Integrability Conditions

Let $\Omega(u, v)$ and $\Lambda(u, v)$ be $n \times n$-matrix valued $C^\infty$-maps defined on a domain $U \subset \mathbb{R}^2$. In this section, we consider an initial value problem of a system of linear partial differential equations

$$
(2.1) \quad \frac{\partial X}{\partial u} = X\Omega, \quad \frac{\partial X}{\partial v} = X\Lambda, \quad X(u_0, v_0) = X_0,
$$

where $(u_0, v_0) \in U$ is a fixed point, $X$ is an $n \times n$-matrix valued unknown, and $X_0 \in \mathrm{M}_n(\mathbb{R})$.

**Proposition 2.1.** If a matrix-valued $C^\infty$-function $X(u, v)$ defined on $U \subset \mathbb{R}^2$ satisfies (2.1) with $X_0 \in \mathrm{GL}(n, \mathbb{R})$, then $X(u, v) \in \mathrm{GL}(n, \mathbb{R})$ for all $(u, v) \in U$. In addition, if $\Omega$ and $\Lambda$ are skew-symmetric and $X_0 \in \mathrm{SO}(n)$, then $X \in \mathrm{SO}(n)$ holds on $U$.

**Proof.** Take a smooth path $\gamma: [0, 1] \to U$ joining $(u_0, v_0)$ and $(u, v)$, and write $\gamma(t) = (u(t), v(t))^4$. Setting $\tilde{X}(t) := X \circ \gamma(t) = (u(t), v(t))^4$, $\tilde{X}(0) = (u(0), v(0))$.

4Since $U$ is connected, there exists a continuous path $\gamma: [0, 1] \to U$ joining $(u_0, v_0)$ and $(u, v)$. Then one can find a smooth curve $\tilde{\gamma}$ joining these points as follows: For each $t \in [0, 1]$, there exists a positive number $\rho_t > 0$ such that $B_{\rho_t}(\gamma(t)) \subset U$. Since $\gamma([0, 1])$ is compact, there exists a finite sequence $0 = t_0 < t_1 < \cdots < t_N = 1$ such that $\gamma([0, 1]) = \bigcup_{j=0}^{N} B_{\rho_j}(\gamma(t_j))$, where $B_\varepsilon(p)$ denotes a disk of radius $\varepsilon$ centered at $p$. Choose $p_j \in B_{\rho_{j-1}}(\gamma(t_{j-1})) \cap B_{\rho_j}(\gamma(t_j))$ ($j = 1, \ldots, N$). Then the polygonal line with vertices $\{\gamma(0), p_1, \ldots, p_N, \gamma(1)\}$ lies on $U$ and a piecewise linear path joining $\gamma(0) = (u_0, v_0)$ and $\gamma(1) = (u, v)$. Modifying such a path at vertices, we have a smooth path joining $\gamma(0)$ and $\gamma(1)$ (cf. see [2-1, Appendix B-5]).

Hence, by Proposition 1.3, $\det \tilde{X}(1) \neq 0$. The latter half of the statement follows from Proposition 1.4.

**Lemma 2.2.** If a matrix-valued $C^\infty$ function $X: U \to \mathrm{GL}(n, \mathbb{R})$ satisfies (2.1), it holds that

$$
(2.2) \quad \Omega_v - A_u = \Omega A - A\Omega.
$$

**Proof.** Differentiating the first (resp. second) equation of (2.1) by $v$ (resp. $u$), we have

$$
X_{uu} = X_u \Omega + X \Omega_v = X (\Lambda \Omega + \Omega v),
$$
$$
X_{vv} = X_u A + X A_u = X (\Omega A + A_u).
$$

These two matrices coincide since $X$ is of class $C^\infty$. Hence we have the conclusion.

The equality (2.2) is called the integrability condition or compatibility condition of (2.1).

**Frobenius’ theorem** In this section, we shall prove the following

**Theorem 2.3.** Let $\Omega(u, v)$ and $\Lambda(u, v)$ be $n \times n$-matrix valued $C^\infty$-functions defined on a simply connected domain $U \subset \mathbb{R}^2$
satisfying (2.2). Then for each \((u_0, v_0) \in U\) and \(X_0 \in M_n(\mathbb{R})\), there exists the unique \(n \times n\)-matrix valued function \(X : U \to M_n(\mathbb{R})\) (2.1). Moreover,

- if \(X_0 \in \text{GL}(n, \mathbb{R})\), \(X(u, v) \in \text{GL}(n, \mathbb{R})\) holds on \(U\),
- if \(\text{tr} \Omega = \text{tr} \Lambda = 0\) holds on \(U\) and \(X_0 \in \text{SL}(n, \mathbb{R})\), \(X(u, v) \in \text{SL}(n, \mathbb{R})\) holds on \(U\),
- if \(\Omega\) and \(\Lambda\) are skew-symmetric matrices, and \(X_0 \in \text{SO}(n)\), \(X(u, v) \in \text{SO}(n)\) holds on \(U\).

To prove Theorem 2.3, it is sufficient to show for the case \(U = \mathbb{R}^2\). In fact, by Lemma 2.4 and Fact 2.5 below, we can replace \(U\) with \(\mathbb{R}^2\) by an appropriate coordinate change.

**Lemma 2.4.** Let \(V \ni (\xi, \eta) \mapsto (u, v) \in U\) be a diffeomorphism between domains \(V, U \subset \mathbb{R}^2\), and let \(\Omega = \Omega(u, v)\) and \(\Lambda = \Lambda(u, v)\) be matrix-valued functions on \(U\). Set

\[
\tilde{\Omega}(\xi, \eta) := \Omega(u(\xi, \eta), v(\xi, \eta)) \frac{\partial u}{\partial \xi} + \Lambda(u(\xi, \eta), v(\xi, \eta)) \frac{\partial v}{\partial \xi},
\]

\[
\tilde{\Lambda}(\xi, \eta) := \Omega(u(\xi, \eta), v(\xi, \eta)) \frac{\partial u}{\partial \eta} + \Lambda(u(\xi, \eta), v(\xi, \eta)) \frac{\partial v}{\partial \eta}.
\]

If a matrix-valued function \(X : U \to M_n(\mathbb{R})\) satisfies (2.1), \(\tilde{X}(\xi, \eta) = X(u(\xi, \eta), v(\xi, \eta))\) satisfies

\[
\frac{\partial \tilde{X}}{\partial \xi} = \tilde{X} \tilde{\Omega}, \quad \frac{\partial \tilde{X}}{\partial \eta} = \tilde{X} \tilde{\Lambda}, \quad \tilde{X}(\xi_0, \eta_0) = X_0,
\]

where \((u(\xi_0, \eta_0), v(\xi_0, \eta_0)) = (u_0, v_0)\). Moreover, the integrability condition (2.2) of (2.1) is equivalent to that of (2.4).

**Proof.** The equation (2.1) can be considered as a equality of 1-forms

\[
dX = X\Theta, \quad \Theta := \Omega du + \Lambda dv,
\]

which does not depend on a choice of coordinate systems. If we write

\[
\Theta = \Omega du + \Lambda dv = \tilde{\Omega} d\xi + \tilde{\Lambda} d\eta,
\]

\(\Omega\), \(\Lambda\), \(\tilde{\Omega}\) and \(\tilde{\Lambda}\) satisfy (2.3). Here, the integrability condition can be rewritten as

\[
d\Theta + \Theta \wedge \Theta = 0,
\]

which is an equality of 2-forms. This does not depend on coordinates, the conclusion follows. \(\square\)

**Fact 2.5.** A simply connected domain in \(\mathbb{R}^2\) is diffeomorphic to \(\mathbb{R}^2\).

In fact, the Riemann mapping theorem yields the fact above.

**Proof of Theorem 2.3.** By Lemma 2.4 and Fact 2.5, we may assume \(U = \mathbb{R}^2\), \((u_0, v_0) = (0, 0)\) without loss of generality.

**Existence:** By the fundamental theorem of linear ordinary differential equations (Corollary 1.7), there exists the unique \(C^\infty\)-map \(F : \mathbb{R} \to M_n(\mathbb{R})\) such that

\[
\frac{dF}{du}(u) = F(u)\Omega(u, 0) \quad F(0) = X_0.
\]

Identifying \(\mathbb{R}^2\) with the complex plane \(\mathbb{C}\), a simply connected domain of \(U = \mathbb{R}^2\) is conformally equivalent to the unit disc \(D := \{z \in \mathbb{C} \mid |z| < 1\}\) or \(\mathbb{C}\), because of the Riemann mapping theorem (cf. [2,3]). Though \(D\) and \(\mathbb{C}\) are not conformally equivalent, \(D\) and \(\mathbb{R}^2\) are diffeomorphic. Then any simply connected domain is diffeomorphic to \(\mathbb{R}^2\).
For each $u \in \mathbb{R}$, we denote by $G^u(v)$ the unique solution of the ordinary differential equation
\[
\frac{dG^u}{dv}(v) = G^u(v)A(u, v), \quad G^u(0) = F(u)
\]
in $v$. Then the function $X(u, v) := G^u(v)$ is the desired one. In fact, the solution of a ordinary differential equation depends smoothly on the initial value, $X(u, v)$ is a matrix-valued $C^\infty$ function defined on $\mathbb{R}^2$. By definition of $G^u(v)$, we have
\[
(2.5) \quad \frac{\partial X}{\partial v}(u, v) = \frac{dG^u}{dv}(v) = G^u(v)A(u, v) = X(u, v)A(u, v).
\]
Since $X$ is $C^\infty$, $X_{uv} = X_{vu}$ holds. Then by the integrability condition (2.2), it holds that
\[
\frac{\partial}{\partial v} \left( \frac{\partial X}{\partial u} - X\Omega \right) = \frac{\partial}{\partial u} \frac{\partial X}{\partial v} - \frac{\partial X}{\partial v} \Omega - X \frac{\partial \Omega}{\partial v}
\]
\[
= \frac{\partial}{\partial u} (XA) - \frac{\partial X}{\partial v} \Omega - X \frac{\partial \Omega}{\partial v}
\]
\[
= \frac{\partial X}{\partial u} A + X \frac{\partial A}{\partial u} - \frac{\partial X}{\partial v} \Omega - X \frac{\partial \Omega}{\partial v}
\]
\[
= X(A_u - \Omega_v) + \frac{\partial X}{\partial u} A - \frac{\partial X}{\partial v} \Omega
\]
\[
= X(A_u - \Omega_v - \Lambda \Omega) + \frac{\partial X}{\partial u} A
\]
\[
= -X\Omega A + \frac{\partial X}{\partial u} A
\]
\[
= \left( \frac{\partial X}{\partial u} - X\Omega \right) A.
\]
That is, for each fixed $u$, the map $H(v) := X_u(u, v) - X\Omega$ satisfies an ordinary differential equation in $v$ as follows:
\[
\frac{dH}{dv}(u, v) = H(u, v)A(u, v).
\]
Letting $v = 0$, we have
\[
H(u, 0) = X_u(u, 0) - X(u, 0)\Omega(u, 0)
\]
\[
= (G^u)_u(u, 0) - G^u(0)\Omega(u, 0)
\]
\[
= F(u) - F(u)\Omega(u, 0) = O
\]
and then, by uniqueness of the solutions of initial value problems for ordinary differential equations, $H(u, v) = 0$ holds. Since $(u, v)$ is arbitrarily taken, we have
\[
\frac{\partial X}{\partial u}(u, v) = X(u, v)\Omega(u, v),
\]
that is, $X(u, v)$ is the solution of (2.1).

\textit{Uniqueness:} Let $X$ and $\bar{X}$ be matrix-valued functions satisfying (2.1). Then $\bar{X} - X$ is a solution of (2.1) with $X_0 = O$ since (2.1) is linear. Hence, to show the uniqueness, it is sufficient to show that the solution $X$ of (2.1) with initial condition $X_0 = O$ is the constant function $X(u, v) = O$.

Let $X$ be such a solution of (2.1). Here, $X(0, 0) = O$ as we have set $(u_0, v_0) = (0, 0)$. For an arbitrary $(u, v) \in \mathbb{R}^2$, let $F(t) := X(tu, tv)$. Then
\[
(2.6) \quad \frac{d}{dt} F(t) = uX_u(tu, tv) + vX_v(tu, tv)
\]
\[
= X(tu, tv)(u\Omega(tu, tv) + v\Lambda(tu, tv)) = F(t)\omega(t)
\]
holds, where \( \omega(t) = u\Omega(tu, tv) + v\Lambda(tu, tv) \). Then the ordinary differential equation (2.6) for \( F(t) \) in \( t \), the uniqueness of solutions of ordinary differential equations yields \( F(t) = 0 \) since \( F(0) = X(0,0) = 0 \). In particular, we have \( X(u,v) = F(1) = 0 \). Since \( (u,v) \) has been taken arbitrarily, \( X(u,v) = 0 \) holds for all \( (u,v) \in \mathbb{R}^2 \). Hence we have the uniqueness.

**Application: Poincaré’s lemma.**

**Theorem 2.6** (Poincaré’s lemma). *If a differential 1-form \( \omega = \alpha(u,v) du + \beta(u,v) dv \) defined on a simply connected domain \( U \subset \mathbb{R}^2 \) is closed, that is, \( d\omega = 0 \) holds, then there exists a \( C^\infty \)-function \( f \) on \( U \) such that \( df = \omega \). Such a function \( f \) is unique up to additive constants.***

**Proof.** Since \( d\omega = (\beta_u - \alpha_v) du \wedge dv \), the assumption is equivalent to \( \beta_u - \alpha_v = 0 \).

Consider a system of linear partial differential equations with unknown a \( 1 \times 1 \)-matrix valued function (i.e. a real-valued function) \( \xi(u,v) \) as

\[
\frac{\partial \xi}{\partial u} = \xi \alpha, \quad \frac{\partial \xi}{\partial v} = \xi \beta, \quad \xi(u_0, v_0) = 1.
\]

Then it satisfies (2.2) because of (2.7). Hence by Theorem 2.3, there exists a smooth function \( \xi(u,v) \) satisfying (2.8). In particular, Proposition 1.3 yields \( \xi = \det \xi \) never vanishes. Since

\[
\xi(u_0, v_0) = 1 > 0, \text{ this means that } \xi > 0 \text{ holds on } U. \text{ Letting } f := \log \xi, \text{ we have the function } f \text{ satisfying } df = \omega.
\]

Next, we show the uniqueness: if two functions \( f \) and \( g \) satisfy \( df = dg = \omega \), it holds that \( d(f - g) = 0 \). Hence by connectivity of \( U \), \( f - g \) must be constant. \( \square \)

**Application: Conjugation of Harmonic functions.** In this paragraph, we identify \( \mathbb{R}^2 \) with the complex plane \( \mathbb{C} \). It is well-known that a function

\[
f: U \ni u + iv \mapsto \xi(u,v) + i\eta(u,v) \in \mathbb{C} \quad (i = \sqrt{-1})
\]

defined on a domain \( U \subset \mathbb{C} \) is holomorphic if and only if it satisfies the following relation, called the Cauchy-Riemann equations:

\[
(2.10) \quad \frac{\partial \xi}{\partial u} = \frac{\partial \eta}{\partial v}, \quad \frac{\partial \xi}{\partial v} = -\frac{\partial \eta}{\partial u}.
\]

**Definition 2.7.** A function \( f: U \ni u + iv \mapsto \xi(u,v) \) defined on a domain \( U \subset \mathbb{R}^2 \) is said to be harmonic if it satisfies

\[
\Delta f = f_{uu} + f_{vv} = 0.
\]

The operator \( \Delta \) is called the Laplacian.

**Proposition 2.8.** *If function \( f \) in (2.9) is holomorphic, \( \xi(u,v) \) and \( \eta(u,v) \) are harmonic functions.*

**Proof.** By (2.10), we have

\[
\xi_{uu} = (\xi_u)_u = (\eta_v)_u = \eta_{uu} = (\eta_u)_v = (-\xi_v)_v = -\xi_{vv}.
\]
Hence $\Delta \xi = 0$. Similarly,

$$\eta_{uu} = (-\xi_v)u = -\xi_{vu} = -(\xi_u)_v = -(\eta_v)_v = -\eta_{vv}.$$ 

Thus $\Delta \eta = 0$.

**Theorem 2.9.** Let $U \subset \mathbb{C} = \mathbb{R}^2$ be a simply connected domain and $\xi(u, v)$ a $C^\infty$-function harmonic on $U^6$. Then there exists a $C^\infty$ harmonic function $\eta$ on $U$ such that $\xi(u, v) + i\eta(u, v)$ is holomorphic on $U$.

**Proof.** Let $\alpha := -\xi_v du + \xi_u dv$. Then by the assumption,

$$d\alpha = (\xi_{uv} + \xi_{uu}) du \wedge dv = 0$$

holds, that is, $\alpha$ is a closed 1-form. Hence by simple connectivity of $U$ and the Poincaré’s lemma (Theorem 2.6), there exists a function $\eta$ such that $d\eta = \eta_{uu} du + \eta_{uv} dv = \alpha$. Such a function $\eta$ satisfies (2.10) for given $\xi$. Hence $\xi + i\eta$ is holomorphic in $u + iv$.

**Example 2.10.** A function $\xi(u, v) = e^u \cos v$ is harmonic. Set

$$\alpha := -\xi_v du + \xi_u dv = e^u \sin v du + e^u \cos v dv.$$ 

Then $\eta(u, v) = e^u \sin v$ satisfies $d\eta = \alpha$. Hence

$$\xi + i\eta = e^u(\cos v + i \sin v) = e^{u+i v}$$

is holomorphic in $u + iv$.

**Definition 2.11.** The harmonic function $\eta$ in Theorem 2.9 is called the conjugate harmonic function of $\xi$.

---

*The theorem holds under the assumption of $C^2$-differentiability.*

**The fundamental theorem for Surfaces.** Let $p: U \to \mathbb{R}^3$ be a parametrization of a regular surface defined on a domain $U \subset \mathbb{R}^2$. That is, $p = p(u, v)$ is a $C^\infty$-map such that $p_u$ and $p_v$ are linearly independent at each point on $U$. Then $\nu := (p_u \times p_v)/|p_u \times p_v|$ is the unit normal vector field to the surface. The matrix-valued function $F := (p_u, p_v, \nu): U \to M_3(\mathbb{R})$ is called the Gauss frame of $p$. We set

$$ds^2 := E \, du^2 + 2F \, du \, dv + G \, dv^2,$$

(2.11)

$$H := L \, du^2 + 2M \, du \, dv + N \, dv^2,$$

where

$$E = p_u \cdot p_u \quad F = p_u \cdot p_v \quad G = p_v \cdot p_v$$

$$L = p_{uu} \cdot \nu \quad M = p_{uv} \cdot \nu \quad N = p_{vv} \cdot \nu.$$ 

We call $ds^2$ (resp. $H$) the first (resp. second) fundamental form. Note that linear independence of $p_u$ and $p_v$ implies

(2.12)  $E > 0, \quad G > 0$ \quad and \quad $EG - F^2 > 0$. 

Set

(2.13)  $$\Gamma_{11}^1 := \frac{GE_u - 2FF_u + FE_v}{2(EG - F^2)},$$

$$\Gamma_{12}^1 := \frac{2EF_v - EE_v - FE_u}{2(EG - F^2)},$$

$$\Gamma_{12}^1 = \Gamma_{21}^1 := \frac{GE_v - FG_u}{2(EG - F^2)}.$$
The functions $\Gamma^k_{ij}$ and the matrix $A$ are called the Christoffel symbols and the Weingarten matrix. We state the following fundamental theorem for surfaces, and give a proof (for a special case) in the following section.

**Theorem 2.12** (The Fundamental Theorem for Surfaces). Let $p: U \supset (u, v) \mapsto p(u, v) \in \mathbb{R}^3$ be a parametrization of a regular surface defined on a domain $U \subset \mathbb{R}^2$. Then the Gauss frame $F := \{p_u, p_v, \nu\}$ satisfies the equations

\[
\begin{align*}
\frac{\partial F}{\partial u} &= F\Omega, & \frac{\partial F}{\partial v} &= F\Lambda, \\
\Omega := \begin{pmatrix} \Gamma^1_{11} & \Gamma^1_{12} & -A^1_1 \\ \Gamma^1_{12} & \Gamma^1_{22} & -A^1_2 \\ L & M & 0 \end{pmatrix}, & \Lambda := \begin{pmatrix} \Gamma^2_{21} & \Gamma^2_{22} & -A^2_2 \\ \Gamma^2_{21} & \Gamma^2_{22} & -A^2_2 \\ M & N & 0 \end{pmatrix},
\end{align*}
\]

where $\Gamma^k_{ij}$ ($i, j, k = 1, 2$), $A^k_i$ and $L, M, N$ are the Christoffel symbols, the entries of the Weingarten matrix and the entries of the second fundamental form, respectively.

Theorem 2.13. Let $U \subset \mathbb{R}^2$ be a simply connected domain, $E, F, G, L, M, N \in C^\infty$-functions satisfying (2.12), and $\Gamma^k_{ij}, A^k_i$ the functions defined by (2.13) and (2.14), respectively. If $\Omega$ and $\Lambda$ satisfy

\[
\Omega_v - \Lambda_u = \Omega\Lambda - \Lambda\Omega,
\]

there exists a parameterization $p: U \rightarrow \mathbb{R}^3$ of regular surface whose fundamental forms are given by (2.11). Moreover, such a surface is unique up to orientation preserving isometries of $\mathbb{R}^3$.

**References**


**Exercises**

2-1 Let $\xi(u, v) = \log \sqrt{u^2 + v^2}$ be a function defined on $U = \mathbb{R}^2 \setminus \{(0, 0)\}$.

1. Show that $\xi$ is harmonic on $U$.

2. Find the conjugate harmonic function $\eta$ of $\xi$ on $V = \mathbb{R}^2 \setminus \{(u, 0) | u \leq 0\} \subset U$.

3. Show that there exists no conjugate harmonic function of $\xi$ defined on $U$. 