3 The Gauss and Codazzi equations

The compatibility conditions. As seen in Theorem 2.5 in Section 2, the Gauss frame $\mathbf{F} = (f_u, f_v, \nu)$ for an immersion $f : D \ni (u, v) \mapsto f(u, v) \in \mathbb{R}^3$ satisfies the equation

\begin{equation}
\frac{\partial \mathbf{F}}{\partial u} = \mathbf{F} \Omega, \quad \frac{\partial \mathbf{F}}{\partial v} = \mathbf{F} \Lambda
\end{equation}

\[ \Omega := \begin{pmatrix} \Gamma^1_{11} & \Gamma^1_{12} & -A^1_1 \\
\Gamma^1_{11} & \Gamma^1_{12} & -A^1_2 \\
L & M & 0 \end{pmatrix}, \quad \Lambda := \begin{pmatrix} \Gamma^2_{21} & \Gamma^2_{22} & -A^2_1 \\
\Gamma^2_{21} & \Gamma^2_{22} & -A^2_2 \\
M & N & 0 \end{pmatrix}, \]

where $\Gamma^i_{jk}$ $(i, j, k = 1, 2)$, $A^k_l$ and $L$, $M$, $N$ are the Christoffel symbols, the entries of the Weingarten matrix and the entries of the second fundamental form, respectively.

Lemma 3.1. The coefficient matrices $\Omega$, $\Lambda$ in (3.1) satisfy

\begin{equation}
\frac{\partial \Omega}{\partial v} - \frac{\partial \Lambda}{\partial u} = \Omega \Lambda - \Lambda \Omega.
\end{equation}

Proof. Differentiating the first equation in (3.1) with respect to $v$, we have

\[ \mathcal{F}_{uv} = (\mathcal{F} \Omega)_v = \mathcal{F}_v \Omega + \mathcal{F} \Omega_v = \mathcal{F} A \Omega + \mathcal{F} \Omega_v = \mathcal{F} (\Omega \Lambda + \Omega_v). \]

Similarly, differentiating the first equation in (3.1) in $u$, it holds that

\[ \mathcal{F}_{vu} = \mathcal{F} (\Omega \Lambda + \Lambda_u). \]

Thus, \[ \mathcal{F} (\Omega \Lambda + \Omega_v) = \mathcal{F} (\Omega \Lambda + \Lambda_u) \]
holds. Noticing that $\mathcal{F}$ is a regular matrix, we have the conclusion. \qed

The equation (3.2) is called the compatibility condition, or the integrability condition of the equation (3.1).

The Gauss and Codazzi equations. The compatibility condition (3.2) consists of nine equations, because it is the equality for $3 \times 3$ matrices. However, they can be reduced three equations:

Lemma 3.2. The compatibility condition (3.2) is equivalent to the equation (Equation (2.8))

\begin{equation}
K = \frac{E (E_v G_v - 2 F_u G_v + G_u^2)}{4 (EG - F^2)^2} + \frac{F (E_u G_v - E_v G_u - 2 F_v F_u - 2 F_u G_u + 4 F_u F_v)}{4 (EG - F^2)^2} + \frac{G (E_u G_u - 2 E_v F_v + E_v^2)}{4 (EG - F^2)^2} - \frac{E_{uv} - 2 F_{uv} + G_{uu}}{2 (EG - F^2)}
\end{equation}

and the following two equations:

\begin{equation}
L_v - M_u = \Gamma^1_{21} L + \Gamma^1_{22} M - \Gamma^1_{11} M - \Gamma^1_{12} N, \quad M_v - N_u = \Gamma^2_{21} L + \Gamma^2_{22} M - \Gamma^1_{12} M - \Gamma^2_{12} N.
\end{equation}
Proof. By a direct computations, we can conclude that the \((1, 1), (1, 2), (2, 1), (2, 2)\)-components of (3.2) are equivalent to (2.8). On the other hand, the first (resp. the second) equation in (3.4) is equivalent to the (3, 1) (resp. (3, 2)) component of (3.2). Moreover, the (1, 3) and (2, 3)-components are equivalent to (3.4) because of the definition of the Weingarten matrix

\[
A = \frac{1}{g} \begin{pmatrix} G & -F \\ -F & E \end{pmatrix} \begin{pmatrix} L & M \\ M & N \end{pmatrix} (g = EG - F^2)
\]

and Lemma 2.3.

The equation (2.8) is called the Gauss equation. On the other hand, the equations (3.4) are called the Codazzi equations, or the Codazzi-Mainardi equations.

**Corollary 3.3.** Let \(f: \mathbb{R}^2 \supset D \to \mathbb{R}^3\) be an immersion with first and second fundamental forms as

\[
\begin{align*}
ds^2 &= E \, du^2 + 2F \, du \, dv + G \, dv^2, \\
II &= L \, du^2 + 2M \, du \, dv + N \, dv^2.
\end{align*}
\]

Then the entries \(E, F, G, L, M\) and \(N\) satisfy the Gauss equation (3.3) and the Codazzi equation (3.4), where \(\Gamma^i_{jk}\)'s are the Christoffel symbols (2.3), and \(K\) is the Gaussian curvature in (1.15).

**References**


Let $f$ be an immersion of the $uv$-plane into $\mathbb{R}^3$. The parameter $(u, v)$ is said to be isothermal or conformal if the first fundamental form is written as
\[ ds^2 = e^{2\sigma}(du^2 + dv^2) \] (i.e. $E = G = e^{2\sigma}, F = 0$),
where $\sigma = \sigma(u, v)$ is a smooth function in $(u, v)$.

Assume that $f$ is parametrized by an isothermal parameter $(u, v)$.

(1) Show that the Gauss frame $F$ satisfies the equation
\[
\frac{\partial F}{\partial u} = F\Omega, \quad \frac{\partial F}{\partial v} = FA
\]
\[
\Omega := \begin{pmatrix} \sigma_u & \sigma_v & -e^{-2\sigma}L \\ -\sigma_v & \sigma_u & -e^{-2\sigma}M \\ L & M & 0 \end{pmatrix},
\]
\[
A := \begin{pmatrix} \sigma_v & -\sigma_u & -e^{-2\sigma}M \\ \sigma_u & \sigma_v & -e^{-2\sigma}N \\ M & N & 0 \end{pmatrix},
\]
where $L$, $M$ and $N$ are the entries of the second fundamental form.

(2) Verify that the Gauss and Codazzi equations are written as
\[
\sigma_{uu} + \sigma_{vv} + e^{-2\sigma}(LN - M^2) = 0
\]
\[
L_v - M_u = \sigma_v(L + N)
\]
\[
N_u - M_v = \sigma_u(L + N).
\]

3-3 Let $f : \Sigma \to \mathbb{R}^3$ be an immersion of an oriented 2-manifold $\Sigma$, and $(u, v)$ be an isothermal coordinate system around $P \in \Sigma$ compatible to the orientation of $\Sigma$, and $(\xi, \eta)$ be another coordinate system around $P$ compatible to the orientation of $\Sigma$.

(1) Show that $(\xi, \eta)$ is isothermal if and only if
\[
u_\xi = \nu_\eta, \quad u_\eta = -v_\xi.
\]
(2) Verify that the above conditions are equivalent to that
\[
\zeta := \xi + i\eta \mapsto z := u + iv
\]
is holomorphic. \(^3\)

\(^2\)Let $(\Sigma, ds^2)$ be an arbitrary 2-dimensional Riemannian manifold. Then, it is known that, for any point $P \in \Sigma$, there exists an isothermal coordinate chart $(u, v)$ containing $P$, that is, the Riemannian metric $ds^2$ is written as $ds^2 = e^{2\sigma}(du^2 + dv^2)$ (cf. Section 15 of [3-1]).

\(^3\)Hence, the existence of isothermal coordinates implies the existence of the structure of a Riemann surface (a 1-dimensional complex manifold) on an oriented Riemannian manifold.
3-4 Let \( f: D \rightarrow \mathbb{R}^3 \) be an immersion with an isothremal parameter \((u, v)\), with fundamental forms

\[
ds^2 = e^{2\sigma}(du^2 + dv^2), \quad II = L\,du^2 + 2M\,du\,dv + N\,dv^2.
\]

(1) Show that the Gauss and Codazzi equations are equivalent to

\[
-e^{-2\sigma}(\sigma_{uu} + \sigma_{vv}) = K, \quad \frac{1}{2} \frac{\partial H}{\partial z} = e^{-2\sigma} \frac{\partial q}{\partial \bar{z}},
\]

where \( z = u + iv \) be an complex coordinate,

\[
q := \frac{1}{4} \left( (L - N) - 2iM \right),
\]

\( K \) is the Gaussian curvature, \( H \) is the mean curvature, and

\[
\frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial u} - i \frac{\partial}{\partial v} \right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial u} + i \frac{\partial}{\partial v} \right).
\]

(2) When \( H \) is constant, verify that the Codazzi equation is equivalent to the holomorphicity of \( q \).