An algorithm to draw external rays of the Mandelbrot set

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Abstract

In this note I explain an algorithm to draw the external rays of the Mandelbrot set with an error estimate. Newton's method is the main tool. [1]

1 Preliminary

We first recall the following definitions and facts: (See [CG] for example.)

(1) For $c \in \mathbb{C}$, set $q_c(z) = z^2 + c$. For given $z \in \mathbb{C}$, its *orbit* $\{q_c^n(z)\}_{n \geq 1}$ is inductively defined by $q_c^{n+1}(z) := q_c(q_c^n(z))$.

(2) The *Mandelbrot set* is defined by:

$$\mathbb{M} \ := \ \Big\{ c \in \mathbb{C} \ : \ \{q_c^n(0)\}_{n \geq 1} \ \textit{is bounded} \Big\}$$

(3) For $c \in \mathbb{C} - \mathbb{M}$, $q_c^n(0) \to \infty$ as $n \to \infty$. In this case, the behavior of this orbit is described as follows: There exists a compact topological disk E_c and a conformal homeomorphism $\phi_c : \mathbb{C} - E_c \to \phi_c(\mathbb{C} - E_c) \subset \mathbb{C} - \overline{\mathbb{D}}$ such that

   (a) $c \in \mathbb{C} - E_c$;

   (b) $\phi_c(q_c(z)) = \phi_c(z)^2$ for any $z \in \mathbb{C} - E_c$; and

   (c) $\phi_c(z)/z \to 1$ as $z \to \infty$

(4) For the Mandelbrot set $\mathbb{M}$ and each $c \in \mathbb{C} - \mathbb{M}$, set $\Phi(c) := \phi_c(c)$. Then Φ is a unique conformal homeomorphism from $\mathbb{C} - \mathbb{M}$ onto $\mathbb{C} - \overline{\mathbb{D}}$ with $\Phi(c)/c \to 1$ as $c \to \infty$.

(5) For $\theta \in \mathbb{R}/\mathbb{Z}$ ("angle"), the set

$$\mathcal{R}_{\mathbb{M}}(\theta) = \mathcal{R}(\theta) \ := \ \big\{ \Phi^{-1}(w) \ : \ \arg w = \theta \big\}$$

is called the *external ray* of angle θ of the Mandelbrot set $\mathbb{M}$.

The aim of this note is to give an algorithm to draw $\mathcal{R}(\theta)$ for given angle $\theta \in \mathbb{R}/\mathbb{Z}$. More precisely, we give finitely many points that enough approximate the set $\mathcal{R}(\theta)$ within a given precision.

[1] I learned its principle by M. Shishikura, but this idea of using Newton's method is probably well-known for many other people working on complex dynamics.

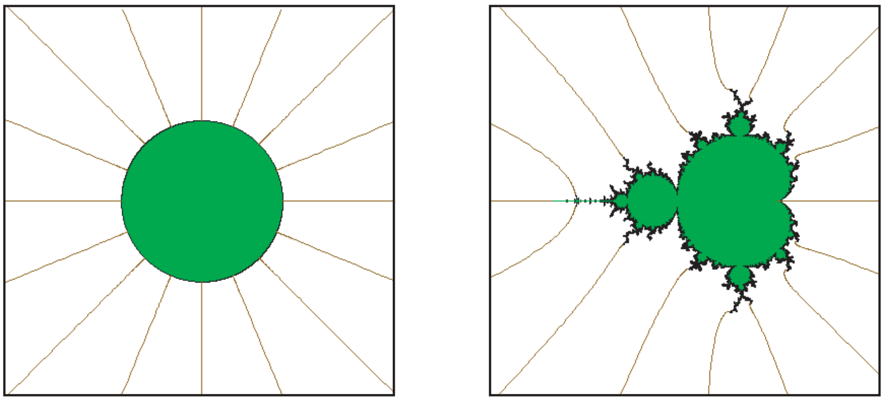

Figure 1: The map Φ sends the radial rays outside the unit disk to the external rays of $\mathbb{M}$. In this figure the rays of angle $m/16$ $(0 \le m < 16)$ are drawn in.

2 The algorithm: Theoretical settings

We first consider an algorithm to calculate $c \in \mathcal{R}_{\mathbb{M}}(\theta)$ with

$$c = \Phi^{-1}(re^{2\pi i\theta}) \iff \Phi(c) = \phi_c(c) = re^{2\pi i\theta}$$

for given $\theta \in \mathbb{R}/\mathbb{Z}$ and $r > 1$. By (3)-(b), we have

$$\phi_c(q_c^n(c)) = (re^{2\pi i\theta})^{2^n} = r^{2^n} e^{2\pi i \cdot 2^n \theta}$$

for any $n \in \mathbb{N}$. Now we assume that n is very large and $q_c^n(c)$ is enough close to infinity. Since we have $\phi_c(z)/z \to 1$ as $z \to \infty$ by (3)-(c), we have a "rough" approximation

$$q_c^n(c) \approx \phi_c(q_c^n(c)) = r^{2^n} e^{2\pi i \cdot 2^n \theta} =: t.$$

Now our task is to solve the equation $q_c^n(c) = t$. (We will later give an error estimate of the root caused by this approximation.)

A bit more generally, for given $n \in \mathbb{N}$ and $t \in \mathbb{C}$, we want to solve the equation

$$P_n(c) := q_c^n(c) - t = 0$$

numerically. Now $P_n(c)$ is a polynomial of degree 2^n in variable c. When n is large, it is impossible to find the roots algebraically.

For this kind of problem, a method which is commonly used is *Newton's method*. It is given as follows:

> **Newton's method.** Let F be a polynomial of degree more than one. We say the function
>
> $$N(w) = N_F(w) := w - \frac{F(w)}{F'(w)}$$
>
> is the *Newton map* of F.
>
> If $F(\alpha) = 0$ and w_0 is sufficiently close to α, then $N^k(w_0) \to \alpha$ as $k \to \infty$ at least exponentially fast. [2]

[2]More precisely, there exist constants $C > 0$ and $0 < \lambda < 1$ such that $|w_k - \alpha| \le C\lambda^k$. When α is a simple root of F, we have $|w_k - \alpha| = O(|w_{k-1} - \alpha|^2)$. In this case the convergence is super-exponentially fast.

See [H] for example. Now we apply this method to $F = P_n$ in variable c instead of w. In this case the Newton map is

$$N(c) \;=\; N_{n,t}(c) \;:=\; c - \frac{P_n(c)}{P_n'(c)}$$

where $P_n'(c) := \dfrac{dP_n}{dc}(c)$, a polynomial of degree $2^n - 1$. If the initial value c_0 is sufficiently close to a zero of $P_n(c)$, the sequence

$$c_0 \overset{N}{\longmapsto} N(c_0) \overset{N}{\longmapsto} N^2(c_0) \overset{N}{\longmapsto} N^3(c_0) \overset{N}{\longmapsto} \cdots$$

will converge to a zero of $P_n(c)$.

To proceed the iteration numerically, we need to calculate $P_n(c)$ and $P_n'(c)$ with given c. The calculation of $P_n(c) = q_c^n(c) - t$ is essentially the same as iteration of $q_c(z) = z^2 + c$. How about $P_n'(c)$?

Let $'$ denote $\dfrac{d}{dc}$. Then we have

$$\begin{aligned} P_n'(c) &= \{q_c^n(c)\}' \\ &= \left\{\left(q_c^{n-1}(c)\right)^2 + c\right\}' \\ &= 2\left\{q_c^{n-1}(c)\right\}' q_c^{n-1}(c) + 1 \\ &= 2P_{n-1}'(c) q_c^{n-1}(c) + 1. \end{aligned}$$

It follows that if we set $C_k := q_c^k(c)$ and $D_k := \left\{q_c^k(c)\right\}'$ for each $1 \le k \le n$, the recursive formulae

$$\begin{cases} C_1 = c, & C_k = C_{k-1}^2 + c \\ D_1 = 1, & D_k = 2D_{k-1}C_{k-1} + 1 \end{cases}$$

will give the values of $P_n(c) = C_n - t$ and $P_n'(c) = D_n$ respectively. Hence the Newton map can be written as

$$N : \; c \;\mapsto\; c - \frac{C_n - t}{D_n}.$$

3 The algorithm: Practical settings

For fixed $R > 1$ and a fixed integer D, consider the subset

$$\mathcal{R} \;:=\; \left\{\Phi^{-1}(re^{2\pi i\theta}) \;:\; R^{1/2^D} \le r < R\right\}$$

of the external ray $\mathcal{R}(\theta)$. If R is sufficiently large, $\mathcal{R}$ reaches enough close to ∞. If D is sufficiently large, $R^{1/2^D}$ is close to 1 and this implies that $\mathcal{R}$ reaches enough close to (the boundary of) $\mathbb{M}$. Hence we call D the *depth* of $\mathcal{R}$. Let us try to approximate this set $\mathcal{R}$ by finitely many points. [3]

For any r with $R^{1/2^D} \le r \le R$, one can approximate $c = \Phi^{-1}(re^{2\pi i\theta})$ by means of Newton's method under a suitable choice of the initial value. (We call this r the *radial parameter*.) Let us fix an integer $S > 0$ and call it the *sharpness*. We will pick up SD radial parameters $\{r_m\}_{m=1}^{SD}$ and calculate (approximate) SD points $\{c_m\}_{m=1}^{SD}$ on $\mathcal{R}$. Then we will

[3] We always draw a bounded domain with a finite number of pixels. Hence drawing the subset $\mathcal{R}$ is reasonable.

join the sequence c_m by segments in the computer display. This is what we mean by "drawing $\mathcal{R}$".

First we divide the interval $[R^{1/2^D}, R)$ into D sub-intervals

$$[R^{1/2^D}, R^{1/2^{D-1}}),\ [R^{1/2^{D-1}}, R^{1/2^{D-2}}), \ldots, [R^{1/2^2}, R^{1/2}),\ [R^{1/2}, R),$$

and we pick up S radial parameters from each sub-intervals as follows: For each $k = 1, 2, \cdots, D$, we define S radial parameters

$$R^{1/2^k},\ R^{1/2^{k-1+(S-1)/S}},\ \ldots\ , R^{1/2^{k-1+1/S}}$$

contained in the sub-interval $[R^{1/2^k}, R^{1/2^{k-1}})$. [4] We enumerate these radial parameters as follows:

$$\begin{cases} m \ := \ (k-1)S + j \quad (1 \le j \le S) \\ r_m \ := \ R^{1/2^{m/S}} \ = \ R^{1/2^{k-1+j/S}} \end{cases}$$

Note that we have $r_1 > r_2 > \cdots > r_{SD}$. [5] Now we are ready to apply Newton's method to calculate $\{c_m = \Phi^{-1}(r_m e^{2\pi i\theta})\}_{m=1}^{SD}$.

When $r_m \in [R^{1/2^k}, R^{1/2^{k-1}})$, we have $r_m^{2^k} \in [R, R^2)$ thus the value

$$\phi_{c_m}(q_{c_m}^k(c_m)) \ = \ r_m^{2^k} e^{2\pi i\theta\cdot 2^k} \ := \ t_m$$

satisfies $|t_m| \ge R$. Hence if R is sufficiently large, we have

$$t_m \ = \ \phi_{c_m}(q_{c_m}^k(c_m)) \ \approx \ q_{c_m}^k(c_m).$$

Under a suitable choice of the initial value $c_{m,0}$, its orbit by the Newton map N_{k,t_m} will give an approximation of c_m with $q_{c_m}^k(c_m) = t_m$. More precisely, we choose $c_{m,0}$ as follows:

- Since R is enough large, we have $\Phi^{-1}(Re^{2\pi i\theta}) \approx Re^{2\pi i\theta}$. (See (4) in the first section.) We set this value $c_0 := Re^{2\pi i\theta}$. [6]

- By using the initial value $c_0 = c_{1,0}$, we iterate the Newton map N_{1,t_1} sufficiently many times, say L_1 times. Set c_1 as its result. That is.

$$c_1 \ := \ N_{1,t_1}^{L_1}(c_0).$$

- Inductively, for any $1 \le m \le DS$ with $m = (k-1)S + j$ $(1 \le j \le S)$, we use c_{m-1} as the initial value $c_{m,0}$ and set
$$c_m \ := \ N_{k,t_m}^{L_m}(c_{m-1})$$
with sufficiently large integer L_m. The value c_{m-1} is presumably a "neighbor" of c_m on $\mathcal{R}$ so it is the best possible initial value for Newton's method.

  We should enlarge L_m when D is large, because better precision would be required when c_m is close to $\mathbb{M}$.

Finally join the set $\{c_m : 1 \le m \le DS\}$ by segments. This will give an approximation of $\mathcal{R}$.

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[4]The boundary of $\mathbb{M}$ is very complicated so it would be reasonable to choose r's in this way.

[5]This enumeration by m would be used only when we plot the segments. When we apply Newton's method to approximate c_m, we use loops by k and j.

[6]This part can be improved by using the expansion $\Phi^{-1}(z) = z - 1/2 + 1/(8z) + \cdots$.

4 Error estimate

In this algorithm we solved the equation $q_c^n(c) = t$ instead of solving $\phi_c(q_c^n(c)) = t$ for given $t \in \mathbb{C}$. Let us establish an error estimate by this approximation.

Let $\mathbb{D}_r$ denote the set $\{z \in \mathbb{C} : |z| < r\}$. It is well-known that $\mathbb{M} \subset \overline{\mathbb{D}_2}$. Hence we fix any $r > 2$ so that $\mathbb{D}_r$ is a neighborhood of $\mathbb{M}$. Now we assume that $|c| \le r$. Then we have:

Theorem 4.1 *Let us fix t with sufficiently large modulus $|t| = R \gg 0$. Let c be a root of $q_c^n(c) = t$. Then there exists a solution $\hat{c}$ of $\phi_{\hat{c}}(q_{\hat{c}}^n(\hat{c})) = t$ such that*

$$|\hat{c} - c| \; = \; O\left(\frac{1}{2^n R^{2-1/2^n}(R^{1/2^n} - 1)}\right).$$

When $n > \log_2 \log R$, we have a uniform estimate

$$|\hat{c} - c| \; = \; O\left(\frac{1}{R^2 \log R}\right).$$

Here "sufficiently large R" means that r/R is sufficiently small. This theorem implies that we would have better approximation of the external rays when R is large. However, note that this estimate does not count the rounding errors coming from Newton's method.

Proof. The equation $\phi_c(q_c^n(c)) = t$ is equivalent to $q_c^n(c) = \psi_c(t)$ where $\psi_c = \phi_c^{-1}$. Let us start with some calculations on ψ_c.

Lemma 4.2 [7] *For any $c \in \mathbb{C} - \mathbb{M}$, the map ϕ_c has the expansion near ∞ as follows:*

$$t \; = \; \phi_c(z) \; = \; z + \frac{c}{2z} - \frac{c(c-2)}{z^3} + O\left(\frac{1}{z^5}\right).$$

Moreover, we have

$$z \; = \; \psi_c(t) \; = \; t - \frac{c}{2t} + \frac{c(3c-8)}{4t^3} + O\left(\frac{1}{t^5}\right).$$

Sketch of the proof. Recall the fact that $\phi_c(z) = \lim_{n\to\infty} \{q_c^n(z)\}^{1/2^n}$, where $\left\{z^{2^n} + \cdots\right\}^{1/2^n} = z + O(1)$ ([CG]). Then it is not difficult to check $\phi_{n+1}(z) - \phi_n(z) = O(1/z^{2^{n+1}-1})$, and this implies that

$$\phi_c(z) \; = \; \phi_n(z) + O(1/z^{2^{n+1}-1}).$$

Now we have the expansion of ϕ_c above by an explicit calculation of $\phi_n(z) = \{q_c^n(z)\}^{1/2^n}$. The expansion of ψ_c follows by using $z = t - c/2z + \cdots$. ■

By this lemma we have

$$|(q_c^n(c) - t) - (q_c^n(c) - \psi_c(t))| \; \le \; \left|-\frac{c}{2t} + O\left(\frac{1}{t^3}\right)\right| \; \le \; \frac{M}{R}.$$

for some constant $M > 0$ independent of $|c| \le r$ and $R = |t| \gg 0$.

Now suppose that c is a root of $q_c^n(c) - t = 0$. We want to apply Rouchè's theorem, so that there exists $\hat{c}$ near c such that $q_{\hat{c}}^n(\hat{c}) - \psi_{\hat{c}}(t) = 0$. It is enough to show that there exists a circle $\{\hat{c} \in \mathbb{C} : |\hat{c} - c| = \rho\}$ with $\rho > 0$ given as in the estimates in the statement such that

$$|q_{\hat{c}}^n(\hat{c}) - t| \; = \; |q_{\hat{c}}^n(\hat{c}) - q_c^n(c)| \; > \; \frac{M}{R}$$

[7] This lemma is true for any $c \in \mathbb{C}$.

for all $\hat{c}$ on the circle. Let us consider the local behavior of the map $\hat{c} \mapsto q_{\hat{c}}^n(\hat{c})$ about c. Since we have

$$q_{\hat{c}}^n(\hat{c}) - q_c^n(c) \;=\; (q_c^n)'(c)(\hat{c}-c) + O(|\hat{c}-c|^2),$$

we need some estimate of $(q_c^n)'(c)$. By the equation $\phi_c(q_c^n(c)) = \{\Phi(c)\}^{2^n} = t$, we have

$$\begin{aligned}(q_c^n)'(c) &= \psi_c'(t) + \frac{\partial \psi_c}{\partial t}(t) \cdot 2^n \cdot \{\Phi(c)\}^{2^n-1} \cdot \Phi'(c) \\ &= \left(-\frac{1}{2t} + O(t^{-3})\right) + \left(1 + O(t^{-2})\right) \cdot 2^n \cdot \frac{t}{\Phi(c)} \cdot \Phi'(c)\end{aligned}$$

By applying the Cauchy integral formula to Φ^{-1}, we have

$$|\Phi'(c)| \;\geq\; \frac{|\Phi(c)|-1}{r}.$$

Since $|t| = |\Phi(c)|^{2^n} = R \gg 0$, it follows that

$$|(q_c^n)'(c)| \;\geq\; C_0 \cdot 2^n R^{1-1/2^n}(R^{1/2^n}-1)$$

for some constant $C_0 > 0$. In particular, the map $\hat{c} \mapsto q_{\hat{c}}^n(\hat{c})$ is locally univalent near c. More precesely, there exists a maximal disk B of radius $\delta = \delta(c)$ centered at c where this map is univalent.

By the Koebe distortion theorem (see [CG] for example), there exist uniform constants $C_1, C_2 > 0$ depending only on the value $|\hat{c}-c|/\delta$ such that

$$C_1|(q_c^n)'(c)||\hat{c}-c| \;\leq\; |q_{\hat{c}}^n(\hat{c}) - q_c^n(c)| \;\leq\; C_2|(q_c^n)'(c)||\hat{c}-c|$$

for $\hat{c} \in B$, and $C_1, C_2 \to 1$ as $|\hat{c}-c|/\delta \to 0$. Hence by the inequality on the left we can take $\rho = |\hat{c}-c|$ as in the first estimate of the statement in order to have $|q_{\hat{c}}^n(\hat{c}) - q_c^n(c)| \;>\; M/R$ when $R \gg 0$.

For the second estimate, recall that $|x|/2 \leq |e^x - 1| \leq 2|x|$ when $|x| \leq 1$. Now the estimate easily follows by setting $x := (\log R)/2^n$. ■

5 Exercises: Some possible improvements

Exercise 1. For calculation with less errors, we need to solve the equation $\phi_c(q_c^n(c)) = t$ more precisely. Now we improve the approximation of $\phi_c(z)$ to degree 3, and consider the equation

$$\phi_c(q_c^n(c)) \;\approx\; q_c^n(c) + \frac{c}{2q_c^n(c)} \;=\; t.$$

In this case, how can we estimate the relative error? Show that the Newton map is

$$N:\; c \;\mapsto\; c - \frac{2C_n^3 - 2tC_n^2 + cC_n}{2C_n^2 D_n + C_n - cD_n},$$

where $C_n = q_c^n(c),\; D_n = \{q_c^n(c)\}'$.

Exercise 2. Next let us solve the equation $q_c^n(c) = \phi_c^{-1}(t)$ by Newton's method.

(1) First show that $\phi_c^{-1}(t)$ can be expanded as

$$\phi_c^{-1}(t) \;=\; t - \frac{c}{2t} + \frac{c(3c-8)}{4t^3} + O\left(\frac{1}{t^5}\right)$$

when t is large enough.

(2) Show that the Newton map is

$$N : \; c \; \mapsto \; c - \frac{2tC_n - 2t^2 + c}{2tD_n - 1}$$

by using the approximation $q_c^n(c) = \phi_c^{-1}(t) \approx t - c/(2t)$.

(3) Compared with the iteration above, can which one have less errors?

References

[CG] L. Carleson and T. Gamelin. *Complex Dynamics* . Springer-Verlag, 1993.

[H] P. Henrici. *Elements of Numerical Analysis.* Wiley, 1964.