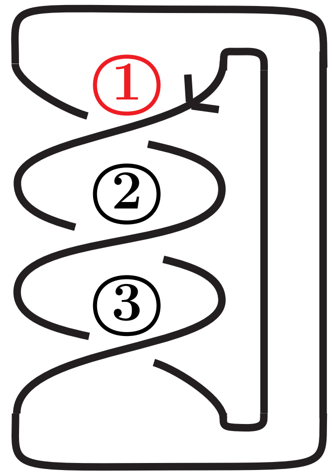

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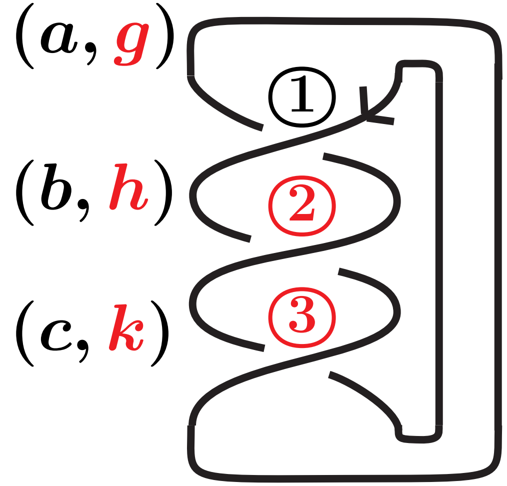

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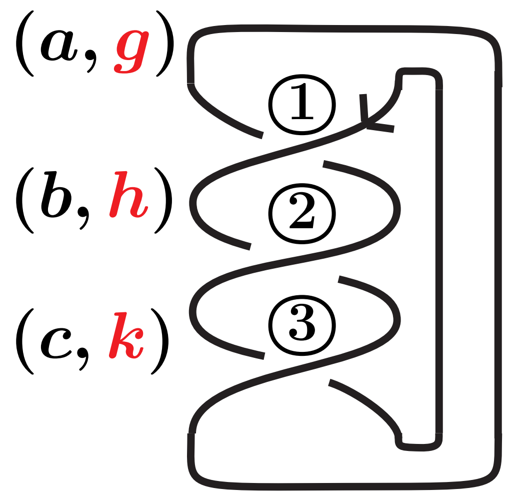

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$$= \cdots\cdot = \phi\Big((a-b)g^{-1},\ (a'-b')\cdot h,\ (a''-b'')\Big) \in A.$$

Summary

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***L*: non-cable knot, or hyperbolic link.**
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The rel. fund. 3-clases are computable from diagrams.

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Conclusion : Many relative fund. 3-classes can be regarded as computable 2-dimension objects.

- The computation is elementary using linear algebra.

HOPE: Continued study of fundamental 3-classes of knots

Thank you