A NOTE ON THE RIEMANN HYPOTHESIS FOR THE CONSTANT TERM OF THE EISENSTEIN SERIES

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joint work with Jeffrey C. Lagarias

In this paper, we give an another proof of Theorem 3 in [4]. Theorem 3 in [4] is the following theorem.

Theorem 1. *Let $\zeta(s)$ be the Riemann zeta function and let $\zeta^*(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$. For each $y \ge 1$ the constant term of the Eisenstein series*

$$a_0(y,s) := \zeta^*(2s)y^s + \zeta^*(2-2s)y^{1-s} \tag{1}$$

is a meromorphic function that satisfies the modified Riemann hypothesis. More precisely, there is a critical value

$$y^* := 4\pi e^{-\gamma} = 7.055507+, \tag{2}$$

such that the following hold:

1. *All zeros of $a_0(y,s)$ lie on the critical line for $1 \le y \le y^*$.*
2. *For $y > y^*$ there are exactly two zeros off the critical line. These are real simple zeros $\rho_y, 1-\rho_y$ with $\frac{1}{2} < \rho_y < 1$. The zero ρ_y is a nondecreasing function of y, and $\rho_y \to 1$ as $y \to \infty$.*

The natural entire function associated to $a_0(y,s)$ is

$$G(y,s) := (2s)(2s-2)a_0(y,s), \tag{3}$$

which behaves similarly to the Riemann ξ-function, satisfying the functional equation $G(y,s) = G(y,1-s)$, being real on the real axis and on $\Re(s) = \frac{1}{2}$. It also has $G(y,\frac{1}{2}) = (\log 4\pi - \gamma - \log y)\sqrt{y}$, where γ is Euler's constant. To establish Theorem 1 it proves useful to study the entire function

$$H(y,s) := \frac{1}{2}(s-\frac{1}{2})G(y,s) = (s-1)\xi(2s)y^s + s\xi(2s-1)y^{1-s}. \tag{4}$$

The function $H(y,s)$ has a zero at $s = \frac{1}{2}$ and satisfies the functional equation $H(y,s) = -H(y,1-s)$, but has the advantage that both terms on the right side of (4) are entire functions. We begin with the analytic part of the proof.

Theorem 2. *For fixed $y \ge 1$, all the zeros of the entire function $H(y,s)$ lie on the critical line $\Re(s) = \frac{1}{2}$, except for possible zeros off the line lying the rectangular box*

$$B := \{s = \sigma + it \mid -19 \le \sigma \le 20,\ |t| \le 20\}.$$

The idea of the proof of Theorem 2 is to show that in the region $\Re(s) > \frac{1}{2}$ outside the box B that the inequality

$$\left|\frac{\xi(2s)}{\xi(2s-1)}\right| \; |\frac{s-1}{s}||y^{2s-1}| > 1 \tag{5}$$

is valid. This inequality implies that two terms on the right in the definition (4) of $H(y,s)$ have differing absolute values in this region, hence must be nonzero there. The functional equation relating $s \to 1-s$ then shows $H(y,s)$ is non-vanishing outside the box B when $\Re(s) < \frac{1}{2}$, and the theorem will follow.

The left side of the inequality (5) increases as y increases, so it suffices to establish it for $y = 1$, to deduce it for all $y \geq 1$. In §3 we established that

$$\left|\frac{\xi(2s)}{\xi(2s-1)}\right| > 1 \text{ when } \Re(s) > 1, \tag{6}$$

by showing that this inequality holds term-by-term in the (modified) Hadamard product on a zero-by-zero basis, where the zero $\rho = \beta + i\gamma$ of $\xi(s)$ in the numerator is paired against the zero $\rho' = 1 - \beta + i\gamma$ in the denominator. That is, we used the fact that

$$\left|\frac{2s-\rho}{2s-1-\rho'}\right| > 1 \text{ if } \Re(s) > \frac{1}{2}. \tag{7}$$

To establish Theorem 2 we show that for each s with $\Re(s) > \frac{1}{2}$ lying outside or on the boundary of the box B, we can locate a finite set of zeros whose products as above already have absolute value exceeding $|\frac{s-1}{s}|$. For this purpose we give a series of preliminary lemmas.

Lemma 1. *(1) For any real $|t| \geq 14$ there exists a zero $\rho = \beta + i\gamma$ of $\xi(s)$ such that $0 < \beta \leq \frac{1}{2}$ and*

$$|t - \gamma| \leq 5. \tag{8}$$

(2) For any real value of t there exist at least three distinct zeros of $\xi(s)$ with $0 < \beta \leq \frac{1}{2}$ and

$$|t - \gamma| \leq 22. \tag{9}$$

Proof. For $|t| \geq 21$ part (1) is a direct consequence of Lemma 3.5 in Lagarias [2]. (This lemma is proved by a method of Turing [5].) For $14 \leq |t| \leq 21$ (1) holds because the smallest zeta zeros have ordinates $\pm 14.13, \pm 21.02$. From (1) we deduce that (2) holds for $|t| \geq 25$, because we get three distinct zeros with $|t-\gamma| \leq 15.1$, by applying (1) to the points $t - 10.1$, t, and $t + 10.1$. Finally (2) holds for $|t| \leq 25$ using the zeta zeros $\pm 14.13, \pm 21.02, \pm 25.01$. To get three distinct zeros at $t = 0$ we cannot make the right side of (9) smaller than 21.02. □

Lemma 2. *Let $s = \sigma + it$ with $\frac{1}{2} < \sigma \le 20$ and $|t| \ge 20$. Then for any $0 < \beta \le \frac{1}{2}$ and $|t_0| \le 5$ there holds*

$$\left| \frac{2\sigma - \beta + it_0}{2\sigma - 2 + \beta + it_0} \right| > \left| \frac{\sigma + it}{\sigma - 1 + it} \right| = \left| \frac{s}{s-1} \right|. \tag{10}$$

Proof. By squaring the inequality (10) is equivalent to

$$\frac{(2\sigma - \beta)^2 + t_0^2}{(2\sigma - 2 + \beta)^2 + t_0^2} > \frac{\sigma^2 + t^2}{(\sigma - 1)^2 + t^2}. \tag{11}$$

For fixed s and β it suffices to prove (10) with $t_0 = 5$, for it would then hold for $|t_0| \le 5$, using the fact that $(2\sigma - \beta)^2 \ge (2\sigma - 2 + \beta)^2$ for $\Re(s) > \frac{1}{2}$. Next, when $t_0 = 5$, it suffices to verify the inequality for $\beta = \frac{1}{2}$ since if it holds there then the left side of (11) with $t_0 = 5$ increases as $\beta > 0$ decreases, while the right side is fixed. To establish (10) it thus suffices to verify the inequality

$$\frac{(2\sigma - \frac{1}{2})^2 + 25}{(2\sigma - \frac{3}{2})^2 + 25} > \frac{\sigma^2 + t^2}{(\sigma - 1)^2 + t^2}. \tag{12}$$

Clearing denominators shows that this inequality is equivalent to

$$\left((2\sigma - \frac{1}{2})^2 + 25\right)((\sigma-1)^2 + t^2) > ((2\sigma - \frac{3}{2})^2 + 25)\left(\sigma^2 + t^2\right).$$

Algebraic simplification reduces this inequality to

$$(2\sigma - 1)(-8\sigma^2 + 8\sigma + 8t^2 - 101) > 0. \tag{13}$$

For $\sigma > \frac{1}{2}$ this holds whenever the second term is positive, i.e.

$$8t^2 > 8\sigma^2 - 8\sigma + 101.$$

For $1/2 < \sigma \le 20$, one has $8\sigma^2 - 8\sigma + 101 \le 3141$, while $8t^2 \ge 3200$ when $|t| \ge 20$. □

Lemma 3. *For all $\sigma \ge 20$ and $0 \le \beta \le \frac{1}{2}$, and $|t_j| \le 22$ for $1 \le j \le 3$ there holds*

$$\prod_{j=1}^{3} \left| \frac{2\sigma - \beta + it_j}{2\sigma - 2 + \beta + it_j} \right| > \frac{\sigma}{\sigma - 1}. \tag{14}$$

Proof. Let $\sigma \ge 20$ be fixed. Since $\frac{\beta}{1-\beta} \le 1$, we have

$$f(\beta) := \frac{2\sigma - \beta}{2\sigma - 2 + \beta} > 1, \tag{15}$$

and $f(\beta)$ decreases as a function of β on $0 \le \beta \le \frac{1}{2}$. The terms in (14) inherit this decreasing property, so it suffices to verify (14) when $\beta = \frac{1}{2}$. It also follows from (15) that

$$f(t_j) := \left| \frac{2\sigma - \beta + it_j}{2\sigma - 2 + \beta + it_j} \right|$$

decreases as $|t_j|$ increases, and it depends only on the value of $|t_j|$. Consequently it suffices to verify (14) in the case $t_1 = t_2 = t_3 = 22$. Thus it suffices to verify that

$$\left|\frac{2\sigma - \frac{1}{2} + 22i}{2\sigma - \frac{3}{2} + 22i}\right|^3 > \frac{\sigma}{\sigma - 1}. \tag{16}$$

Squaring both sides converts this to the equivalent inequality

$$\left(\frac{(2\sigma - \frac{1}{2})^2 + 484}{(2\sigma - \frac{3}{2})^2 + 484}\right)^3 > \left(\frac{\sigma}{\sigma - 1}\right)^2,$$

and algebraic simplification reduces it to

$$\left(1 + \frac{4\sigma - 2}{4\sigma^2 - 6\sigma + \frac{1945}{4}}\right)^3 > 1 + \frac{2\sigma - 1}{\sigma^2 - 2\sigma + 1}. \tag{17}$$

To establish (17) we note that

$$\left(1 + \frac{4\sigma - 2}{4\sigma^2 - 6\sigma + \frac{1945}{4}}\right)^3 \geq 1 + \frac{3(4\sigma - 2)}{4\sigma^2 - 6\sigma + \frac{1945}{4}},$$

since $\frac{4\sigma-2}{4\sigma^2-6\sigma+\frac{1945}{4}} > 0$ when $\sigma \geq 20$.

Now (17) follows by verifying that

$$\frac{3(4\sigma - 2)}{4\sigma^2 - 6\sigma + \frac{1945}{4}} > \frac{2(4\sigma - 2)}{4\sigma^2 - 8\sigma + 4}$$

holds for $\sigma \geq 20$. Here we have

$$\frac{3(4\sigma - 2)}{4\sigma^2 - 6\sigma + \frac{1945}{4}} - \frac{2(4\sigma - 2)}{4\sigma^2 - 8\sigma + 4} = \frac{(2\sigma - 1)(8\sigma^2 - 24\sigma - 1921)}{(\sigma - 1)^2(16\sigma^2 - 24\sigma + 1945)}.$$

For $\sigma \geq 20$, the factor $8\sigma^2 - 24\sigma - 1921$ increases as σ increases and is larger than 799. Hence the inequality (17) holds. □

Proof of Theorem 2. We will show that for all $y \geq 1$ there holds

$$\left|\frac{\xi(2s)}{\xi(2s - 1)}\right| \; |\frac{s - 1}{s}| \; |y^{2s-1}| > 1 \text{ for } \Re(s) > \frac{1}{2}, \; x \notin B. \tag{18}$$

This inequality establishes there are no zeros outside the box for $\Re(s) > \frac{1}{2}$, and the result for $\Re(s) < \frac{1}{2}$ then follows using the functional equation, since the box B is invariant under $s \mapsto 1 - s$.

It suffices to prove (18) for $y = 1$ since $|y^{2s-1}| \geq 1$ on the region in question. Thus we must show

$$\left|\frac{\xi(2s)}{\xi(2s - 1)}\right| |\frac{s - 1}{s}| > 1 \text{ for } \Re(s) > \frac{1}{2}, \; x \notin B. \tag{19}$$

In terms of the modified Hadamard product for $\frac{\xi(2s)}{\xi(2s-1)}$ the proof of Theorem 2 in [4] (via Theorem 4 in [4]) showed that if we paired the zero $\rho = \beta + i\gamma$ of $\xi(2s)$ with that of the zero $\rho' = 1-\beta+i\gamma$ of $\xi(2s-1)$, then each term $|\frac{2s-\rho}{2s-1-\rho'}|$ separately exceeded one in absolute value. Now let $s=\sigma+it$. First suppose it lies in the region $\frac{1}{2}<\sigma\leq 20$ and $|t|\geq 20$. Then Lemma 1 gives a zero $\rho=\beta+i\gamma$ with $0<\beta\leq\frac{1}{2}$ and $|2t-\gamma|\leq 5$. Now Lemma 2 produces a term

$$\left|\left(\frac{2s-\rho}{2s-1-\rho'}\right)\left(\frac{s-1}{s}\right)\right|>1.$$

It follows that (19) holds in this case. If $\sigma\geq 20$ then Lemma 1 produces at least three zeros with $0<\beta_i\leq\frac{1}{2}$ and $|2t-\gamma_i|\leq 22$. Now Lemma 3 gives for $\sigma\geq 20$ that

$$\left|\left(\prod_{i=1}^{3}\frac{2s-\rho_i}{2s-1-\rho_i'}\right)\left(\frac{s-1}{s}\right)\right|>\frac{\sigma}{\sigma-1}\left|\frac{s-1}{s}\right|\geq\left(\frac{\sigma}{\sigma-1}\right)\left(\frac{\sigma-1}{\sigma}\right)=1,$$

where we used $|\frac{s-1}{s}|\geq|\frac{\sigma-1}{\sigma}|$ when $\Re(s)>\frac{1}{2}$. It follows that (19) holds in this case, and Theorem 2 follows. $\Box$

In order to prove Theorem 1 we establish two additional preparatory lemmas.

Lemma 4. *(Barrier Lemma) Let $y\geq 7.5$. Then for all $s=(\frac{1}{2}+\epsilon)+it$ with $0<\epsilon\leq 0.005$ and any real t there holds*

$$|\frac{s-1}{s}||y^{2s-1}|>1. \tag{20}$$

Proof. We have $\log y = 2.0149+$. Now

$$\begin{aligned}|\frac{s-1}{s}| &= \left(\frac{(-\frac{1}{2}+\epsilon)^2+t^2}{(\frac{1}{2}+\epsilon)^2+t^2}\right)^{\frac{1}{2}}\\ &= \left(1-\frac{2\epsilon}{\frac{1}{4}+\epsilon+\epsilon^2+t^2}\right)^{\frac{1}{2}}.\end{aligned}$$

For $0\leq\delta\leq\frac{1}{2}$, we have

$$\sqrt{1-\delta}\geq 1-\frac{1}{2}\delta-\frac{1}{8}\frac{\delta^2}{1-\delta}\geq 1-\frac{1}{2}\delta-\frac{1}{4}\delta^2,$$

which yields

$$\left|\frac{s-1}{s}\right|>1-\frac{4\epsilon}{1+4\epsilon+4\epsilon^2+4t^2}-\frac{16\epsilon}{(1+4\epsilon+4\epsilon^2+4t^2)^2}>1-4\epsilon-16\epsilon^2.$$

On the other hand,

$$|y^{2s-1}|=y^{2\epsilon}=e^{2\epsilon\log y}\geq 1+2\epsilon\log y>1+4.0298\epsilon.$$

For $\log y\geq 2.0149$ we have for $0<\epsilon<0.0005$ that

$$|\frac{s-1}{s}||y^{2s-1}|>(1-4\epsilon-16\epsilon^2)(1+4.0298\epsilon)>1,$$

which completes the argument. □

Lemma 5. *For $y > 0$ the function*

$$H'(y, \frac{1}{2}) := \frac{d}{ds} H(y, s)|_{s=\frac{1}{2}}$$

is a strictly decreasing function of y. It satisfies satisfies $H'(y^*, \frac{1}{2}) = 0$ at the unique value $y^* = 4\pi e^{-\gamma} = 7.055507+$. For each $y > y^*$ the function $H(y, s)$ has at least one real zero ρ_y satisfying $\frac{1}{2} < \rho_y < 1$.

Proof. For $y > 0$ we have

$$\begin{aligned} \frac{d}{ds} H(y, s) &= \Big(\xi(2s) y^s + \xi(2s-1) y^{1-s}\Big) + \Big(2(s-1)\xi'(2s) y^s + 2s\xi'(2s-1) y^{1-s}\Big) \\ &\quad + (\log y)\Big((s-1)\xi(2s) y^s - s\xi(2s-1) y^{1-s}\Big). \end{aligned}$$

This yields

$$\begin{aligned} H'(y, \frac{1}{2}) := \frac{d}{ds} H(y, s)|_{s=\frac{1}{2}} &= \sqrt{y}\,(-\xi(1)\log y + 2\xi(1) - 2\xi'(1)) \\ &= -\xi(1)\sqrt{y}\left(\log y + 2(\frac{\xi'(0)}{\xi(0)} - 1)\right), \end{aligned} \tag{21}$$

using $\xi(0) = \xi(1)$ and $\xi'(0) = -\xi'(1)$. Since $\xi(1) > 0$ it follows that $H'(y, \frac{1}{2})$ is a strictly decreasing function of y, for all $y > 0$. We recall the fact that

$$-\frac{\xi'(0)}{\xi(0)} = \sum_{\rho} \frac{1}{\rho} = 1 + \frac{1}{2}\gamma - \frac{1}{2}\log 4\pi = 0.0230957+,$$

where $\gamma = 0.57721+$ is Euler's constant. The unique value y^* where $H'(y^*, \frac{1}{2}) = 0$ is given by

$$\log y^* = 2(1 - \frac{\xi'(0)}{\xi(0)}) = \log 4\pi - \gamma,$$

so that $y^* = 4\pi e^{-\gamma}$.

For $y > 0$ the function $H(y, s)$ is real on the real axis. For $y > y^*$ we have $H(y, \frac{1}{2}) = 0$ and $H'(y, \frac{1}{2}) < 0$, so that $H(y, \frac{1}{2} + \epsilon) < 0$ for sufficiently small positive ϵ. Since $H(1, y) = \xi(1) > 0$ there is a sign change in $(\frac{1}{2}, 1)$ so there is at least one zero in the interval. □

Proof of Theorem 1. Theorem 2 shows that we need only determine the locations of zeros of $H(y, s)$ inside the box $B = \{z = x + iy \mid |s - \frac{1}{2}| \le 20,\ |y| \le 20\}$.

We establish the following two facts by numerical computation.

(1) For $y = 1$ all the zeros of the function $H(y, s)$ inside the box B fall on the critical line. There are 7 such zeros, counting the zero at $s = \frac{1}{2}$.

(2) For $y = 7.5$ all but two of the zeros of the function inside the box B fall on the critical line, and these two lie on the real axis in the open unit interval $(0, 1)$. There are 25 such zeros in total.

Fact (1) is established by first counting zeros inside B by computing the change in argument of $\log H(y, s)$ counterclockwise around the boundary of the box, finding 13 zeros in the box. (Symmetries of the problem show that this is four times the argument change in the first quadrant, and this argument change is $\frac{7}{2}\pi$.) One finds simple zeros on the critical line by counting sign changes of $\Im H(y, s)$ and locates seven zeros, i.e. three zeros $\frac{1}{2} + i\gamma$ with $0 < \gamma < 14$, see Table 1.

Fact (2) is established similarly by showing the by first counting zeros of $H(y, s)$ inside B to be 39 by computing the change in argument of $\log H(y, s)$ counterclockwise around the boundary of the box. Then one locates 37 simple zeros on the critical line by counting sign changes of $\Im H(y, s)$,i.e. a zero $\rho = \frac{1}{2}$ and 11 zeros with $0 < \gamma < 20$, see Table 1. There is at least one real zero ρ_y of $H(y, s)$ in $(\frac{1}{2}, 1)$, for $H(y, \frac{1}{2}) = 0$ with negative derivative there, by Lemma 5, while $H(y, 1) > 0$. The functional equation produces another real zero $1 - \rho_y$ in $(0, \frac{1}{2})$. This completes the count of zeros, and they are all simple zeros. Numerically the two real zeros of $H(y, s)$ for $y = 7.5$ not on the critical line are $\rho_y = 0.6473174359$ and $1 - \rho_y = 0.3526825641$.

	$y = 1$	$y = y^*$	$y = 7.5$
1	6.974683133	2.244794235	2.175129987
2	10.40228756	3.851296383	3.736361075
3	12.42264167	5.404657031	5.249487964
4	15.08382464	6.732441081	6.604132477
5	16.40456028	7.383718196	7.271363142
6	18.68201963	8.670185248	8.433916576
7	20.34995710	10.02271471	9.798649742
8	21.60499108	10.69728308	10.57499116
9	23.85087057	11.78575276	11.50885682
10	24.83364580	12.56610869	12.44012066
11	26.40277087	13.53142535	13.20288515
12	28.11180718	14.79167003	14.47834099
13	29.54150449	15.42550847	15.25340970
14	30.39424164	16.28902291	16.04689820
15	32.41487455	16.93621484	16.66329033
16			17.72678899
17			18.69205671
18			19.38268940

TABLE 1. Zeros of constant term $a_0(y, s)$ on critical line

We now deduce the conclusion of the theorem for all $y \geq 7.5$, using Lemma 4. This lemma permits the inference that if we vary from $y_0 = 7.5$ to y, no zeros on the entire critical line can move off the to the right half-plane, and conversely, that no zero in the half-plane $\{\Re(s) > \frac{1}{2}\}$ can move onto the critical line. The functional equation of $H(y,s)$ then shows no zeros can escape off the critical line to the half-plane $\{\Re(s) < \frac{1}{2}\}$ or vice-versa. For the value $y_0 = 7.5$ fact (2) gives one zero each in the open half plane $\{\Re(s) < \frac{1}{2}\}$ and in the open right-half plane $\{\Re(s) > \frac{1}{2}\}$ By Theorem 2 these zeros are trapped inside the half-planes and the box B; each open half-plane thus contains one zero for all such values of y, necessarily in the box. Lemma 5 accounts for these zeros, showing that for all $y > y^*$ there is at least one real zero ρ_y with $\frac{1}{2} < \rho_y < 1$. This zero must therefore be the unique zero in the right half-plane. Similarly $1 - \rho_y$ is then the unique zero in the left half plane. It remains to show that $\rho_y \to 1$ as $y \to \infty$. This follows because as $y \to \infty$ the region with $\Re(s) > \frac{1}{2}$ and

$$f(y,s) := \left|\frac{\xi(2s)}{\xi(2s-1)}\right| |\frac{s-1}{s}||y^{2s-1}| \leq 1$$

monotonically shrinks down to the point $s = 1$. This is due to the the increase in size of $|y^{2s-1}|$. Here we use Lemma 4 for $y_0 = 7.5$ and write $|y^{2s-1}| = |y_0^{2s-1}||(\frac{y}{y_0})^{2s-1}|$. For any point $s \neq 1$ the term $|(\frac{y}{y_0})^{2s-1}|$ eventually grows large enough as $y \to \infty$ to force $f(y,s) > 1$.

It remains to establish the theorem when $1 \leq y \leq 7.5$. Here further numerical calculations are required. We analyze the movement of zeros in the box B as the parameter y is varied over $1 \leq y \leq 7.5$, starting from $y = 1$. Recall that each zero moves continuously as the parameter y is varied (though not necessarily differentiably, at points where multiple zeros occur.) The functional equation ensures that zeros on the critical line cannot move off it without first combining as a multiple zero, and we start with all zeros in the box B start as simple zeros on the critical line. By numerical calculations on the critical line alone, taking sufficiently small steps in the y-parameter, one can establish the following fact.

(3) For $1 \leq y \leq 7.5$, all zeros ρ of $H(y,s)$ on the critical line with $1 \leq |\Im(\rho)| \leq 15$ are simple. In addition, for each such y there is at most one zero in the region $0 < \Im(\rho) \leq 1$, resp. $-1 \leq \Im(\rho) < 0$.

Simplicity of zeros in (3) is established by establishing a separation bound between zeros. Extra zeros migrate into the box along the critical line, and the general movement of zeros on the critical line is to approach the origin as y increases. [In fact their ordinates decrease in monotone fashion as y increases in this range, but we do not prove this.] (To do these calculations rigorously we must first establish an analytic inequality establishing an upper bound on how fast these zeros can move as functions of y.)

We conclude from fact (3) that at most one pair of zeros can escape from the critical line, in $1 \leq y \leq 7.5$, and that the multiple zero involved must be at the origin. Lemma 5 shows a multiple zero can occur at the origin only at the value $y = y^*$, where one gets a triple zero by the functional equation. But for $y > y^*$ the behavior of these zeros is dictated by Lemma 5; two of them are forced onto the real axis away from $s = \frac{1}{2}$, where they must stay for $y^* < y \leq 7.5$, since to escape the real axis requires four zeros, and Lemma 5 doesn't permit a second multiple zero to allow them to return to the origin. □

Remark. The calculation (2) is not essential to the proof of Theorem 1. One can avoid it using (1) and (3) alone. However, once the calculation of (1) is set up, it is useful to perform (2) as a reliability check, and the resulting proof is easier to follow.

We conclude with a remark about the constant $y^* = 4\pi e^{-\gamma}$ at which real zeros appear in Theorem 1. A similar phenomonon was noted long ago for individual Epstein zeta functions, where there is not quite a sharp cutoff. In 1964 Bateman and Grosswald [1] gave a criterion for Epstein zeta functions to have a real zero, whose main term was $4\pi e^{-\gamma}$ with a very small error term depending on a parameter k. Their parameter $k = y$, and one of their results can be reprhased as saying that the function $E(z, s)$ for $z = x + iy$ always has a real zero when $y \geq 7.0556$ and never has a real zero when $k \leq 7.0554$. Note that the constant term $a_0(y, s)$ is obtained by averaging over x of $E(z, s)$, for fixed y.

References

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