The relative cup products and quandle cocycle invariants of knots

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§0 Introduction; Two Motivations and Results.

1. **Develop knot theory from** $\left(\begin{array}{l} \textbf{rel. cup products on } \boldsymbol{H^1}, \\ \textbf{Bilinear forms} \end{array} \right.$

2. **Find applications of "quandles".**

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Ex.1 (Higher) Blanchfield pairing.

Ex.2 [Cochran-Orr-Teiechner] "The solvable filtration"

$$\cdots \subset \mathcal{F}_{(n.5)} \subset \mathcal{F}_{(n)} \subset \cdots \subset \mathcal{F}_{(1.5)} \subset \mathcal{F}_{(1)} \subset \mathcal{F}_{(0.5)} \subset \mathcal{F}_{(0)} \subset \mathcal{C}$$

used by Ordinary homology & Intersection forms.

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Q'dl cocycle invariant [Carter-Jelsovsky-Kamada-Langford-Saito' 99]

$$\left(\begin{array}{l} X : \text{ quandle} \\ \phi : X^2 \to A : \text{ "2-cocycle"} \end{array} \right) \Longrightarrow \text{'}\mathrm{Col}_X(D) \to A\text{'}.$$

Rem. Few top. meaning so far.

Results (Outlined)

GOAL: show rel. cup products of $S^3 \setminus L$ are pretty useful.

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Every rel. cup product can be diagrammatically described in terms of quandle theory.

$\Longrightarrow$

We may abstractly research only the relative cup products.

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Recover → **Classical subjects**

- Blanchfield paring of knots.
- Casson-Gordon's related signatures

Def. → **∞-ld coverings**

Twisted version of Blanchfield parings

w/ non-degeneracy (cf. Milnor duality)

Application. → **4-dim objects**

Def. invariants of 4-dim Lefschetz fibrations over S^2.

(Not told today)

Four topological Applications from this work (Brief)

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Thm. [N. '14] (in 1-knots cases)

Top. meaning of 2-cocycle inv. w.r.t. Alexander q'dls.

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I first compute the classical Bl. pairing of the torus knots.

Thm. [N. '15]

A method for computing the "related" CG signatures.

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Thm. [Ishii-Iwakiri-Jang-Oshiro. '12]

They found new handlebody knots which are achiral.

(Here they used q'dl cocycles in [N.'12])

Thm. [N. '14]

$^{\exists}$closed 4-mfd ($\overset{\text{homeo}}{\cong}$ K3 surface) has

$\infty^{\exists}$ Lefschetz fibration structures with the same SW inv.

Contents

Rem. This is a survey, without proofs (17 sheets). Several statements today hold for links in S^3. However, we focus on only knots.

Main Result on the relative cup product (Keep this in mind)

(Invariant of hom's $f : \pi_1(Y) \longrightarrow G$. Here $Y := S^3 \setminus K$)

Input M : right G-module/ a comm. ring A

$\psi : M \otimes M \longrightarrow A$: bilinear s.t. $\psi(a \cdot g, b \cdot g) = \psi(a, b)$.

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$$\smile_\psi : H^1(Y, \partial Y; M)^{\otimes 2} \xrightarrow{\smile} H^2(Y, \partial Y; M^{\otimes 2}) \longrightarrow$$

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can be described as a "quandle cocycle inv." diagrammatically.

Rem.

$\smile_\psi$ seems *un*computable (cf. Σ & longitude $\in \pi_1(\partial Y)$).

Description of the q'dl cocycle inv. in 1-page

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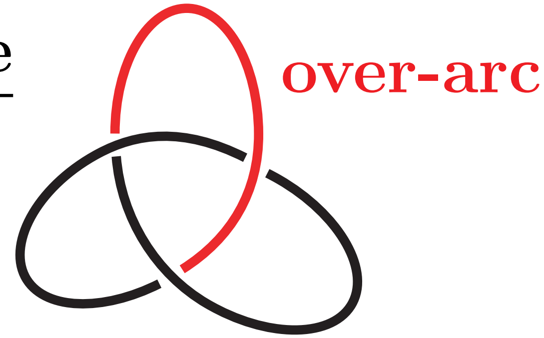
Given $f : \pi_1(S^3 \setminus K) \longrightarrow G$,

Wirtinger present f as $\{$ over-arcs $\} \xrightarrow{\text{map}} G$.



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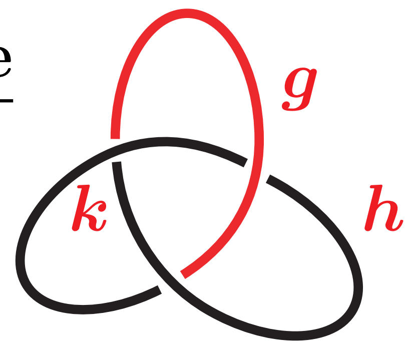

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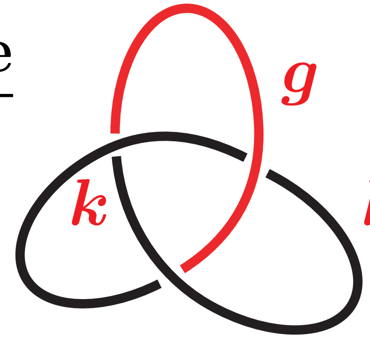

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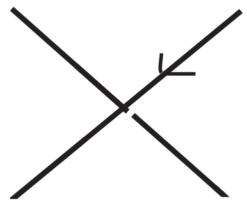
s.t. (x, g)  $(y, h) \in M \times G$

$\Downarrow$

$(y + (x - y) \cdot h,\ h^{-1}gh\ )$

$\mathrm{Col}(D_f) \overset{\text{def}}{=} \{$ Coloring $\mathcal{C}$ (over f) $\}$.

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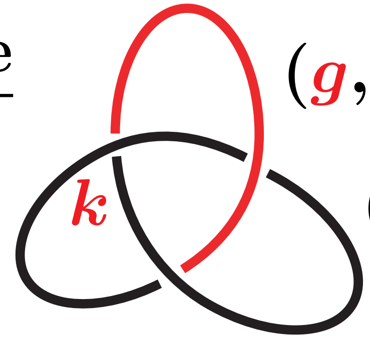

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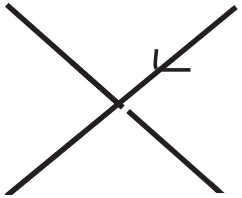

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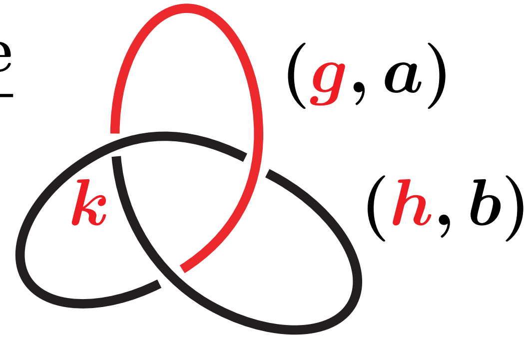

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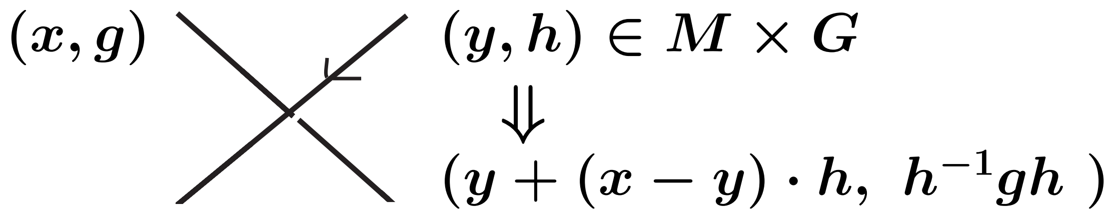
s.t. 

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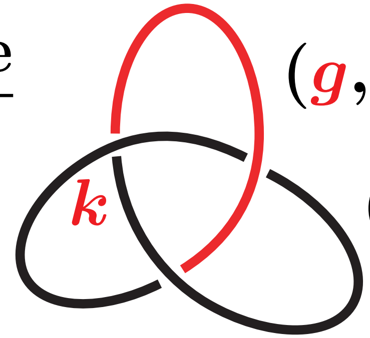

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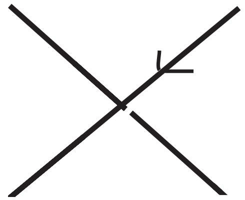
s.t. (x, g)  $(y, h) \in M \times G$

$$\Downarrow$$

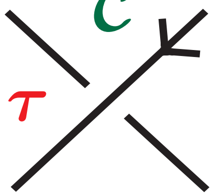
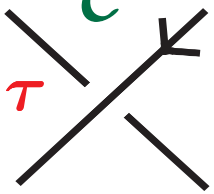
$$(y + (x - y) \cdot h,\ h^{-1}gh\ )$$

Def.[N.] cf. diagonally [CJKLS].)

Quandle cocycle inv. of f is the binary map

$$\mathcal{Q}_\psi : (\mathrm{Col}(D_f))^2 \longrightarrow A$$

$$\begin{array}{ccc} \cup\!\!\!\cup & & \cup\!\!\!\cup \\ (\mathcal{C},\ \mathcal{C}') & \longmapsto & \displaystyle\sum_{\tau:\ \text{crossing}} (-1)^{\epsilon_\tau}\psi(x_\tau - y_\tau,\ y'_\tau(1 - h_\tau^{-1})). \end{array}$$

(x_τ, g_τ)  (y_τ, h_τ) $\quad$ (x'_τ, g_τ)  (y'_τ, h_τ)

Main Result; q'dl inv $\mathcal{Q}_\psi$ VS the relative cup product $\smile_\psi$

(Invariant of hom's $f : \pi_1(Y) \longrightarrow G$. Here $Y := S^3 \setminus K$. (Again.)

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$\Sigma \subset S^3 \setminus K$: a Seifert surface $\in H_2(Y, \partial Y; \mathbb{Z})$

Thm.[N.] $\mathrm{Col}(D_f)^{\exists} \cong H^1(Y, \partial Y; M) \oplus M$

s.t. $\mathrm{res}(\mathcal{Q}_\psi)|_{H^1}$ coincides with

$$\smile_\psi : H^1(Y, \partial Y; M)^2 \xrightarrow{\smile} H^2(Y, \partial Y; M^{\otimes 2}) \xrightarrow{\psi(\bullet \cap \Sigma)} A.$$

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- $\smile_\psi$ become computable $\Longrightarrow$ We may study only $\smile_\psi$.
- We can compute $\mathcal{Q}_\psi$, describing no Seifert surface.

Q.

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Q. Does the above setting include classical subjects?

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§1 Relative cup products

§2 Classical Blanchfield pairings

The pairing of a knot K is roughly

$$\underbrace{H_1(S^3 \setminus K; \mathbb{Z}[t^{\pm 1}])}_{\uparrow}{}^{\otimes 2} \xrightarrow{\text{bilinear}} \mathbb{Z}[t^{\pm 1}]/(\Delta_K).$$

Alexander module

§3 Twisted Blanchfield pairings

Main thm (again). Apply $M = \Lambda/(\Delta_K)$ to the thm. :

(Invariant of hom's $f : \pi_1(Y) \longrightarrow G$. Here $Y := S^3 \setminus K$. (Again.)

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Recover of Bl_K. The thm. with $M = \Lambda/(\Delta_K)$ ($\Lambda := \mathbb{Z}[t^{\pm 1}]$):

(Invariant of hom's $f : \pi_1(Y) \longrightarrow \mathbb{Z}$. Here $Y := S^3 \setminus K$, $\mathbb{Z} = \langle t \rangle$)

Input $M := \Lambda/(\Delta_K)$

$\psi : M \otimes M \longrightarrow \Lambda/(\Delta_K)$ defined by $\psi(a,b) = \overline{a}b$.

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Key (Poincaré duality) [Trotter, Levine]

$H^1(Y, \partial Y; M) \overset{\exists}{\cong} H_1(S^3 \setminus K; \mathbb{Z}[t^{\pm 1}])$ explicitly.

Recover of the classical Blanchfield pairing

Recall the pairing of a knot K is

$$\mathrm{Bl}_K : H_1(S^3 \setminus K; \mathbb{Z}[t^{\pm 1}])^{\otimes 2} \xrightarrow{\text{bilinear}} \mathbb{Z}[t^{\pm 1}]/(\Delta_K).$$

Thm(N.) $\mathrm{Col}(K) \overset{\exists}{\cong} H_1(S^3 \setminus K; \mathbb{Z}[t^{\pm 1}]) \oplus X \ni \mathcal{C}_i = (x_i, 0)$

s.t. $$\mathcal{Q}_\psi(\mathcal{C}_1,\ \mathcal{C}_2) = \frac{1+t}{1-t} \cdot \mathrm{Bl}_K(x_1 \otimes x_2).$$

p.f. Describe $\smile_\psi$ by a Seifert surface. $\square$

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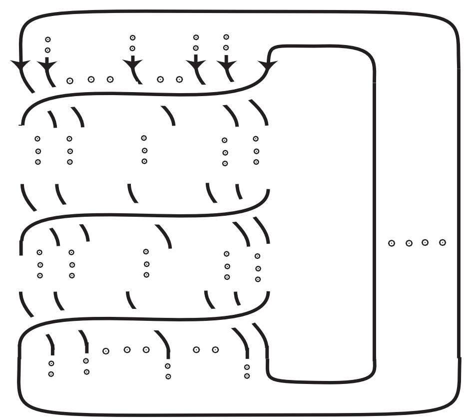
$\Downarrow$ with some discussion in quandle theory.

Cor. (topological meaning)

$^\forall$the 2-cocycle inv. w.r.t $^\forall$Alexander q'dl recover from Bl_K

i.e., the inv. is a complete inv. of the "S-equivalence".

Corollary : Bl_K of the torus knot $K = T_{m,n}$.



Fact : Alexander module $\mathbb{Z}[t^{\pm 1}]/\Delta_K$,

where $\Delta_K = \dfrac{(t^{nm}-1)(t-1)}{(t^n-1)(t^m-1)}$.

Thm. (N.) Let $(n, m, a, b) \in \mathbb{Z}^4$ be $an + bm = 1$.

$$\mathrm{Bl}_K(x, y) = \frac{nm}{(1+t^{-1})(1-t^{bm})(1-t^{an})} \cdot \bar{x}y \in \mathbb{Z}[t^{\pm 1}]/(\Delta_K),$$

for $x, y \in \mathbb{Z}[t^{\pm 1}]/(\Delta_K)$.

Rem.

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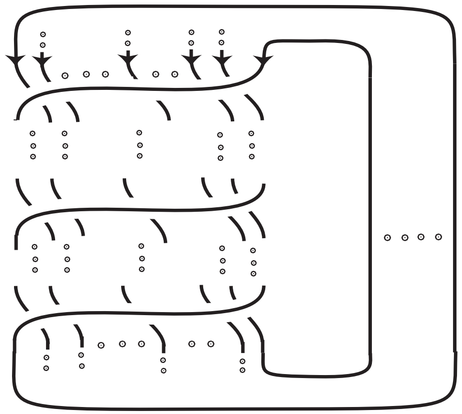

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for $x, y \in \mathbb{Z}[t^{\pm 1}]/(\Delta_K)$.

Rem. (I) The proof is done within One-page.

(II) Seifert martix is of rank $g_* = (n-1)(m-1)/2$.

(cf. signature [T. Matsumoto, '77])

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§3 Twisted Blanchfield pairings

I introduce bilinear forms on the twisted Alexander modules.

$\Longrightarrow$

- **Infinite cyclic covering & Non-degeneracy**
- **Casson-Gordon "related" signatures.**

Bilinear form on the twisted Alexander module

Given $f : \pi_K \longrightarrow GL_n(\mathbb{F})$, $\quad \rho : \pi_K \xrightarrow{\text{Ab.}} \mathbb{Z} = \langle t^{\pm 1} \rangle$.

$\Longrightarrow$

$\rho \otimes f : \pi_K \longrightarrow GL_n(\mathbb{F}[t^{\pm 1}])$ as a local coefficient.

The twisted Alexander module is $H_1(S^3 \setminus K; \mathbb{F}[t^{\pm 1}]^n)$.

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Result[N.] (rough) $(G := GL_n(\mathbb{F}))$ Here $\mathbb{F}$ is a field of Char. 0.

Input $f : \pi_1(S^3 \setminus K) \longrightarrow GL_n(\mathbb{F})$.

$\psi : \mathbb{F}^n \otimes \mathbb{F}^n \longrightarrow \mathbb{F}$: bilinear s.t. $\psi(a \cdot g, b \cdot g) = \psi(a, b)$.

Output

$H_1(S^3 \setminus K; (\mathbb{F}[t^{\pm 1}])^n)^{\otimes 2} \longrightarrow \mathbb{F}[t^{\pm 1}]/\Delta_f$. bilinear.

The twisted Alex. poly.[Wada, Lin] is the order of the H_1,

i.e., $\Delta_f :=$ L.C.M.$\{ \Delta \in \mathbb{F}[t^{\pm 1}] \mid \Delta \cdot x = 0, \ {}^\forall x \in H_1 \}$.

least common multiple

Assumption (technical, but a week condi. You may forget)

1 **The poly $\Delta_f \neq 0$. that is, H_1 is of fin. dim over $\mathbb{F}$.**

2 **Δ_f is relatively prime to $\det(\mathrm{id} - \mathrm{f} \otimes \mathrm{Ab}(\mathrm{meridian}))$.**

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Thm.[N.] (cf. Universal coeff. thm.)

$A := \mathbb{F}[t^{\pm 1}]/\Delta_f, \qquad M = A^n \curvearrowleft GL_n(A)$.

$\Longrightarrow$

$H^1(Y, \partial Y; M) \cong H_1(Y; \mathbb{F}[t^{\pm 1}])$ by an Adjoint matrix.

Cor.[N.] We get

$\mathrm{twBl}_K : H_1(Y; \mathbb{F}[t^{\pm 1}])^{\otimes 2} \cong H^1(Y, \partial Y; M)^{\otimes 2} \xrightarrow{\smile_\psi} A$.

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Rem.[N] I can define such twBl_K without the assumption;

Further, $\mathbb{F}$ can be replaced by any Noetherian UFD.

Topological degeneracy of twBl_K under the assumption.

Observation (Bilinear form considered [Kirk-Livingston])

$$H^1(Y_\infty, \partial Y_\infty; \mathbb{F}^n)^2 \xrightarrow{\smile} H^2(Y_\infty, \partial Y_\infty; \mathbb{F}^n) \xrightarrow{\psi(\bullet \cap \Sigma)} \mathbb{F}.$$

where Y_∞ is the ∞-cyclic cover of $S^3 \setminus K$.

Thm.[N.]

$$H^1(Y, \partial Y; (\mathbb{F}[t]/\Delta_f)^n)^{\otimes 2} \xrightarrow{\mathcal{Q}_\psi} \mathbb{F}[t]/\Delta_f \quad \text{in } (-1)\text{-page}$$

$$H^1(Y_\infty, \partial Y_\infty; \mathbb{F}^n)^{\otimes 2} \xrightarrow{\smile_{KL}} \mathbb{F} \qquad \text{as above}$$

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\end{tikzcd}$$

in (-1)-page

as above

Rem. Conversely, $\mathcal{Q}_\psi$ can recover from $\smile_{KL}$ as

$$\mathcal{Q}_\psi(x, y)/\Delta_f = \sum_{j\geq 0} (\tau_*^{-j} x) \smile_{KL} y \cdot t^j \in \mathbb{F}((t))/\mathbb{F}[t].$$

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Cor.[N.] ("Twisted" Milnor duality of Y_∞).

If the $f : \pi_K \longrightarrow GL_n(\mathbb{F})$ is unitary or symplectic,

then, the twBl_K is non-degeneracy.

Result 1.5 Casson-Gordan signature $\sigma_{\omega}^{\mathrm{CG}}(K,\chi)$

Announce. (N.) Let $\omega \in \mathbb{C}$ be "generic" and $|\omega| = 1$.

Using no 4-mfds, we get a diagrammatic computation for

$$\sigma_{\omega}^{\mathrm{CG}}(K,\chi) - \sigma_{1}^{\mathrm{CG}}(K,\chi) \in \mathbb{Q}.$$

Here, we set $f : \pi_K \to GL_q(\mathbb{C}[t]), \quad M = (\mathbb{C}[t]/\Delta_f)^q.$

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Casson-Gordan (meta-abelian) signature (Rough review)

Input $(q, d) \in \mathbb{Z}^2$: prime powers, $\quad w \in \mathbb{C}$

$\chi : H_1(S^3 \setminus K; \Lambda) \longrightarrow \mathbb{Z}/d$ with "Gilmer's condi.".

Output

$\mathrm{Sign}_w(\mathrm{Int}(W_{\mathrm{ab}\otimes\hat{\chi}}) - \mathrm{Int}(W_{\mathrm{tri}})) \in \mathbb{Q}.$ concordance inv.

Rem.
- W: 4-mfd s.t. $\partial W = \sharp^q$ (q-fold cover bra. over K)
- [Kirk-Livingston] Cup product without p.f. & Exa.

Summary

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GOAL: show rel. cup products of $S^3 \setminus K$ are pretty useful.

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Every rel. cup product can be diagrammatically described in terms of quandle theory.

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Twisted version of Blanchfield parings

w/ non-degeneracy (cf. Milnor duality)

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Thank You