Tutte's polynomial for hypergraphs and polymatroids

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From matroids to polymatroids

Generalizing

$$\text{matroid} \mapsto \text{base polytope} \mapsto \text{Tutte polynomial } T(x,y)$$

$$\mapsto \begin{cases} T(x,1) \ (h\text{-vector}) \\ T(1,y), \end{cases}$$

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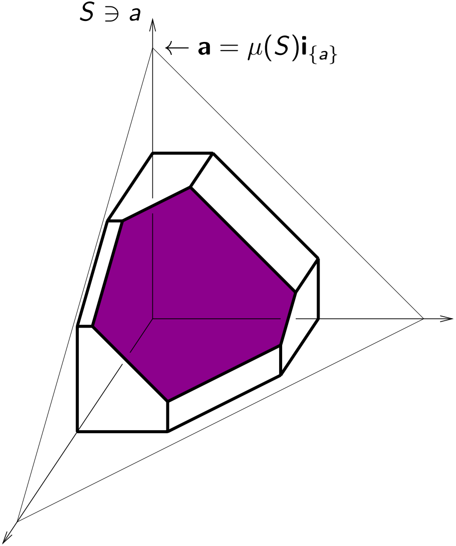
we define

$$\text{integer polymatroid} \mapsto \text{base polytope} \mapsto \begin{cases} \text{interior polynomial } I(\xi) \\ \text{exterior polynomial } X(\eta). \end{cases}$$

All coefficients will be non-negative integers.

Polymatroids



S: finite ground set.

$$P_\mu = \left\{ \mathbf{x} \in \mathbf{R}^S \;\middle|\; \begin{array}{l} \mathbf{x} \geq \mathbf{0}; \\ \mathbf{x} \cdot \mathbf{i}_U \leq \mu(U) \\ \text{for all } U \subset S \end{array} \right\},$$

where $\mu \colon \mathcal{P}(S) \to \mathbf{Z}$ is a submodular and non-decreasing set function.

Base polytope:

$$B_\mu = \{\, \mathbf{x} \in P_\mu \mid \mathbf{x} \cdot \mathbf{i}_S = \mu(S) \,\}.$$

Activities

We say that the base $\mathbf{x} \in B_\mu \cap \mathbf{Z}^S$ is such that a *transfer* is possible from $s_1 \in S$ to $s_2 \in S$ if by decreasing the s_1-component of $\mathbf{x}$ by 1 and increasing its s_2-component by 1, we get another base.

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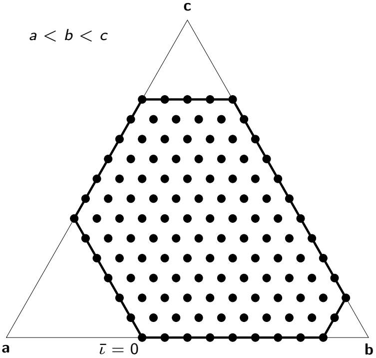
Order S arbitrarily.

Call an element $s \in S$ internally active with respect to the base $\mathbf{x} \in B_\mu \cap \mathbf{Z}^S$ if $\mathbf{x}$ is such that no transfer is possible from s to a smaller element of S.

We say that s is externally active with respect to $\mathbf{x}$ if it is such that no transfer is possible to s from a smaller element of S.

Interior and exterior polynomials



For $\mathbf{x} \in B_\mu \cap \mathbf{Z}^S$, let

$$\bar{\iota}(\mathbf{x}) = \# \left\{ s \in S \;\middle|\; \begin{array}{l} s \text{ is internally} \\ \text{inactive with} \\ \text{respect to } \mathbf{x} \end{array} \right\}$$

Define

$$I_\mu(\xi) = \sum_{\mathbf{x} \in B_\mu \cap \mathbf{Z}^S} \xi^{\bar{\iota}(\mathbf{x})}$$

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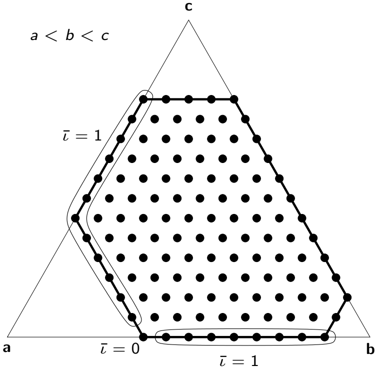

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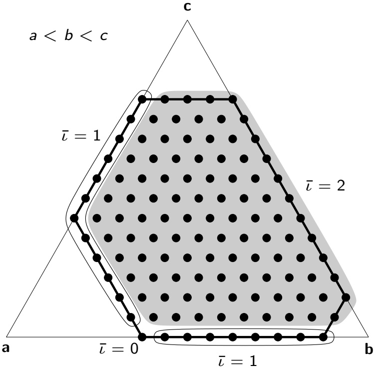

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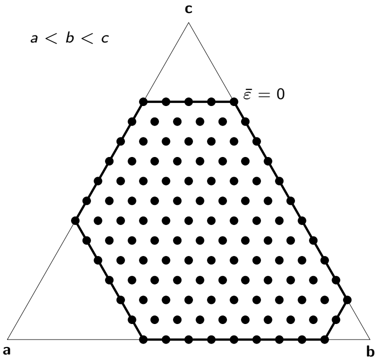

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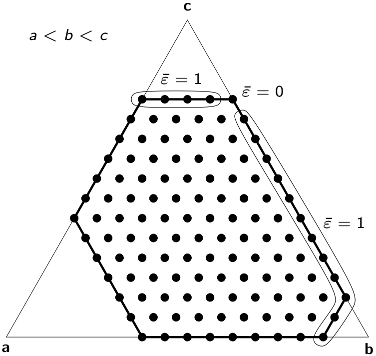
$$\bar{\varepsilon}(\mathbf{x}) = \#\left\{ s \in S \;\middle|\; \begin{array}{l} s \text{ is externally} \\ \text{inactive with} \\ \text{respect to } \mathbf{x} \end{array} \right\}.$$

Define

$$I_\mu(\xi) = \sum_{\mathbf{x} \in B_\mu \cap \mathbf{Z}^S} \xi^{\bar{\iota}(\mathbf{x})} \quad \text{and} \quad X_\mu(\eta) = \sum_{\mathbf{x} \in B_\mu \cap \mathbf{Z}^S} \eta^{\bar{\varepsilon}(\mathbf{x})}.$$



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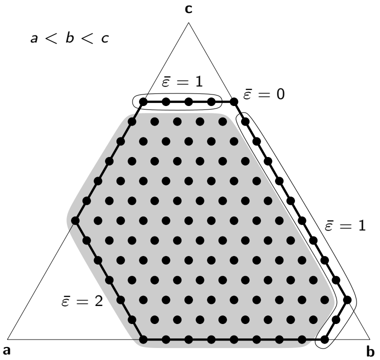

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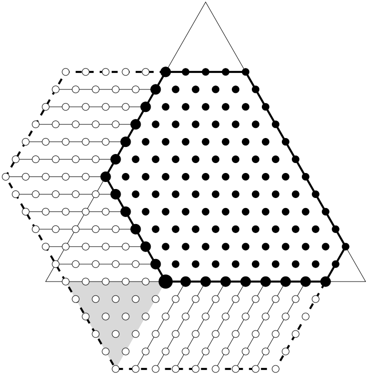
Order-independence

Theorem

I_μ and X_μ do not depend on the way S was ordered.



$I_\mu(\xi)$, in the basis $1, \xi, \xi^2, \ldots$, has the same coefficients as

$$\#\left((B_\mu + k\nabla) \cap \mathbf{Z}^S\right)$$

in the basis

$$\binom{k+|S|-1}{|S|-1}, \ldots, \binom{k+2}{2}, k+1, 1.$$

Here $\nabla = -\Delta_S$ is the inverted unit simplex.

X_μ has a similar relation to the Minkowski sum $B_\mu + k\Delta_S$.

Special case

Theorem

Let M be a rank r matroid on the ground set S with base polytope B_M and Tutte polynomial $T_M(x,y)$. Then the lattice point count

$$\#\left((B_M + k\nabla)\cap \mathbf{Z}^S\right)$$

is a polynomial function of k which, in the basis

$$\binom{k+|S|-1}{|S|-1},\ldots,\binom{k+2}{2},k+1,1, \tag{1}$$

has the same coefficients as $T_M(x,1)$ in the basis $x^r, x^{r-1},\ldots,x,1$.

Likewise $\#\left((B_M + k\Delta)\cap \mathbf{Z}^S\right)$, in the basis (1), has the same coefficients as $T_M(1,y)$ in the basis $y^{|S|-r}, y^{|S|-r-1},\ldots,y,1$.

From graphs to hypergraphs

Generalizing

$$\text{graph} \mapsto \text{cycle matroid} \mapsto \text{Tutte polynomial } T(x,y) \mapsto \begin{cases} T(x,1) \\ T(1,y), \end{cases}$$

we define

$$\text{hypergraph} \mapsto \textcolor{red}{\text{cycle polymatroid}} \mapsto \begin{cases} \text{interior polynomial } I(\xi) \\ \text{exterior polynomial } X(\eta). \end{cases}$$

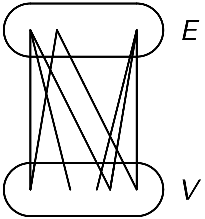
The independent sets of the cycle matroid are the cycle-free subgraphs.

Abstract duality

Let $\mathcal{H} = (V, E)$ be a hypergraph. The two-to-one correspondence

$$\mathcal{H} \mapsto \operatorname{Bip}\mathcal{H}$$

associates a bipartite graph to it. (We always assume $\operatorname{Bip}\mathcal{H}$ is connected.)



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associates a bipartite graph to it. (We always assume $\operatorname{Bip} \mathcal{H}$ is connected.) The other hypergraph with the same image is the *abstract dual*

$$\overline{\mathcal{H}} = (E, V).$$

Cycle polymatroid / Hypertree polytope

Given $\mathcal{H}$, the lattice points in its cycle polymatroid are vectors

$$\mathbf{f}\colon E \to \mathbf{Z} = \{\, 0, 1, 2, \dots \,\}$$

so that $\operatorname{Bip}\mathcal{H}$ has a cycle-free subgraph with valence $\mathbf{f}(e) + 1$ at every $e \in E$.

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The integer bases are valence distributions on E (minus $\mathbf{1}$) of spanning trees of $\operatorname{Bip}\mathcal{H}$. We call these *hypertrees* and refer to the base polytope as the *hypertree polytope* $B_{\mathcal{H}}$ of $\mathcal{H}$.

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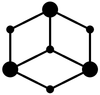
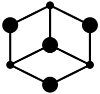
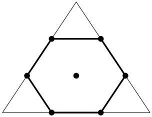
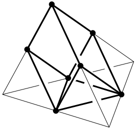
Example: for $\mathcal{H} =$  (three hyperedges)

and $\overline{\mathcal{H}} =$  (four hyperedges), we get

$B_{\mathcal{H}}$



$B_{\overline{\mathcal{H}}}$



Hypergraph polynomials

$I_{\mathcal{H}}$ and $X_{\mathcal{H}}$ are defined as the interior and exterior polynomial of the cycle polymatroid of $\mathcal{H}$.

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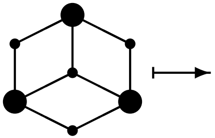

$$I = 1 + 3\xi + 3\xi^2$$

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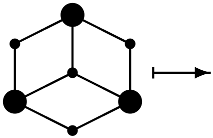

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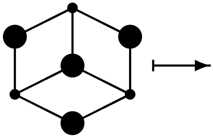

$$I = 1 + 3\xi + 3\xi^2$$

$$X = 1 + 2\eta + 3\eta^2 + \eta^3$$

Properties

- Both I and X have constant term 1.
- The linear coefficient in I is the fist Betti number (nullity) of Bip $\mathcal{H}$.

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  1. $I_{\mathcal{H}}(\xi) = I_{\mathcal{H}-e}(\xi) + \xi I_{\mathcal{H}/e}(\xi)$ and $X_{\mathcal{H}}(\eta) = \eta X_{\mathcal{H}-e}(\eta) + X_{\mathcal{H}/e}(\eta)$
  2. $I_{\overline{\mathcal{H}}}(\xi) = I_{\overline{\mathcal{H}-e}}(\xi) + \xi I_{\overline{\mathcal{H}/e}}(\xi)$.

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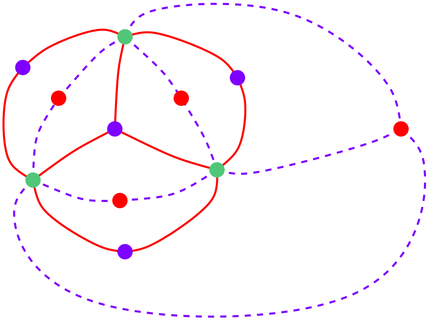
Theorem (A. Postnikov)

$B_{\mathcal{H}}$ and $B_{\overline{\mathcal{H}}}$ have the same number of lattice points. ($\Rightarrow I_{\mathcal{H}}(1) = I_{\overline{\mathcal{H}}}(1)$.)

Planar hypergraphs

We call $\mathcal{H}$ planar if $\operatorname{Bip}\mathcal{H}$ is planar.

Plane hypergraphs form dual pairs.



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Plane hypergraphs form dual pairs.

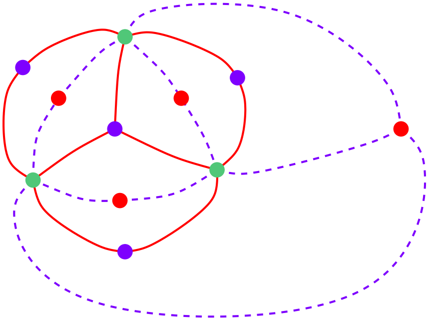
For such a pair $\mathcal{H}, \mathcal{H}^*$, we have

$$B_{\mathcal{H}^*} \cong -B_{\mathcal{H}}$$

and consequently,

$$I_{\mathcal{H}^*} = X_{\mathcal{H}} \quad \text{and} \quad X_{\mathcal{H}^*} = I_{\mathcal{H}}.$$

This generalizes
$T_{G^*}(x, y) = T_G(y, x)$.



Trinities

Applying both planar and abstract duality generates trinities. These are triangulations of the sphere.

Trinities contain three bipartite graphs and six hypergraphs with altogether three polynomials.

Tutte's Tree Trinity Theorem: The (classical) planar dual graphs of the three bipartite graphs are directed and have the same arborescence number.

This number is also the sum of the coefficients in all three polynomials.