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Meridian twisting of closed braids and the Homfly polynomial

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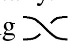
Abstract

Let β be a braid on n strands, with exponent sum w. Let Δ be the Garside half-twist braid. We prove that the coefficient of v^{w-n+1} in the Homfly polynomial of the closure of β agrees with $(-1)^{n-1}$ times the coefficient of v^{w+n^2-1} in the Homfly polynomial of the closure of $\beta\Delta^2$. This coincidence implies that the lower Morton–Franks–Williams estimate for the v–degree of the Homfly polynomial of $\widehat{\beta}$ is sharp if and only if the upper MFW estimate is sharp for the v–degree of the Homfly polynomial of $\widehat{\beta\Delta^2}$.

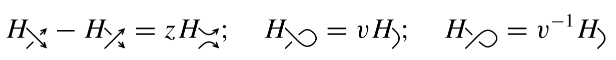
In this paper, parts of an old story are told again in a way that yields a surprising new result (Figure 2 shows a representative example). Indeed, the only items that could not have been written twenty years ago are some speculation about Khovanov–Rozansky homology and Remark 4·1, which do not belong to the proof, and the proof of Proposition 2·1. The latter is a well known fact and in particular it too can be established using skein theory. The new proof is included for completeness and because it is short.

The main theorem (Theorem 1·3) is stated in the introduction and proven in section 2. Section 3 contains an alternative argument suggested by the referee that puts our formula in the context of the Hecke algebra. We conclude the paper with several examples and remarks.

1. *Introduction*

Braids will be drawn horizontally. In particular, the standard generators σ_i of the braid group B_n [**2**] appear as a crossing  sandwiched between groups of $i-1$ and $n-i-1$ trivial strands, respectively. 1_n is the trivial braid on n strands. The usual closure of the braid β will be denoted by $\widehat{\beta}$. It gets its orientation from directing the strands of β from left to right.

The *framed Homfly polynomial* H_D is a two-variable, integer-coefficient Laurent polynomial in the indeterminates v and z. It is associated to any oriented link diagram D, and it is uniquely defined by the skein relations



and the requirement (normalization) that for the crossingless diagram of the unknot (with either orientation), $H_{\bigcirc}(v,z)=1$. H is invariant under regular isotopy of knot diagrams.

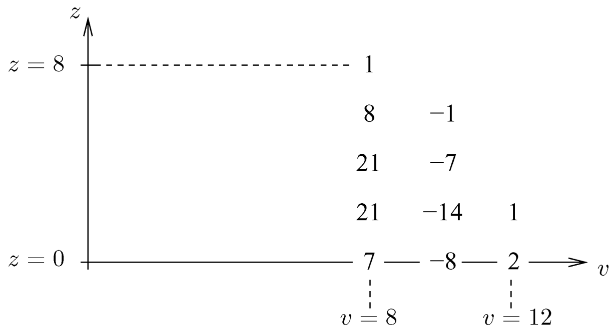


Fig. 1. The Homfly polynomial of the (3, 5) torus knot.

The Homfly polynomial itself is

$$P_D(v, z) = v^w H_D(v, z),$$

where w is the writhe of D. Unlike H, it is an oriented link invariant: $P(v, z)$ takes the same value for diagrams that represent isotopic oriented links.

We will record Homfly coefficients on the vz–plane. (We hope that our blurring of the distinction between the indeterminates and their exponents will not lead to confusion.) We will often speak about *columns* of the Homfly polynomial. These can be equivalently thought of as polynomials in z that appear as the coefficients of various powers of v.

Example 1·1. The Homfly polynomial of the torus knot $T(3, 5)$ is

$$\begin{aligned}P_{T(3,5)}(v, z) &= z^8v^8 + 8z^6v^8 - z^6v^{10} + 21z^4v^8 - 7z^4v^{10} + 21z^2v^8 - 14z^2v^{10} + z^2v^{12}\\ &\quad + 7v^8 - 8v^{10} + 2v^{12}\\ &= (z^8 + 8z^6 + 21z^4 + 21z^2 + 7)v^8 - (z^6 + 7z^4 + 14z^2 + 8)v^{10} + (z^2 + 2)v^{12},\end{aligned}$$

and we will write it as shown in Figure 1.

Next, recall the famous Morton–Franks–Williams inequality [**6**, **11**] which (put somewhat sloppily) says that

$$\text{braid index} \geqslant \text{number of non-zero columns in } P \tag{1·1}$$

for any oriented link type. (For knot theoretical definitions such as braid index, we refer the reader to [**4**].) In its standard proof, this is derived from the following pair of inequalities. Let β be a braid word on n strands, with exponent sum w. Then,

$$w - n + 1 \leqslant \text{lowest } v\text{–degree of } P_{\widehat{\beta}} \tag{1·2}$$

and

$$\text{highest } v\text{–degree of } P_{\widehat{\beta}} \leqslant w + n - 1. \tag{1·3}$$

We will refer to (1·2) as the *lower MFW estimate* and to (1·3) as the *upper MFW estimate*. Similarly, we might call $w - n + 1$ the *lower MFW bound* (for β) and $w + n - 1$ the *upper MFW bound*. Graphically, these inequalities mean that the left column of the Homfly polynomial $P_{\widehat{\beta}}$ is on or to the right of the vertical line $v = w - n + 1$ while the right column is on or to the left of $v = w + n - 1$.

Example 1·2. If we represent $T(3, 5)$ with the braid word

$\beta = (\sigma_1\sigma_2)^5 =$,

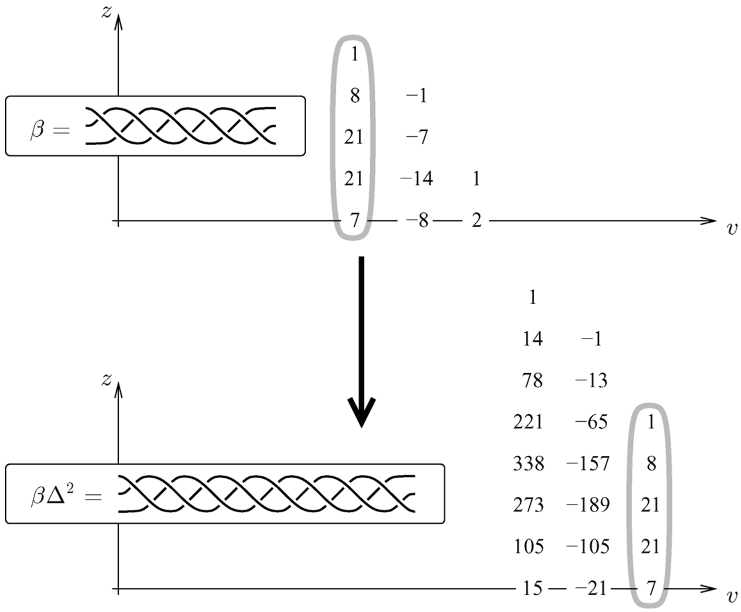


Fig. 2. The effect of adding a full twist on the Homfly polynomial.

then $n = 3$, $w = 10$, $w - n + 1 = 8$, $w + n - 1 = 12$, and we see that both the lower and the upper MFW estimates are sharp for this braid. Indeed, the Homfly polynomial of Example 1·1 has 3 columns.

We will denote the Garside braid (positive half twist) on n strands by Δ_n or simply by Δ. Then Δ^2 represents a positive full twist. (For example, $\Delta_3 =$  and $\Delta_3^2 =$ .) Since Δ^2 contains $n(n-1)$ positive crossings, the braid $\beta\Delta^2$ still has n strands but exponent sum $w + n(n-1)$. Thus,

$$\text{the upper MFW bound for } \beta\Delta^2 \text{ is } w + n(n-1) + n - 1 = w + n^2 - 1. \tag{1·4}$$

Now we are ready to state our main theorem.

THEOREM 1·3. *For any braid β on n strands, the lower MFW estimate is sharp if and only if the upper MFW estimate is sharp for the braid $\beta\Delta^2$. If this is the case, then*

$$\textit{left column of } P_{\widehat{\beta}} = (-1)^{n-1} \textit{ right column of } P_{\widehat{\beta\Delta^2}}. \tag{1·5}$$

Example 1·4. If we add a full twist to the braid of Example 1·2, we observe a change in the Homfly polynomial as shown in Figure 2 (cf. Corollary 1·9).

Somewhat more precisely, we can claim the following.

THEOREM 1·5. *For any braid β of n strands and exponent sum w,*

$$\textit{the coefficient of } v^{w-n+1} \textit{ in } P_{\widehat{\beta}} = (-1)^{n-1} \cdot \textit{the coefficient of } v^{w+n^2-1} \textit{ in } P_{\widehat{\beta\Delta^2}}. \tag{1·6}$$

From this latter statement, Theorem 1·3 follows immediately. Indeed, comparing Theorem 1·5 with (1·2), (1·3) and (1·4), we see that (1·6) either says that $0 = 0$, or the more meaningful formula (1·5), depending on whether or not the sharpness condition is met.

We give yet another formulation of our result which fits best with the methods of both proofs we will present. It is important to stress that whenever the framed Homfly polynomial of a closed braid is taken, $\widehat{\beta}$ means the diagram that is the union of a diagram of β and n *disjoint*, *simple* curves *above* it to form its closure.

THEOREM 1·6. *For any braid β of n strands and exponent sum w,*

$$\text{the coefficient of } v^{-n+1} \text{ in } H_{\widehat{\beta}} = (-1)^{n-1} \cdot \text{the coefficient of } v^{n-1} \text{ in } H_{\widehat{\beta\Delta^2}}. \qquad (1\cdot7)$$

It is easy to see that (1·6) and (1·7) are equivalent. The left-hand side of (1·7) agrees with the coefficient of $v^w v^{-n+1}$ in $v^w H_{\widehat{\beta}} = P_{\widehat{\beta}}$, which is the left-hand side of (1·6). Similarly, the right-hand side of (1·7) equals (plus or minus) the coefficient of $v^{w+n(n-1)}v^{n-1}$ in $v^{w+n(n-1)} H_{\widehat{\beta\Delta^2}} = P_{\widehat{\beta\Delta^2}}$, i.e. the right-hand sides agree, too.

By reading our formulae from right to left, we get versions of our claims for the case of a full negative twist. (Equally trivial proofs can be derived from the fact that the Homfly polynomials of an oriented link and its mirror image are related by the change of variable $v \mapsto -v^{-1}$.) In particular, corresponding to Theorem 1·3, we have

COROLLARY 1·7. *For any braid β on n strands, the upper MFW estimate is sharp if and only if the lower MFW estimate is sharp for the braid $\beta\Delta^{-2}$. If this is the case, then the right column of $P_{\widehat{\beta}}$ coincides with $(-1)^{n-1}$ times the left column of $P_{\widehat{\beta\Delta^{-2}}}$.*

Example 1·8. One recognizes the right column in Figure 1, $z^2 + 2$, as the left column in the Homfly polynomial, $z^2v^2 + 2v^2 - v^4$, of the trefoil knot $T(3, 2)$.

The two equivalent sharpness conditions of Theorem 1·3 are both well known to hold for positive braids [**6**]. (The sharpness of (1·2) for positive braids is easier to see (cf. the footnote to Remark 4·1.) With that, we can use Theorem 1·3 to give a new proof of Franks and Williams's claim that for positive braids with a full twist, (1·1) is sharp, and therefore they realize the braid index of their closure. This argument however would not be very different from the original one.) Thus we have:

COROLLARY 1·9. *For any positive braid β on n strands, the left column of its Homfly polynomial agrees with $(-1)^{n-1}$ times the right column of the Homfly polynomial of $\beta\Delta^2$.*

Note however that the Morton–Franks–Williams inequalities are sharp for many non-braid-positive knots, too. Up to 10 crossings, there are only five knots that do *not* possess braid representations with a sharp (lower) MFW estimate [**8**]. Thus, (1·6) is informative for many non-positive braids as well (cf. Example 4·2).

For all three of the inequalities, (1·1), (1·2) and (1·3), that we quoted from [**6**] and [**11**], it makes sense to say that they are *sharp for a braid* (like we did throughout most of this introduction) and also to discuss (as in the previous paragraph) whether they are *sharp for a link*. The latter means that a given oriented link type has a braid representative that turns the inequality into an equality.

We are unaware of a version of the 'twist phenomenon' for the Kauffman polynomial. Likewise, there do not seem to be noteworthy consequences of Theorem 1·5 for the various other knot polynomials that are derived from the Homfly polynomial. (Let us mention though Yokota's work [**15**] concerning the Jones polynomial here.) However we do anticipate there to be a twisting formula for the generalization (categorification) of the Homfly polynomial known as Khovanov–Rozansky homology [**10**] (cf. Example 4·2). For instance, the MFW inequalities extend to the categorification [**5**, **14**].

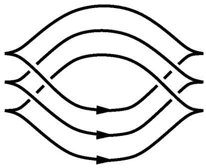


Fig. 3. A Legendrian representative of the link $\widehat{\Delta^{-2}}$.

2. *Proof of the main theorem*

We will give a detailed proof of Theorem 1·6. It has already been explained how our other claims follow from it. The proof is based on skein theory. In Section 3, we shall sketch a more algebra- and representation theory-oriented version of essentially the same argument.

2·1. *Review of known techniques*

First, note that for the framed Homfly polynomial the MFW estimates take the form

$$-n+1 \leqslant \text{ any exponent of } v \text{ in } H_{\widehat{\beta}}(v,z) \leqslant n-1. \tag{2·1}$$

We will need the following fact about the link $\widehat{\Delta_n^2}$ (formed by n fibres of the Hopf fibration). It is a tiny portion of a known formula.

PROPOSITION 2·1. *The highest v-exponent in $H_{\widehat{\Delta_n^2}}(v,z)$ is $n-1$ and the single term in which it occurs is $(-1)^{n-1}z^{1-n}v^{n-1}$.*

Proof. This statement can be proven by induction based on the formula

$$\Delta_n^2 = (\sigma_{n-1}\cdots\sigma_1)(\sigma_1\cdots\sigma_{n-1})\Delta_{n-1}^2,$$

skein relations, inequality (1·3), and repeated use of arguments similar to Lemma 2·2. It can also be inferred from [**8**, section 9]. We will present a third proof, based on a different set of notions. All relevant definitions can be found in [**13**].

Figure 3 shows the front diagram of a Legendrian representative of the link $\widehat{\Delta_n^{-2}}$. The number of its left cusps is $c=n$ and it has Thurston–Bennequin number $-n^2$. As all crossings are negative, the only oriented ruling of this diagram is the one without switches. Rutherford associates to it the number $j = \text{number of switches} - c + 1 = -n+1$.

Now [**13**, Theorem 4·3] says that the so-called oriented ruling polynomial, in this case $\sum z^j = z^{-n+1}$, agrees with the coefficient of $a^{c-1} = v^{1-c} = v^{1-n}$ in the framed Homfly polynomial of the diagram obtained by smoothing the cusps. Furthermore, the Homfly estimate on the Thurston–Bennequin number (which is a consequence of (1·2) and Bennequin's results [**1**] on transverse knots via the 'transverse push-off' trick) implies that this $(1-n)$ is the lowest exponent of v that occurs in $H_{\widehat{\Delta^{-2}}}(v,z)$. Finally, to obtain information on $\widehat{\Delta^2}$, we substitute $-v^{-1}$ for v. We find that the highest exponent of v in $H_{\widehat{\Delta^2}}(v,z)$ is indeed $n-1$ and that it occurs with the coefficient $(-1)^{1-n}z^{-n+1}$.

Note that since the Homfly polynomial of the n-component unlink is $((v^{-1}-v)/z)^{n-1}$, which has left column (coefficient of v^{1-n}) z^{1-n}, Proposition 2·1 already confirms Theorem 1·6 for the special case $\beta = 1_n$.

Next, recall that for any braid, a *computation tree* can be built using the four types of moves discussed below [**6**]. Each vertex of the tree carries a *label*, which either means a braid (word) or some polynomial associated to the (closure of) the braid. At first, labels will be interpreted as braids. The moves are as follows:

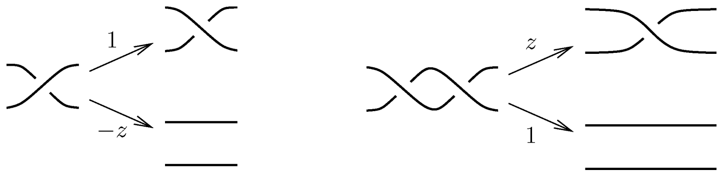


Fig. 4. Conway splits.

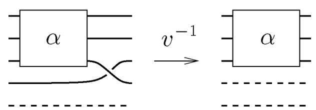


Fig. 5. Positive Markov destabilization changes H by a factor of v.

(i) isotopy (braid group relations);
(ii) conjugation: $\beta_1\beta_2 \mapsto \beta_2\beta_1$;
(iii) positive Markov destabilization: $\alpha\sigma_i \in B_{i+1}$ becomes $\alpha \in B_i$;
(iv) Two types of Conway splits, as shown in Figure 4.

The tree is rooted and oriented away from the root. The *terminal nodes* of the tree are labelled with unlinks on various numbers of strands. To be precise, we should note that isotopy and conjugation take place 'within the vertices' of the tree; that is to say, by labels we really mean certain equivalence classes of braids. This does not lead to ambiguity because these two operations on braids correspond to regular isotopies of the closures. A Markov destabilization also has a very controlled effect, but for us it will be convenient to treat it already as an edge of the tree (the tail of such an edge is a 2-valent vertex). Clearly, Conway splits are mostly responsible for the structure of the graph.

A computation tree can be used to evaluate various knot polynomials. In particular, to obtain the *framed* Homfly polynomial H of the closure of the (label of the) root, we proceed as follows. Label edges resulting from a Conway split as already shown in Figure 4. Any edge representing a Markov destabilization carries the label v^{-1} (see Figure 5). If a terminal node x is labelled with 1_k, then also label it with the polynomial $((v^{-1} - v)/z)^{k-1}$. Then, this terminal node contributes

$$\left(\frac{v^{-1} - v}{z}\right)^{k-1} h(x), \tag{2·2}$$

where $h(x)$ is the product of the edge labels that appear along the path that connects x to the root. Finally, the framed Homfly polynomial of the root is the sum of these contributions from all terminal nodes.

It is not essential to insist on trivial braids as terminal nodes. The point is that we must know what the framed Homfly polynomials of the labels of the terminal nodes are. Also, note that the subtree generated by any vertex of a computation tree is a computation tree for (the label of) that vertex.

2·2. *Proof of Theorem* 1·6

Starting from a computation tree Γ for the braid β, we build a computation tree $\widetilde{\Gamma}$ for $\beta\Delta^2$. Of course, we will try to imitate Γ as much as possible. In particular, we will attempt to delay altering the full twist in the braid word until after the crossings of β are all removed. (To see to what extent this is possible, we will have to analyse the four basic moves.) Doing so, we will be able to realize Γ as a subtree of $\widetilde{\Gamma}$ and to read off our result.

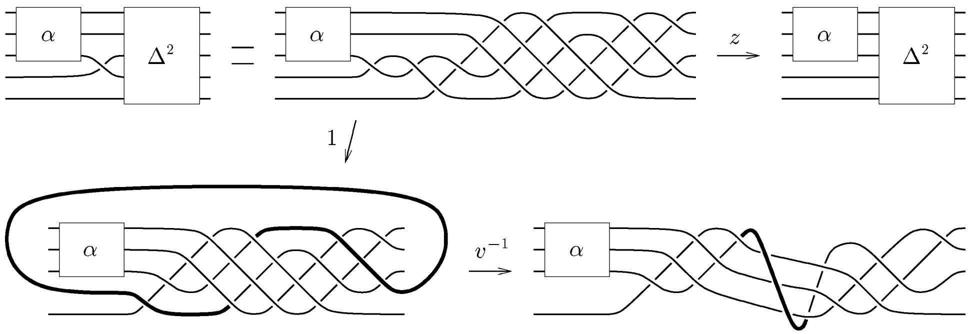


Fig. 6. The Conway split (and consequent destabilization) in $\widetilde{\Gamma}$ that replaces the Markov destabilization in Γ.

An important difference between Γ and $\widetilde{\Gamma}$ is that whenever a Markov destabilization in Γ reduces the number of strands, the corresponding vertices of $\widetilde{\Gamma}$ will still be labelled with braids on n strands. (This is simply because as long as Δ_n^2 is in the braid word, we need n strands.) One may think that the reduced strands live on as trivial "ghost strands" shown as dotted lines in Figure 5.

Two of the four basic steps, namely isotopy and Conway splitting, are completely local and thus they can be carried out unchanged when Δ^2 is attached to the end of β. Conjugation is easy too, since Δ^2 belongs to the centre of B_n. Thus, the conjugation move

$$\beta_1\beta_2 \mapsto \beta_2\beta_1 \quad \text{in } \Gamma$$

can be replaced by an isotopy followed by a conjugation

$$\beta_1\beta_2\Delta^2 \mapsto \beta_1\Delta^2\beta_2 \mapsto \beta_2\beta_1\Delta^2 \quad \text{in } \widetilde{\Gamma}. \tag{2·3}$$

Markov destabilization however requires more work. If a step $\alpha\sigma_i \mapsto \alpha$, as shown in Figure 5, occurs in the computation tree Γ, then we will realize it in $\widetilde{\Gamma}$ as part of a Conway split; see Figure 6. Of course this gives rise to a new, 'unnecessary' subtree of $\widetilde{\Gamma}$ that does not have a counterpart in Γ.

A precise description is as follows. Δ can be represented by positive braid words that either start or end with arbitrarily chosen braid group generators [**2**, lemma 2·4·1]. Thus we may write $\Delta^2 = \sigma_i\Lambda\sigma_i$ and hence $\alpha\sigma_i\Delta^2 = \alpha\sigma_i^2\Lambda\sigma_i$. We perform the Conway split at σ_i^2. One of the two resulting braids is $\alpha\sigma_i\Lambda\sigma_i = \alpha\Delta^2$, as desired. Note that contrary to Γ, in $\widetilde{\Gamma}$ the number of strands did not decrease.

The other edge of the Conway split points to the 'side-product' word $\alpha\Lambda\sigma_i$.

LEMMA 2·2. *The closure of the braid $\alpha\Lambda\sigma_i$ is also the closure of a braid on $n-1$ strands.*

Proof. We explain the move pictured at the bottom of Figure 6: the thick part of the left hand side diagram is pulled tight to appear as on the right. The braid $\Lambda\sigma_i$ is almost a full twist. Its strands can be bundled together into three groups: (i) the $i-1$ strands whose endpoints are on the top; (ii) the 2 strands right below them; (iii) the remaining $n-i-1$ strands. In other words, we think of $\Lambda\sigma_i$ as the union of three smaller braids. (In Figure 6, $n=5$ and $i=3$, hence we are talking about 2, 2 and 1 strands, respectively.) One by one, these are isotopic to (i) a full twist on $i-1$ strands; (ii) a single crossing on two strands, represented by σ_i in the word; (iii) a full twist on $n-i-1$ strands. These three smaller braids in turn are braided together to form $\Lambda\sigma_i$ in the fashion of Δ_3^2.

This time, we consider the pattern Δ_3^2 as the braid word $\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2\sigma_1 =$ . Note that its middle strand crosses under only twice, at the third and fourth crossings. Thus the part of the diagram that is thickened in the lower left of Figure 6 contains no undercrossings. Hence it can be lifted and rearranged as shown in the lower right.

LEMMA 2·3. *Any "extra branch" of $\widetilde{\Gamma}$ created in our process (starting as shown in the lower part of Figure* 6*) does not contribute to the coefficient of v^{n-1} in $H_{\widehat{\beta\Delta^2}}$.*

Proof. Lemma 2·2 and Markov's theorem [**2**] imply that isotopy, conjugation, and a single Markov destabilization reduce $\alpha\Lambda\sigma_i$ to a braid word $\eta \in B_{n-1}$. Now by (2·1), any exponent of v in $H_{\widehat{\eta}}(v, z)$ is $n-2$ or less. The labels v^{-1} along the path from η to the root $\beta\Delta^2$ can only reduce that exponent.

We shall now concentrate on the part of $\widetilde{\Gamma}$ that is isomorphic to Γ. Note that at the terminal nodes, where the trivial braids of Γ used to be, now there are copies of Δ^2.

LEMMA 2·4. *The contribution of a terminal node $\widetilde{x}$ of $\widetilde{\Gamma}$ to the coefficient of v^{n-1} in $H_{\widehat{\beta\Delta^2}}$ agrees (up to sign) with the contribution of the corresponding terminal node x of Γ to the coefficient of v^{1-n} in $H_{\widehat{\beta}}$. (In fact, both contributions are a single power of z.)*

Proof. Let us assume that x is labelled with 1_k. To arrive (in Γ) from β to 1_k, there had to be exactly $n-k$ Markov destabilizations. Therefore, $h(x) = \pm z^m v^{k-n}$, where m is a natural number. Thus, the coefficient of v^{1-n} in (2·2) becomes $\pm z^{m-k+1}$.

Now in $\widetilde{\Gamma}$, all $n-k$ of those v^{-1} labels have been changed to z (compare Figures 5 and 6), while other labels along the path remained the same. Hence $h(\widetilde{x}) = \pm z^{m+n-k}$. By Proposition 2·1, we see that the coefficient of v^{n-1} in the contribution of $\widetilde{x}$ is $\pm z^{m+n-k} \cdot (-1)^{n-1}z^{1-n} = \pm(-1)^{n-1}z^{m-k+1}$, as desired.

Put together, Lemmas 2·3 and 2·4 conclude the proof of Theorem 1·6.

3. *The Hecke algebra*

The proof presented in the previous section was largely made possible by the simple geometric observation that Δ_n^2 commutes with any other element of B_n (cf. (2·3)), the reduction in strand number observed in Figure 6, and the MFW inequalities themselves. In this section, we outline a second argument that gives Theorem 1·6. In it, similar simple principles will be at play, but instead of computation trees, we will rely on the Hecke algebra $\mathcal{H}_n(z)$. It is of course the central object in Jones's seminal paper [**8**]; see also [**12**] for a concise survey. We thank the referee for suggesting this alternative line of proof.

As an algebra, $\mathcal{H}_n(z)$ is generated by $\sigma_1, \ldots, \sigma_{n-1}$. In addition to the standard relations of B_n (that is, $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i-j| \geqslant 2$ and $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for all i) we impose

$$\sigma_i - \sigma_i^{-1} = z \text{ for all } i. \tag{3·1}$$

The key ingredient in the Homfly polynomial is Ocneanu's trace functional $\mathrm{Tr}\colon \mathcal{H}_n(z) \to \mathbf{Z}[z, T]$. For $\beta \in B_n$, take its natural representation $\beta \in \mathcal{H}_n(z)$ (a common abuse of notation) and its trace, substitute $T = z/(1-v^2)$ in it, and normalize suitably. The result,

$$H(\beta) = \left(\frac{v^{-1} - v}{z}\right)^{n-1} \cdot \mathrm{Tr}(\beta)\Big|_{T=\frac{z}{1-v^2}}, \tag{3·2}$$

is the framed Homfly polynomial of $\widehat{\beta}$.

As a $\mathbf{Z}[z]$–module, $\mathcal{H}_n(z)$ is free of rank $n!$. We will need two of its well known bases: the set of positive permutation braids, $\mathbf{\Omega} = \{\omega_\pi\}_{\pi\in S_n}$, and that of negative permutation braids, $\mathbf{N} = \{\nu_\pi\}_{\pi\in S_n}$. Now, β can be expressed in these bases with coefficients in $\mathbf{Z}[z]$:

$$\beta = \sum_{\pi\in S_n} a_\pi(z)\omega_\pi = \sum_{\pi\in S_n} b_\pi(z)\nu_\pi. \tag{3·3}$$

PROPOSITION 3·1. *If δ is the underlying permutation of Δ (and of Δ^{-1}), then $a_\delta = b_\delta$.*

Proof. This is because (positive and negative) permutation braids have standard (positive and negative) braid words representing them and among those, $\omega_\delta = \Delta$ and $\nu_\delta = \Delta^{-1}$, respectively, have the unique longest words (of exponent sum $\binom{n}{2}$). When we convert from the ω_π to the ν_π, we may do so by substituting $\sigma_i = \sigma_i^{-1} + z$ for each letter, multiplying out and then further reducing each word using $\sigma_i^{-2} = 1 - z\sigma_i^{-1}$ (cf. (3·1)). Clearly, ν_δ only appears when we convert ω_δ, and with the same coefficient.

Of course, $\nu_{\mathrm{id}} = \omega_{\mathrm{id}} = 1_n$. Note that as sets, $\{\,\nu_\pi\Delta\,\} = \{\,\omega_\pi\,\}$ and $\{\,\omega_\pi\Delta^{-1}\,\} = \{\,\nu_\pi\,\}$. In particular, $\nu_\delta\Delta = \omega_{\mathrm{id}}$ and $\omega_\delta\Delta^{-1} = \nu_{\mathrm{id}}$. Thus, when we use β written in $\mathbf{N}$ to expand the braid $\beta\Delta$ in $\mathbf{\Omega}$, the polynomial b_δ resurfaces as the coefficient of ω_{id}. Similarly, the same polynomial a_δ becomes the coefficient of ν_{id} when $\beta\Delta^{-1}$ is written in terms of $\mathbf{N}$.

It turns out that this is exactly the coincidence noted in Theorem 1·6. Other than the fact that $\beta\Delta^{-1}$ and $\beta\Delta$ differ by a full twist, we need one more observation for this. Going back to our general braid β, we have

PROPOSITION 3·2. *The polynomial $a_{\mathrm{id}}(z)$ equals $(-z)^{n-1}$ times the coefficient of v^{n-1} in $H(\beta)$, and $b_{\mathrm{id}}(z)$ is z^{n-1} times the coefficient of v^{1-n}.*

Proof. Along with being linear, conjugation-invariant, and normalized at $\mathrm{Tr}(1_n) = 1$, the key property of the trace is its behaviour under Markov destabilization: $\mathrm{Tr}(x\sigma_{n-1}) = T\,\mathrm{Tr}(x)$ for all $x \in \mathcal{H}_{n-1}(z)$. Similarly, $\mathrm{Tr}(x\sigma_{n-1}^{-1}) = (T - z)\mathrm{Tr}(x)$.

Also, note that other than 1_n, any permutation braid admits a Markov destabilization. Hence, as we substitute the expressions of (3·3) in (3·2) and distribute, each term contributed by non-trivial positive (resp. negative) permutation braids contains at least the first power of $T = zv^{-1}/(v^{-1} - v)$ (resp. $T - z = zv/(v^{-1} - v)$). Comparing these with the normalizing factor $((v^{-1} - v)/z)^{n-1}$ makes the result apparent.

Remark 3·3. In light of the last Proposition, we may say that the MFW estimate (1·2) (resp. (1·3)) is not sharp for the braid β if and only if β is "orthogonal" to the unit braid as an element of the basis $\mathbf{N}$ (resp. $\mathbf{\Omega}$) of the Hecke algebra.

4. *Remarks and examples*

There are several indications that coefficients in the left and right columns of the Homfly polynomial are more geometrically significant than others. For starters, these numbers persist under certain standard changes of variables and/or normalizing conditions. For example, if we require $H'_{\bigcirc} = (v^{-1} - v)/z$ instead of $H_{\bigcirc} = 1$, then the left and right columns, up to sign and a downward shift, stay the same. The numbers (up to sign) do not change if we use $a = v^{-1}$, $l = -\sqrt{-1}\cdot v^{-1}$, $m = \sqrt{-1}\cdot z$ etc. either.

The next remark explains why the author examined these columns in the first place. The pattern of Theorem 1·3 emerged while using Knotscape [**7**] to browse through many examples.

Remark 4·1. Rutherford [**13**] showed that if the oriented link type K contains Legendrian representatives with sufficiently high Thurston–Bennequin number, then the coefficients in the left column of the Homfly polynomial $P_K(v, z)$ represent numbers of so-called oriented rulings (of various genera) of these Legendrian links. Similarly, the right column may speak of oriented rulings of the mirror of K.

Theorems 1·3 and 1·5 however are not about oriented rulings. In particular, the method of the proof of Proposition 2·1 does not extend to the general case. For instance, if β is a positive braid (as in Example 1·4) then the left-hand side of (1·5) always contains counts of rulings[1][**9**], whereas the right-hand side almost never does.

There is also the intriguing possibility that our results could be used to verify Jones's conjecture [**8**] that the writhe of a braid representative with minimum strand number is an oriented link invariant. Assume that the braids β_1 and β_2 are both on n strands and they have isotopic closures but their exponent sums disagree, say $w_1 < w_2$. Suppose that the lower MFW bound $w_2 - n + 1$ from β_2 is sharp. Then by (1·5), $P_{\widehat{\beta_2\Delta^2}}(v, z)$ contains non-zero coefficients at $v = w_2 + n^2 - 1$, whereas by (1·6), $P_{\widehat{\beta_1\Delta^2}}$ vanishes at $v = w_1 + n^2 - 1$ and above. This is a contradiction if we also hypothesize that the closures of $\beta_1\Delta^2$ and $\beta_2\Delta^2$ are isotopic. There are too many ifs here, however. The last assumption certainly fails, as demonstrated in the next example. (Such counterexamples do not seem to be too common among small knots.)

Example 4·2. The 9-crossing mirrored alternating knots 9^*_{27}, 9^*_{30} and 9^*_{33} (Rolfsen's numbering) each have 4-braid representatives that make the MFW inequality (1·1) sharp. For example, the braids listed in KnotInfo [**3**]

$$\sigma_1^{-2}\sigma_2\sigma_1^{-1}\sigma_2^2\sigma_3^{-1}\sigma_2\sigma_3^{-1},\ \sigma_1^{-2}\sigma_2^2\sigma_1^{-1}\sigma_2\sigma_3^{-1}\sigma_2\sigma_3^{-1},\ \text{and}\ \sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2^2\sigma_1^{-1}\sigma_3^{-1}\sigma_2\sigma_3^{-1},$$

respectively have this property. Adding the full twist Δ_4^2 to these braids, in each case we obtain a braid representative of the same knot $12n_{187}$ (from the Hoste–Thistlethwaite table). (In other words, $12n^*_{187}$ has three 4-braid representatives so that adding a full positive twist to each, the three different knots 9_{27}, 9_{30}, and 9_{33} are obtained. Let us also mention that these braid representatives of $12n_{187}$ are mutually non-isotopic within the solid torus which is the complement of the braid axis.)

9^*_{27} is a slice knot while 9^*_{30} and 9^*_{33} are not. But even though they represent different concordance classes, all three have signature 0. Their Homfly polynomials are distinct, but each has a left column (coefficient of v^{-4}) of $-z^2-1$. Of course, the right column (coefficient of v^{14}) in the Homfly polynomial of $12n_{187}$ is $z^2 + 1$.

To further underline that Theorem 1·3 is more about braids than knots, we end the paper with the following extension of Example 1·4.

Example 4·3. We examine the effect of changing the braid $(\sigma_1\sigma_2)^5$ by positive and negative Markov stabilizations. Let us denote the resulting four-strand braids by β and β', respectively. Of course, the isotopy type of the closure of both is still the (3, 5) torus knot, but the lower MFW inequality (1·2) remains sharp only in the first case. See Figure 7 for the Homfly polynomials after adding a full twist to each braid. Note how (1·5) fails to work for β'. This is because in this case, (1·6) states that the columns of zeros indicated in Figure 7 agree. (In fact, the closure of $(\sigma_1\sigma_2)^5\sigma_3^{-1}\Delta_4^2$ is the torus knot $T(3, 10)$.)

[1] From this and [**13**, Theorem 4·3], the sharpness of (1·2) for positive braids follows.

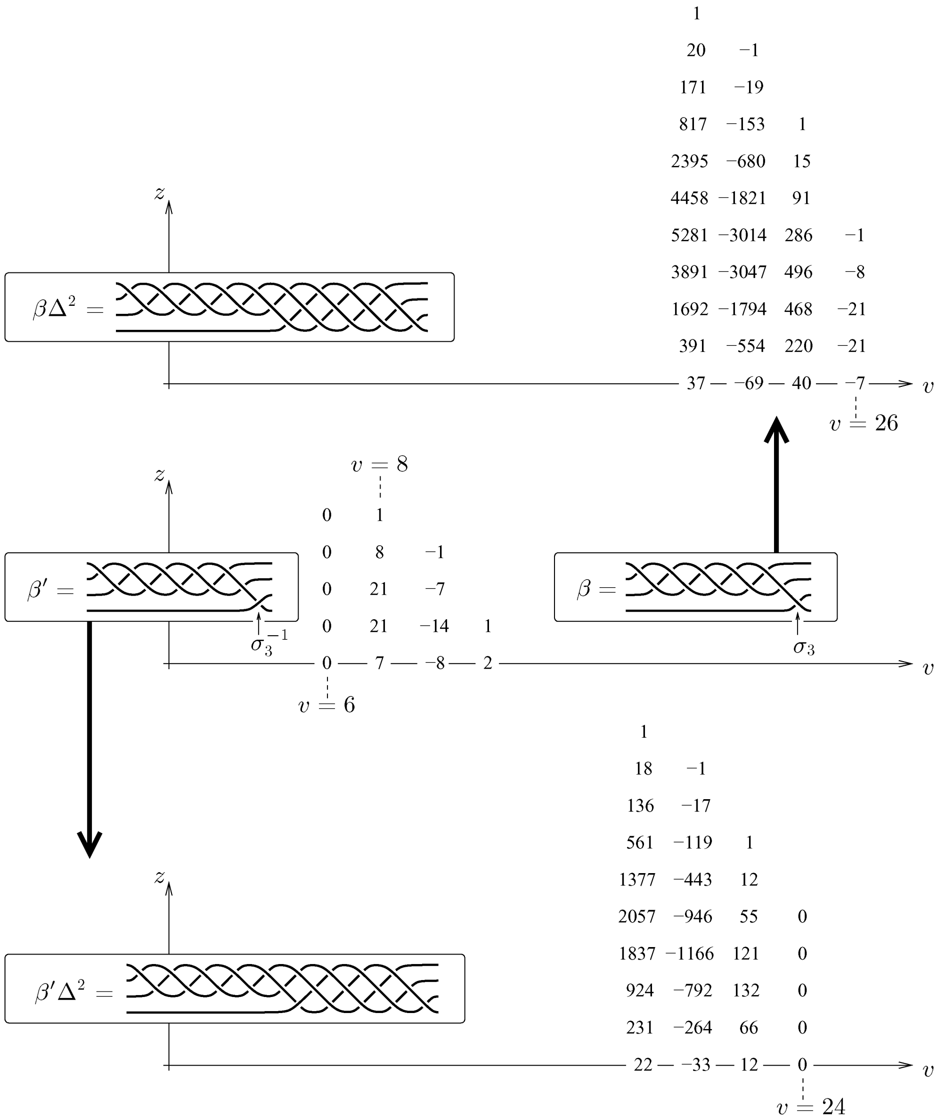


Fig. 7. Stabilizing a braid in two different ways before adding a full twist.

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