AN ADDITIONAL WRITING ON LAGARIAS–SUZUKI (2006)

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This is a note on the zeros of functions

$$H(y;s) = p(s)\zeta^*(2s)y^s + p(1-s)\zeta^*(2-2s)y^{1-s} \tag{0.1}$$

which was mentioned in (24) of Lagarias–Suzuki [4].

1. A basic fact

Let $s=\sigma+it$ ($i=\sqrt{-1}$, $\sigma, t \in \mathbb{R}$) be a complex variable. Let $\zeta^*(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ and $\xi(s) = \frac{1}{2}s(s-1)\zeta^*(s)$ and let $p(s)$ be a nonzero polynomial with real coefficients.

In order to study the zeros of (0.1), we introduce the entire function

$$\begin{aligned} H^*(y;s) :&= (2s)(2s-1)(2s-2)H(y;s) \\ &= 2(s-1)p(s)\,\xi(2s)y^s + 2s\,p(1-s)\,\xi(2-2s)y^{1-s}. \end{aligned}$$

At first, we show the following matter.

Theorem 1. *Let $H^*(y;s)$ be as above. Suppose that $y \geq 1$ and $p(s)$ has N many zeros counted with multiplicity in the right half-plane $\Re(s) > 1/2$. Then $H^*(y;s)$ has at most $N+1$ many zeros in the right half-plane $\Re(s) > 1/2$ counting with multiplicity, and the same thing holds in the left half-plane $\Re(s) < 1/2$.*

The trivial functional equation $H(y;s) = H(y;1-s)$ implies

$$H^*(y;1-s) = -H^*(y;s).$$

Therefore, it suffices to study the zeros of $H^*(y;s)$ in the right half-plane $\Re(s) > 1/2$. In what follows, we suppose that $\sigma = \Re(s) > 1/2$.

A key gradient of the proof of Theorem 1 is the following fact.

Proposition 1. *Let $W(z)$ be an entire function. Suppose that it has a product formula*

$$W(z) = H(z)e^{\alpha z}\prod_{n=1}^{\infty}(1-z/\lambda_n)(1+z/\bar{\lambda}_n),$$

where $H(z)$ is a nonzero polynomial having N many zeros in the lower half-plane counting with multiplicity, $\Im(\lambda_n) \geq 0$ ($n=1,2,3,\cdots$) and the product converges uniformly in any compact subset of $\mathbb{C}$. In addition, suppose that α is real or $\alpha = i\alpha'$ for some positive real number α'. Then $W(z)+\overline{W(\bar{z})}$ and $W(z)-\overline{W(\bar{z})}$ have at most N pair of conjugate complex zeros counting with multiplicity.

Proof. If α is real, the proposition is Proposition 3.1 of [2]. To prove the case $\alpha = i\alpha'$ for some positive real α', we recall the result in [1, p. 215]: Let $U(z)$ and $V(z)$ be real polynomials. Assume that $U \not\equiv 0$ and that $W(z) = U(z)+iV(z)$ has exactly n zeros counted with multiplicity in the lower half-plane. Then $U(z)$ can have at most n pairs of conjugate complex zeros counted with multiplicity.

We define the polynomials $w_n(z)$ ($n=1,2,\cdots$) by

$$w_n(z) = H(z)\left(1+\frac{i\alpha' z}{n}\right)^n\prod_{k=1}^{n}(1-z/\lambda_k)(1+z/\bar{\lambda}_k).$$

Then each $w_n(z)$ has at most N many zeros in the lower half-plane, since $\alpha' > 0$. By the above fact in [1, p. 215], $w_n(z)+\overline{w_n(\bar{z})}$ has at most N pairs of conjugate complex zeros. Since $w_n(z)+\overline{w_n(\bar{z})}$ converges uniformly to $W(z)+\overline{W(\bar{z})}$ in any compact subset of $\mathbb{C}$, $W(z)+\overline{W(\bar{z})}$ has at most N pairs of conjugate complex zeros. Similarly, we prove the proposition for $W(z)-\overline{W(\bar{z})}$. □

Proof of Theorem 1. In order to apply Proposition 1 to $H^*(y,s)$, we define

$$W_{p,y}(z) = 2(s-1)p(s)\,\xi(2s)y^s \quad \text{with} \quad s = \frac{1}{2}+iz.$$

Then, we have

$$W_{p,y}(z) = -2\left(\frac{1}{2}-iz\right)p\left(\frac{1}{2}+iz\right)\xi(1+2iz)y^{\frac{1}{2}+iz}$$

and

$$\overline{W_{p,y}(\bar{z})} = -2\left(\frac{1}{2}+iz\right)p\left(\frac{1}{2}-iz\right)\xi(1-2iz)y^{\frac{1}{2}-iz} = -2sp(1-s)\,\xi(2-2s)y^{1-s},$$

since $p(s)$ has real coefficients. Hence, we obtain

$$H^*(y,s) = W_{p,y}(z) - \overline{W_{p,y}(\bar{z})} \quad \text{with} \quad s = \frac{1}{2}+iz.$$

Here $(s-1)p(s)$ $(s=1/2+iz)$ is a polynomial of z having $N+1$ many zeros in the lower half-plane $\Im(z)<0$ counting with multiplicity. Therefore, by Proposition 1, Theorem 1 will be established if the following lemma is proved, since $\log y \geq 0$ if $y \geq 1$. □

Lemma 1. *We have*

$$\xi(1+2iz) = \prod_{\Re(\lambda)>0}(1-z/\lambda)(1+z/\bar{\lambda}),$$

where

$$\lambda = \frac{\gamma}{2}+i\frac{1-\beta}{2}$$

for a zero $\rho=\beta+i\gamma$ of $\xi(s)$. In particular, any λ is in the upper-half plane $\Im(z)>0$.

Proof. Put $F(z)=\xi(1+2iz)$. Then $F(z)$ is an entire function of order one. A complex number λ is a zeros of $F(z)$ if and only if $1+2i\lambda=\rho$ for some zero $\rho=\beta+i\gamma$ of $\xi(s)$. Therefore, if λ is a zero of $F(z)$, $-\bar{\lambda}$ is also a zero of $F(z)$, since if

$$\lambda = \frac{\gamma}{2}+i\frac{1-\beta}{2}$$

for some zero $\rho=\beta+i\gamma$ of $\xi(s)$,

$$-\bar{\lambda} = -\frac{\gamma}{2}+i\frac{1-\beta}{2}.$$

Hence we have the factorization

$$F(z) = e^{B'z}\prod_{\Re(\lambda)>0}(1-z/\lambda)(1+z/\bar{\lambda})\exp\left(z(1/\lambda-1/\bar{\lambda})\right).$$

We find that

$$\sum_{\Re(\lambda)>0}(1/\lambda-1/\bar{\lambda})$$

converges absolutely by a standard way. Thus

$$F(z) = e^{Bz}\prod_{\Re(\lambda)>0}(1-z/\lambda)(1+z/\bar{\lambda})$$

for

$$B = B' + \sum_{\Re(\lambda)>0}(1/\lambda-1/\bar{\lambda}).$$

Finally, we show $B = 0$. We have

$$F(z) = \xi(-2iz) = \xi(1 + 2i(-z + i/2)) = F(-z + i/2).$$

This implies

$$\frac{F'}{F}(0) = -\frac{F'}{F}(i/2).$$

On the left-hand side, we have

$$\frac{F'}{F}(0) = B + \sum_{\Re(\lambda)>0} \left(-\frac{1}{\lambda} + \frac{1}{\overline{\lambda}}\right).$$

On the right-hand side, we have

$$\frac{F'}{F}(i/2) = B - \sum_{\Re(\lambda)>0} \left(-\frac{1}{\lambda} + \frac{1}{\overline{\lambda}}\right)$$

by

$$\frac{i}{2} - \lambda = \frac{i}{2} - \frac{\gamma}{2} - i\frac{1-\beta}{2} = -\frac{\gamma}{2} + i\frac{1-(1-\beta)}{2}$$

and the symmetry between β and $1-\beta$ for the zeros of $\xi(s)$. Hence $B = 0$ by $(F'/F)(0) = -(F'/F)(i/2)$. □

2. Narrowing regions for off-line zeros

Theorem 1 does not mention where off-line zeros exist. In this part, we study a region where off-line zeros of (0.1) exists by restricting the following three cases

(i) $p(s)$ has no zeros in $\Re(s) > 1/2$,
(ii) $p(s)$ has one zero in $\Re(s) > 1/2$,
(iii) $p(s)$ has two zeros in $\Re(s) > 1/2$.

We can deal with general cases in a similar way by generalizing Lemma 5 and 9 below.

2.1. **Case (i).**

We have

$$H^*(y;s) = 2(s-1)p(s)\xi(2s)y^s \left(1 + \frac{s \cdot p(1-s)}{(s-1)\cdot p(s)} \cdot \frac{\xi(2-2s)}{\xi(2s)} \cdot y^{1-2s}\right). \tag{2.1}$$

Because the factor $(s-1)p(s)\zeta^*(2s)y^s$ has no zeros in the right-half plane $\Re(s) > 1/2$ except for the simple zero $s = 1$, we study the zeros of

$$1 + \frac{p(1-s)}{p(s)} \cdot \frac{s}{s-1} \cdot \frac{\xi(2-2s)}{\xi(2s)} \cdot y^{1-2s} = 1 + R_{p,y}(s), \tag{2.2}$$

say.

Lemma 2. *There exists computable $\sigma_1 > 1/2$ which does not depend on $p(s)$ and $y \geq 1$ such that $1 + R_{p,y}(s)$ has no zeros (and poles) in the right-half plane $\Re(s) \geq \sigma_1$.*

Proof. Put

$$R_1(s) = \frac{s}{s-1} \cdot \frac{\xi(2-2s)}{\xi(2s)} = \frac{s}{s-1} \cdot \frac{\xi(2s-1)}{\xi(2s)}.$$

We have

$$|R_1(s)| = \sqrt{\pi}\left|\frac{\Gamma(s-1/2)}{\Gamma(s)}\right| \left|\frac{\zeta(2s-1)}{\zeta(2s)}\right|.$$

Using the Stirling formula

$$\Gamma(z) = \sqrt{\frac{2\pi}{z}}\left(\frac{z}{e}\right)^z \left(1 + O_\varepsilon(|z|^{-1})\right) \quad (|z| \geq 1,\ |\arg z| < \pi - \varepsilon),$$

we have

$$\left|\frac{\Gamma(s-1/2)}{\Gamma(s)}\right| = O(|s|^{-1/2})$$

as $|s| \to +\infty$ and $\Re(s) \to +\infty$. On the other hand,

$$\frac{\zeta(2s-1)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{\phi(n)}{n^{2s}} = 1 + O\left(\frac{1}{4^{\sigma}}\right)$$

for $\Re(s) > 1$, where $\phi(n)$ is Euler's totient function. Hence

$$|R_1(s)| = O(|s|^{-1/2})$$

as $|s| \to +\infty$ and $\Re(s) \to +\infty$.

On the other hand, by

$$\left|\frac{1-s-(\mu+i\lambda)}{s-(\mu+i\lambda)}\right|^2 = \frac{(1-\sigma-\mu)^2+(t-\lambda)^2}{(\sigma-\mu)^2+(t-\lambda)^2} = 1 - \frac{(2\sigma-1)(1-2\mu)}{(\sigma-\mu)^2+(t-\lambda)^2} \le 1, \quad (2.3)$$

we have

$$\left|\frac{p(1-s)}{p(s)}\right| \le 1 \quad (2.4)$$

for $\Re(s) > 1/2$, and

$$0 < |y^{1-2s}| = y^{1-2\sigma} \le 1 \quad (2.5)$$

for $\Re(s) > 1/2$ and $y \ge 1$.

Therefore, there exists computable $\sigma_1 > 1/2$ such that $|R_{p,y}(s)| < 1$ for any s with $\Re(s) \ge \sigma_1$. □

Lemma 3. *Let σ_1 be the number of Lemma 2. There exists computable $T_1 > 0$ which does not depend on $p(s)$ and $y \ge 1$ such that $1 + R_{p,y}(s)$ has no zeros (and poles) in the region $1/2 < \Re(s) < \sigma_1$ with $|\Im(s)| \ge T_1$.*

Proof. For a zero $\rho = \beta + i\gamma$ of $\xi(s)$, we have

$$\left|\frac{2s-1-(1-\bar{\rho})}{2s-\rho}\right|^2 = 1 - \frac{4(2\sigma-1)(1-\beta)}{(2\sigma-\beta)^2+(2t-\gamma)^2} < 1,$$

since $\beta < 1$. Thus,

$$\left|\frac{2s-\rho}{2s-1-(1-\bar{\rho})}\frac{\xi(2s-1)}{\xi(2s)}\right| = \left|\frac{\xi(2s-1)/(2s-1-(1-\bar{\rho}))}{\xi(2s)/(2s-\rho)}\right| < 1$$

for any zero ρ of $\xi(s)$ (Note that if ρ is a zero of $\xi(s)$, $1-\bar{\rho}$ is also a zeros of $\xi(s)$ by functional equations $\xi(s) = \xi(1-s)$ and $\overline{\xi(s)} = \xi(\bar{s})$). Therefore, by (2.5) and (2.4), the proof of Lemma 3 is reduced to Lemma 4 below. □

Lemma 4. *Let σ_1 be the number of Lemma 2. There exists computable $T_1 > 0$ which does not depend on $p(s)$ and $y \ge 1$ such that there exists at least one zero ρ of $\xi(s)$ satisfying*

$$\left|\frac{s}{s-1} \cdot \frac{2s-1-(1-\bar{\rho})}{2s-\rho}\right| < 1 \quad (2.6)$$

if $1/2 < \sigma \le \sigma_1$ and $t \ge T_1$.

We prove Lemma 4 by using the following lemma:

Lemma 5 (Lemma 5 of [5], Lemma 3.5 of [3])**.** *For any real $|t| \ge 14$ there exists a zero $\rho = \beta + i\gamma$ of $\xi(s)$ such that $0 < \beta \le 1/2$ and $|t-\gamma| \le 5$.*

Proof of Lemma 4. Inequality (2.6) is equivalent to

$$1 - \frac{4(2\sigma-1)(1-\beta)}{(2\sigma-\beta)^2+(2t-\gamma)^2} = \left|\frac{2s-1-(1-\bar{\rho})}{2s-\rho}\right|^2 < \left|\frac{s-1}{s}\right|^2 = 1 - \frac{2\sigma-1}{\sigma^2+t^2},$$

where $s = \sigma + it$ and $\rho = \beta + i\gamma$. This inequality is equivalent to

$$\frac{(\sigma-\beta/2)^2+(t-\gamma/2)^2}{\sigma^2+t^2} < 1-\beta$$

when $\sigma > 1/2$. On the right-hand side, we have

$$\frac{(\sigma-\beta/2)^2+(t-\gamma/2)^2}{\sigma^2+t^2} \le \frac{\sigma^2+(t-\gamma/2)^2}{\sigma^2+t^2},$$

since $\sigma^2 \ge (\sigma-\beta/2)^2$ if $\sigma \ge \beta/4$ $(0<\beta<1)$. Moreover, if $|t| \ge 7$, there exists a zero $\rho = \beta + i\gamma$ of $\xi(s)$ such that $0 < \beta \le 1/2$ and $|t-\gamma/2| \le 5/2$ by Lemma 5. Therefore, for such a zero, we have

$$\frac{\sigma^2+(t-\gamma/2)^2}{\sigma^2+t^2} < \frac{\sigma^2+9}{\sigma^2+t^2} \quad \text{and} \quad \frac{1}{2} \le 1-\beta.$$

Here, $(\sigma^2+9)/(\sigma^2+t^2)$ is an increasing function of σ if $|t|>3$. In particular,

$$\frac{\sigma^2+9}{\sigma^2+t^2} \le \frac{\sigma_1^2+9}{\sigma_1^2+t^2}$$

if $1/2 < \sigma \le \sigma_1$ and $|t| \ge 7$. Hence, if we take $T_1 \ge 7$ so that $(\sigma_1^2+9)/(\sigma_1^2+t^2) < 1/2$ holds for any $|t| \ge T_1$, we obtain (2.6). □

Conclusion: By Lemma 2 and 3, we find that off-line zeros of $H^*(y;s)$ must be in the region

$$D_1 = \{s \,:\, \Re(s) \ne 1/2,\ 1-\sigma_1 \le \Re(s) \le \sigma_1,\ |\Im(s)| \le T_1\},$$

where σ_1 and T_1 are computable numbers independent of $p(s)$ and $y \ge 1$. The numbers σ_1 and T_1 are determined by $\xi(s)$.

2.2. **Case (ii).** Let $s=\mu$ be the zero of $p(s)$ in the right-half plane$\Re(s) > 1/2$. The zero $s=\mu$ should be real, since $p(s)$ has real coefficients. In this case, the factor $(s-1)p(s)\zeta^*(2s)y^s$ in (2.1) has no zeros in the right-half plane $\Re(s)>1/2$ except for the double zero $s=1$ (if $\mu=1$) or two simple zeros $s=1$ and $s=\mu$ (if $\mu \ne 1$). As in the case (i), we study the zeros of $1+R_{p,y}(s)$ in (2.2).

Lemma 6. *There exists computable $\sigma_2 > 1/2$ which does not depend on $y \ge 1$ such that $1+R_{p,y}(s)$ has no zeros (and poles) in the right-half plane $\Re(s) \ge \sigma_2$.*

Proof. We have

$$\frac{p(1-s)}{p(s)} \cdot \frac{s}{s-1} \cdot \frac{\xi(2-2s)}{\xi(2s)} \cdot y^{1-2s} = O(|s|^{-1/2})$$

as $|s| \to \infty$ and $\Re(s) \to \infty$ in a way similar to the proof of Lemma 2, where the implied constant does not depend on $y \ge 1$ but may depend on $p(s)$. The above estimate implies Lemma 6. □

Lemma 7. *Let σ_2 be the number of Lemma 6. There exists computable $T_2 > 0$ which does not depend on $y \ge 1$ such that $1+R_{p,y}(s)$ has no zeros (and poles) in the region $1/2 < \Re(s) < \sigma_2$ and $|\Im(s)| \ge T_2$.*

Proof. For a zero $\rho = \beta + i\gamma$ of $\xi(s)$, we have

$$\left|\frac{2s-1-(1-\bar{\rho})}{2s-\rho}\right|^2 = 1 - \frac{4(2\sigma-1)(1-\beta)}{(2\sigma-\beta)^2+(2t-\gamma)^2} < 1,$$

since $\beta < 1$. Thus,

$$\left|\frac{2s-\rho}{2s-1-(1-\bar{\rho})}\frac{2s-\rho'}{2s-1-(1-\bar{\rho}')}\frac{\xi(2s-1)}{\xi(2s)}\right| < 1$$

for any zeros ρ, ρ' of $\xi(s)$. On the other hand,

$$\left|\frac{s-\mu}{1-s-\mu} \cdot \frac{p(1-s)}{p(s)}\right|^2 \le 1$$

by (2.3). Therefore, the proof of Lemma 7 is reduced to Lemma 8 below. □

Lemma 8. *Let σ_2 be the number of Lemma 6. There exists computable $T_2 > 0$ which does not depend on $y \geq 1$ such that there exists at least two distinct zeros ρ and ρ' of $\xi(s)$ satisfying*

$$\left| \frac{s}{s-1} \cdot \frac{2s-1-(1-\bar{\rho})}{2s-\rho} \right| < 1 \quad \textit{and} \quad \left| \frac{1-s-\mu}{s-\mu} \cdot \frac{2s-1-(1-\bar{\rho}')}{2s-\rho'} \right| < 1 \tag{2.7}$$

if $1/2 < \Re(s) \leq \sigma_2$ and $|\Im(s)| \geq T_2$.

We prove Lemma 8 by using the following lemma:

Lemma 9 (Lemma 9 of [6])**.** *For any real value of t there exist at least three distinct zeros $\rho = \beta + i\gamma$ of $\xi(s)$ such that $0 < \beta \leq 1/2$ and $|t - \gamma| \leq 22$.*

Proof of Lemma 8. The first inequality of (2.7) is equivalent to

$$1 - \frac{4(2\sigma-1)(1-\beta)}{(2\sigma-\beta)^2+(2t-\gamma)^2} = \left| \frac{2s-1-(1-\bar{\rho})}{2s-\rho} \right|^2 < \left| \frac{s-1}{s} \right|^2 = 1 - \frac{2\sigma-1}{\sigma^2+t^2},$$

where $s = \sigma + it$ and $\rho = \beta + i\gamma$. This inequality is equivalent to

$$\frac{(\sigma-\beta/2)^2+(t-\gamma/2)^2}{\sigma^2+t^2} < 1-\beta$$

when $\sigma > 1/2$. On the right-hand side, we have

$$\frac{(\sigma-\beta/2)^2+(t-\gamma/2)^2}{\sigma^2+t^2} \leq \frac{\sigma^2+(t-\gamma/2)^2}{\sigma^2+t^2},$$

since $\sigma^2 \geq (\sigma-\beta/2)^2$ if $\sigma \geq \beta/4$ $(0 < \beta < 1)$. Moreover, there exists a zero $\rho = \beta + i\gamma$ of $\xi(s)$ such that $0 < \beta \leq 1/2$ and $|t - \gamma/2| \leq 11$ by Lemma 9. Therefore, for such a zero, we have

$$\frac{\sigma^2+(t-\gamma/2)^2}{\sigma^2+t^2} < \frac{\sigma^2+121}{\sigma^2+t^2} \quad \text{and} \quad \frac{1}{2} \leq 1-\beta.$$

Here, $(\sigma^2+121)/(\sigma^2+t^2)$ is an increasing function of σ if $|t| > 11$. In particular,

$$\frac{\sigma^2+121}{\sigma^2+t^2} \leq \frac{\sigma_1^2+121}{\sigma_1^2+t^2}$$

if $1/2 < \sigma \leq \sigma_1$ and $|t| \geq 11$.

The second inequality of (2.7) is equivalent to

$$1 - \frac{4(2\sigma-1)(1-\beta')}{(2\sigma-\beta')^2+(2t-\gamma')^2} = \left| \frac{2s-1-(1-\bar{\rho}')}{2s-\rho'} \right|^2 < \left| \frac{s-\mu}{1-s-\mu} \right|^2 = 1 - \frac{(2\sigma-1)(2\mu-1)}{(\sigma-1+\mu)^2+t^2},$$

where $s = \sigma + it$ and $\rho' = \beta' + i\gamma'$. This inequality is equivalent to

$$(2\mu-1)\frac{(\sigma-\beta'/2)^2+(t-\gamma'/2)^2}{(\sigma-1+\mu)^2+t^2} < 1-\beta'$$

when $\sigma > 1/2$. On the right-hand side, we have

$$\frac{(\sigma-\beta'/2)^2+(t-\gamma'/2)^2}{(\sigma-1+\mu)^2+t^2} \leq \frac{(\sigma-\kappa)^2+(t-\gamma'/2)^2}{(\sigma-\kappa)^2+t^2},$$

for $\sigma > 1/2$, where $\kappa = 1-\mu$ if $\beta' - 2(1-\mu) \geq 0$ and $\kappa = \beta'/2$ if $\beta' - 2(1-\mu) < 0$, since $(\sigma-1+\mu)^2 \geq (\sigma-\beta'/2)^2$ if $\sigma \geq (\beta'+2-2\mu)/4$ and $\beta' - 2(1-\mu) \geq 0$, $(\sigma-1+\mu)^2 < (\sigma-\beta'/2)^2$ if $\sigma \geq (\beta'+2-2\mu)/4$ and $\beta' - 2(1-\mu) < 0$, and $(\beta'+2-2\mu)/4 < 1/2$ by $\beta' < 1$ and $\mu > 1/2$. Moreover, a zero $\rho' = \beta' + i\gamma'$ of $\xi(s)$ can be taken as $0 < \beta' \leq 1/2$ and $|t - \gamma'/2| \leq 11$ by Lemma 9. Therefore, for such a zero, we have

$$\frac{(\sigma-\kappa)^2+(t-\gamma'/2)^2}{(\sigma-\kappa)^2+t^2} < \frac{(\sigma-\kappa)^2+121}{(\sigma-\kappa)^2+t^2} \quad \text{and} \quad \frac{1}{2} \leq 1-\beta.$$

Here, $((\sigma-\kappa)^2+121)/((\sigma-\kappa)^2+t^2)$ is an increasing function of σ if $\sigma > \kappa$ and $|t| > 11$. Note that $\kappa < 1/2$ by $\beta' < 1$ and $\nu > 1/2$. In particular,

$$\frac{(\sigma-\kappa)^2+121}{(\sigma-\kappa)^2+t^2} \le \frac{(\sigma_2-\kappa)^2+121}{(\sigma_2-\kappa)^2+t^2}$$

if $1/2 < \sigma \le \sigma_2$ and $|t| \ge 11$.

Hence, if we take $T_2 \ge 11$ so that $(2\mu-1)((\sigma_2-\kappa)^2+121)/((\sigma_2-\kappa)^2+t^2) < 1/2$ holds for any $|t| \ge T_2$, we obtain (2.7). □

Conclusion: By Lemma 6 and 7, we find that off-line zeros of $H^*(y;s)$ must be in the region

$$D_2 = \{s : \Re(s) \ne 1/2,\ 1-\sigma_2 \le \Re(s) \le \sigma_2,\ |\Im(s)| \le T_2\},$$

where σ_2 and T_2 are computable numbers independent of $y \ge 1$. The numbers σ_2 and T_2 are determined by $\xi(s)$ and μ.

2.3. **Case (iii).** Let $\mu+i\lambda$ and $\nu-i\lambda$ be two zeros of $p(s)$ in the right-half plane$\Re(s) > 1/2$. They should satisfy $(\mu-\nu)\lambda = 0$, since $p(s)$ has real coefficients. In this case, the factor $(s-1)p(s)\zeta^*(2s)y^s$ in (2.1) has no zeros in the right-half plane $\Re(s) > 1/2$ except for the triple zeros $s = 1$ ($\mu = \nu = 1$, $\lambda = 0$); or the simple zero $s = 1$ and the double zero $s = \mu$ ($1/2 < \mu = \nu \in \mathbb{R}$, $\mu \ne 1$); or three simple zeros $s = 1$, $s = \mu+i\lambda$ and $s = \mu-i\lambda$ ($1/2 < \mu = \nu \in \mathbb{R}$, $\lambda \ne 0$). As in the case (i), we study the zeros of $1+R_{p,y}(s)$ in (2.2).

Lemma 10. *There exists computable $\sigma_3 > 1/2$ which does not depend on $y \ge 1$ such that $1+R_{p,y}(s)$ has no zeros (and poles) in the right-half plane $\Re(s) \ge \sigma_3$.*

Proof. We have

$$\frac{p(1-s)}{p(s)} \cdot \frac{s}{s-1} \cdot \frac{\xi(2-2s)}{\xi(2s)} \cdot y^{1-2s} = O(|s|^{-1/2})$$

as $|s| \to \infty$ and $\Re(s) \to \infty$ in a way similar to the proof of Lemma 2, where the implied constant does not depend on $y \ge 1$ but may depend on $p(s)$. The above estimate implies Lemma 10. □

Lemma 11. *Let σ_3 be the number of Lemma 10. There exists computable $T_3 > 0$ which does not depend on $y \ge 1$ such that $1+R_{p,y}(s)$ has no zeros (and poles) in the region $1/2 < \Re(s) < \sigma_3$ and $|\Im(s)| \ge T_3$.*

Proof. For a zero $\rho = \beta + i\gamma$ of $\xi(s)$, we have

$$\left|\frac{2s-1-(1-\bar\rho)}{2s-\rho}\right|^2 = 1 - \frac{4(2\sigma-1)(1-\beta)}{(2\sigma-\beta)^2+(2t-\gamma)^2} < 1,$$

since $\beta < 1$. Thus,

$$\left|\frac{2s-\rho_1}{2s-1-(1-\bar\rho_1)}\frac{2s-\rho_2}{2s-1-(1-\bar\rho_2)}\frac{2s-\rho_3}{2s-1-(1-\bar\rho_3)}\frac{\xi(2s-1)}{\xi(2s)}\right| < 1$$

for any zeros ρ_1, ρ_2, ρ_3 of $\xi(s)$. On the other hand,

$$\left|\frac{s-(\mu+i\lambda)}{1-s-(\mu+i\lambda)} \cdot \frac{s-(\nu-i\lambda)}{1-s-(\nu-i\lambda)} \cdot \frac{p(1-s)}{p(s)}\right|^2 \le 1$$

by (2.3). Therefore, the proof of Lemma 11 is reduced to Lemma 12 below. □

Lemma 12. *Let σ_3 be the number of Lemma 10. There exists computable $T_3>0$ which does not depend on $y\geq 1$ such that there exists at least three distinct zeros ρ_1, ρ_2, ρ_3 of $\xi(s)$ satisfying*

$$\begin{aligned}&\left|\frac{s}{s-1}\cdot\frac{2s-1-(1-\bar\rho_1)}{2s-\rho_1}\right|<1,\\ &\left|\frac{1-s-\mu-i\lambda}{s-\mu-i\lambda}\cdot\frac{1-s-\nu+i\lambda}{s-\nu+i\lambda}\cdot\frac{2s-1-(1-\bar\rho_2)}{2s-\rho_2}\cdot\frac{2s-1-(1-\bar\rho_3)}{2s-\rho_3}\right|<1,\end{aligned}\tag{2.8}$$

if $1/2<\Re(s)\leq\sigma_3$ and $|\Im(s)|\geq T_3$.

We prove Lemma 12 by using Lemma 9 as in case (ii).

Proof of Lemma 12. The first inequality of (2.8) is proved as in the proof of Lemma 8.

First, we deal with the case $\mu=\nu$. In this case, we have

$$\begin{aligned}&\left|\frac{s-\mu-i\lambda}{1-s-\mu-i\lambda}\frac{s-\nu+i\lambda}{1-s-\nu+i\lambda}\right|^2\\ &\quad=\left(1-\frac{(2\sigma-1)(2\mu-1)}{(\sigma-1+\mu)^2+(t+\lambda)^2}\right)\left(1-\frac{(2\sigma-1)(2\mu-1)}{(\sigma-1+\mu)^2+(t-\lambda)^2}\right).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&\left|\frac{2s-1-(1-\bar\rho_2)}{2s-\rho_2}\frac{2s-1-(1-\bar\rho_3)}{2s-\rho_3}\right|^2\\ &\quad=\left(1-\frac{4(2\sigma-1)(1-\beta_2)}{(2\sigma-\beta_2)^2+(2t-\gamma_2)^2}\right)\left(1-\frac{4(2\sigma-1)(1-\beta_3)}{(2\sigma-\beta_3)^2+(2t-\gamma_3)^2}\right).\end{aligned}$$

Therefore, to prove the second inequality (2.8), it is sufficient to show

$$1-\frac{4(2\sigma-1)(1-\beta_2)}{(2\sigma-\beta_2)^2+(2t-\gamma_2)^2}<1-\frac{(2\sigma-1)(2\mu-1)}{(\sigma-1+\mu)^2+(t+\lambda)^2}$$

and

$$1-\frac{4(2\sigma-1)(1-\beta_3)}{(2\sigma-\beta_3)^2+(2t-\gamma_3)^2}<1-\frac{(2\sigma-1)(2\mu-1)}{(\sigma-1+\mu)^2+(t-\lambda)^2},$$

where $s=\sigma+it$, $\rho_2=\beta_2+i\gamma_2$ and $\rho_3=\beta_3+i\gamma_3$. These inequalities are equivalent to

$$(2\mu-1)\frac{(\sigma-\beta_2/2)^2+(t-\gamma_2/2)^2}{(\sigma-1+\mu)^2+(t+\lambda)^2}<1-\beta_2$$

and

$$(2\mu-1)\frac{(\sigma-\beta_3/2)^2+(t-\gamma_3/2)^2}{(\sigma-1+\mu)^2+(t-\lambda)^2}<1-\beta_3$$

when $\sigma>1/2$. On the right-hand side, we have

$$\frac{(\sigma-\beta_2/2)^2+(t-\gamma_2/2)^2}{(\sigma-1+\mu)^2+(t+\lambda)^2}\leq\frac{(\sigma-\kappa_2)^2+(t-\gamma_2/2)^2}{(\sigma-\kappa_2)^2+(t+\lambda)^2},$$

and

$$\frac{(\sigma-\beta_3/2)^2+(t-\gamma_3/2)^2}{(\sigma-1+\mu)^2+(t-\lambda)^2}\leq\frac{(\sigma-\kappa_3)^2+(t-\gamma_3/2)^2}{(\sigma-\kappa_3)^2+(t-\lambda)^2},$$

for $\sigma>1/2$, where $\kappa_j=1-\mu$ if $\beta_j-2(1-\mu)\geq 0$ and $\kappa_j=\beta_j/2$ if $\beta_j-2(1-\mu)<0$, since $(\sigma-1+\mu)^2\geq(\sigma-\beta_j/2)^2$ if $\sigma\geq(\beta_j+2-2\mu)/4$ and $\beta_j-2(1-\mu)\geq 0$, $(\sigma-1+\mu)^2<(\sigma-\beta_j/2)^2$ if $\sigma\geq(\beta_j+2-2\mu)/4$ and $\beta_j-2(1-\mu)<0$, and $(\beta_j+2-2\mu)/4<1/2$ by $\beta_j<1$ and $\mu>1/2$. Moreover, two zeros $\rho_j=\beta_j+i\gamma_j$ $(j=2,3)$ of $\xi(s)$ can be taken as $0<\beta_j\leq 1/2$ and $|t-\gamma_j/2|\leq 11$ by Lemma 9. Therefore, for such choice of zeros, we have

$$\frac{(\sigma-\kappa_j)^2+(t-\gamma_j/2)^2}{(\sigma-\kappa_j)^2+(t\pm\lambda)^2}<\frac{(\sigma-\kappa_j)^2+121}{(\sigma-\kappa_j)^2+(t\pm\lambda)^2}\quad\text{and}\quad\frac{1}{2}\leq 1-\beta_j,$$

Here, $((\sigma-\kappa_j)^2+121)/((\sigma-\kappa_j)^2+(t\pm\lambda)^2)$ are increasing functions of σ if $\sigma>\kappa$ and $|t\pm\lambda|>11$. Note that $\kappa_j<1/2$ by $\beta_j<1$ and $\mu>1/2$. In particular,

$$\frac{(\sigma-\kappa_j)^2+121}{(\sigma-\kappa_j)^2+(t\pm\lambda)^2}\le\frac{(\sigma_3-\kappa_j)^2+121}{(\sigma_3-\kappa_j)^2+(t\pm\lambda)^2}$$

if $1/2<\sigma\le\sigma_3$ and $|t\pm\lambda|\ge 11$.

Hence, if we take $T_3\ge 11$ so that $(2\mu-1)((\sigma_3-\kappa_j)^2+121)/((\sigma_3-\kappa_j)^2+(t\pm\lambda)^2)<1/2$ holds for any $|t|\ge T_3$, we obtain (2.8) for the case $\mu=\nu$.

Next, we deal with the case $\mu\ne\nu$. In this case, it must be $\lambda=0$. We have

$$\begin{aligned}&\left|\frac{s-\mu-i\lambda}{1-s-\mu-i\lambda}\frac{s-\nu+i\lambda}{1-s-\nu+i\lambda}\right|^2\\&\quad=\left(1-\frac{(2\sigma-1)(2\mu-1)}{(\sigma-1+\mu)^2+t^2}\right)\left(1-\frac{(2\sigma-1)(2\nu-1)}{(\sigma-1+\nu)^2+t^2}\right).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&\left|\frac{2s-1-(1-\bar\rho_2)}{2s-\rho_2}\frac{2s-1-(1-\bar\rho_3)}{2s-\rho_3}\right|^2\\&\quad=\left(1-\frac{4(2\sigma-1)(1-\beta_2)}{(2\sigma-\beta_2)^2+(2t-\gamma_2)^2}\right)\left(1-\frac{4(2\sigma-1)(1-\beta_3)}{(2\sigma-\beta_3)^2+(2t-\gamma_3)^2}\right).\end{aligned}$$

Therefore, to prove the second inequality (2.8), it is sufficient to show

$$1-\frac{4(2\sigma-1)(1-\beta_2)}{(2\sigma-\beta_2)^2+(2t-\gamma_2)^2}<1-\frac{(2\sigma-1)(2\mu-1)}{(\sigma-1+\mu)^2+t^2}$$

and

$$1-\frac{4(2\sigma-1)(1-\beta_3)}{(2\sigma-\beta_3)^2+(2t-\gamma_3)^2}<1-\frac{(2\sigma-1)(2\nu-1)}{(\sigma-1+\nu)^2+t^2},$$

where $s=\sigma+it$, $\rho_2=\beta_2+i\gamma_2$ and $\rho_3=\beta_3+i\gamma_3$. These inequalities are equivalent to

$$(2\mu-1)\frac{(\sigma-\beta_2/2)^2+(t-\gamma_2/2)^2}{(\sigma-1+\mu)^2+t^2}<1-\beta_2$$

and

$$(2\nu-1)\frac{(\sigma-\beta_3/2)^2+(t-\gamma_3/2)^2}{(\sigma-1+\nu)^2+t^2}<1-\beta_3$$

when $\sigma>1/2$. On the right-hand side, we have

$$\frac{(\sigma-\beta_2/2)^2+(t-\gamma_2/2)^2}{(\sigma-1+\mu)^2+t^2}\le\frac{(\sigma-\kappa_2)^2+(t-\gamma_2/2)^2}{(\sigma-\kappa_2)^2+t^2},$$

and

$$\frac{(\sigma-\beta_3/2)^2+(t-\gamma_3/2)^2}{(\sigma-1+\nu)^2+t^2}\le\frac{(\sigma-\kappa_3)^2+(t-\gamma_3/2)^2}{(\sigma-\kappa_3)^2+t^2},$$

for $\sigma>1/2$, where $\kappa_2=1-\mu$ if $\beta_2-2(1-\mu)\ge0$ and $\kappa_2=\beta_2/2$ if $\beta_2-2(1-\mu)<0$; $\kappa_3=1-\nu$ if $\beta_3-2(1-\nu)\ge0$ and $\kappa_3=\beta_3/2$ if $\beta_3-2(1-\nu)<0$ by a reason similar to the case $\mu=\nu$. Moreover, two zeros $\rho_j=\beta_j+i\gamma_j$ $(j=2,3)$ of $\xi(s)$ can be taken as $0<\beta_j\le1/2$ and $|t-\gamma_j/2|\le11$ by Lemma 9. Therefore, for such choice of zeros, we have

$$\frac{(\sigma-\kappa_j)^2+(t-\gamma_j/2)^2}{(\sigma-\kappa_j)^2+t^2}<\frac{(\sigma-\kappa_j)^2+121}{(\sigma-\kappa_j)^2+t^2}\quad\text{and}\quad\frac{1}{2}\le1-\beta_j\quad(j=2,3).$$

Here, $((\sigma-\kappa_j)^2+121)/((\sigma-\kappa_j)^2+t^2)$ are increasing functions of σ if $\sigma>\kappa$ and $|t|>11$. Note that $\kappa_j<1/2$ by $\beta_j<1$ and $\mu,\nu>1/2$. In particular,

$$\frac{(\sigma-\kappa_j)^2+121}{(\sigma-\kappa_j)^2+t^2}\le\frac{(\sigma_3-\kappa_j)^2+121}{(\sigma_3-\kappa_j)^2+t^2}$$

if $1/2 < \sigma \le \sigma_3$ and $|t| \ge 11$.

Hence, if we take $T_3 \ge 11$ so that $(2\mu - 1)((\sigma_3 - \kappa_2)^2 + 121)/((\sigma_3 - \kappa_2)^2 + t^2) < 1/2$ and $(2\nu - 1)((\sigma_3 - \kappa_3)^2 + 121)/((\sigma_3 - \kappa_3)^2 + t^2) < 1/2$ hold for any $|t| \ge T_3$, we obtain (2.8) for the case $\mu \ne \nu$. Now we complete the proof. □

Conclusion: By Lemma 10 and 11, we find that off-line zeros of $H^*(y; s)$ must be in the region

$$D_3 = \{s : \Re(s) \ne 1/2,\ 1 - \sigma_3 \le \Re(s) \le \sigma_3,\ |\Im(s)| \le T_3\},$$

where σ_3 and T_3 are computable numbers independent of $y \ge 1$. The numbers σ_3 and T_3 are determined by $\xi(s)$, $\mu + i\lambda$ and $\nu - i\lambda$.

References

[1] de Bruijn, N.G. The roots of trigonometric integrals, *Duke Mathematical Journal* **17**, (1950), 197–226.

[2] Ki, H. On the Zeros of Weng's Zeta Functions, *Int. Math. Res. Not. IMRN* **2010**, no. 13, 2367–2393.

[3] Lagarias, J.C. On a positivity property of the Riemann ξ-function, *Acta Arithmetica* **89** (1999), no.3, 217–234.

[4] Lagarias, J.C.; Suzuki, M. The Riemann hypothesis for certain integrals of Eisenstein series, *J. Number Theory* **118** (2006), no. 1, 98–122.

[5] Suzuki, M. A proof of the Riemann hypothesis for the Weng zeta function of rank 3 for the rationals, *The Conference on L-Functions*, 175–199, *World Sci. Publ., Hackensack, NJ*, 2007.

[6] Suzuki, M.; Weng, L. Zeta functions for G_2 and their zeros, *Int. Math. Res. Not. IMRN*, **2009** no. 2, 241–290.