Legendrian knots and exact Lagrangian cobordisms

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Overview of contact homology

Legendrian knots

Combinatorial relative contact homology

Lagrangian cobordisms

Credits

Everything in this talk is joint work with Tobias Ekholm and Ko Honda.

Goals

1. Build Lagrangian surfaces with a given boundary condition.
2. Distinguish them by the holomorphic curves whose boundaries lie on the Lagrangians.
3. Make this all readily computable.

Context

Homology theories where the differential is defined by a count of holomorphic curves (so called "Floer theories") are hard to work with in practice.

There is much effort to reduce the analysis in these theories down to combinatorics.

Eliashberg and Hofer's *symplectic field theory (SFT)* is an instance of Floer theory. It associates invariants to contact manifolds and to their Legendrian submanifolds.

Chekanov provided the combinatorial formulation in a very special (but also the most classical) case: contact homology of Legendrian knots in $\mathbf{R}^3$ (the Darboux ball).

Cobordisms

We will focus on a certain piece in the "SFT package," namely the idea that cobordisms should induce maps between the homologies associated to their ends.

Cobordisms between contact manifolds will be symplectic and cobordisms between Legendrians will be (exact) Lagrangian.

For certain cobordisms, we will explicitly write down the associated map.

Contact geometry

A *contact manifold* is a smooth $(2n+1)$–dimensional manifold M along with a maximally non-integrable field ξ of hyperplanes.

In other words, if ξ is given (locally) as the kernel of the 1–form α, then $\alpha \wedge (\mathrm{d}\alpha)^n$ is a volume form.

Note that for the same ξ to be a foliation by $2n$–dimensional submanifolds, the condition would be that $\alpha \wedge (\mathrm{d}\alpha)^n$ is identically zero.

Indeed, maximal non-integrability implies that the largest possible dimension of a submanifold $L \subset M$ that is everywhere tangent to ξ is $\dim L = n$. Such submanifolds are called *Legendrian*.

Symplectic field theory (SFT)

SFT is an invariant of contact manifolds up to contactomorphism (absolute version) or of Legendrian submanifolds within contact manifolds up to isotopy through Legendrians (relative version).

By restricting the types of holomorphic curves considered, we arrive at the "sub-theories" known as (absolute or relative) *contact homology*.

Reeb trajectories

Let the contact structure ξ be globally defined as $\xi = \ker \alpha$. Then, the *Reeb vector field* R_α is determined by the conditions

$$\mathrm{d}\alpha(R_\alpha, \cdot\,) = 0 \text{ and } \alpha(R_\alpha) = 1.$$

SFT is defined in terms of integral curves of R_α. However the end result does not depend on α, only on its kernel ξ.

The generators of SFT are the closed Reeb orbits in the absolute case and the so-called *Reeb chords* in the relative case.
Reeb chord: integral trajectory of R_α which starts and ends on the Legendrian L.

For contact homology, the chain complex is a polynomial algebra freely generated by these Reeb trajectories. In the relative case, we consider *non-commutative* polynomials in the Reeb chords. For coefficients, today we will only use $\mathbf{Z}/2\mathbf{Z}$.

A class of holomorphic curves

The holomorphic curves we need live in the *symplectization*

$$M \times \mathbf{R}_t \text{ with symplectic form } \omega = \mathrm{d}(e^t \alpha).$$

(We choose an **R**–invariant compatible almost complex structure too. This choice does not influence the invariants.)

For contact homology, the domains of the curves are punctured spheres (absolute case) or boundary-punctured disks (relative case). In the relative case, their boundaries lie on the Lagrangian cylinder $L \times \mathbf{R} \subset M \times \mathbf{R}$.

As $t \to \infty$, exactly one of the punctures (the *positive puncture*) is mapped asymptotically to a cylinder over a Reeb orbit/chord. All other punctures are *negative*: they are asymptotic to similar cylinders/strips, but as $t \to -\infty$.

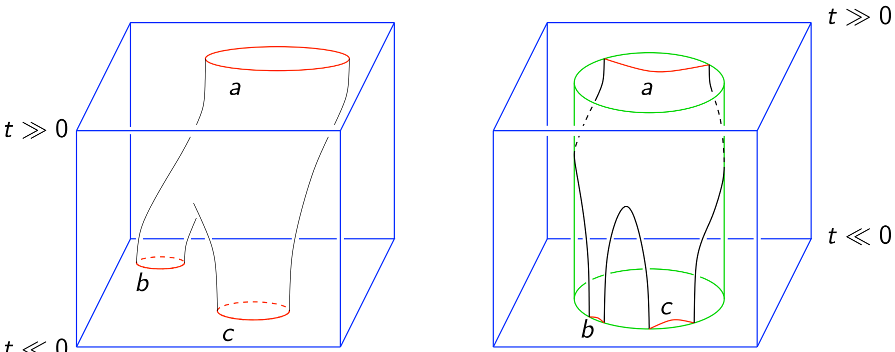
The differential

To compute the differential ∂a of a generator a (Reeb orbit/chord), we enumerate all those curves whose positive puncture goes to a and that are *rigid*: that is, their only moduli is from the $\mathbf{R}$–symmetry.

Each such curve contributes the product of (the images of) its negative punctures to ∂a and we sum these contributions.

To the rest of the algebra (generated by the orbits/chords) ∂ is extended by linearity and the Leibniz rule.

A schematic picture of the differential



In blue: M and $M \times \mathbf{R}$. In green: L and $L \times \mathbf{R}$.
In red: Reeb trajectories. In black: holomorphic curve.

In both cases, a is a positive puncture and b and c denote negative punctures. If the curve is rigid, then $\partial a = bc + \cdots$.

Contact homology

Hope: $\partial^2 = 0$ and the homology $CH = \ker \partial / \operatorname{im} \partial$ is an invariant of M or of the pair (M, L). This is not quite true without restrictions on (the topology of) M. In many cases when we do expect it to be true, there are still analytical difficulties with the proof.

In the relative case which is most interesting to us, Ekholm showed:

Theorem

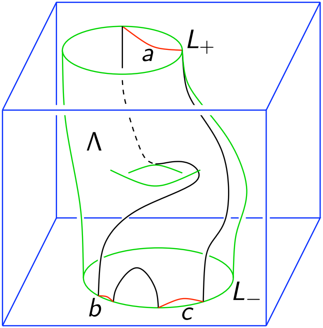
Let N be an n–dimensional manifold. If L is a Legendrian submanifold of the contact manifold J^1N, then the corresponding contact homology differential does satisfy $\partial^2 = 0$ and contact homology is an invariant of L.

Lagrangian cobordisms

It is also "expected" that if an exact Lagrangian surface Λ connects the Legendrians L_+ and L_-, then by counting rigid holomorphic disks we can define a map

$$\Phi_\Lambda \colon CH(L_+) \to CH(L_-)$$

(from the chain map $\varphi_\Lambda(a) = bc + \cdots$).



Φ_Λ "should be" invariant under deformations of Λ relative to its boundary.

Exactness is crucial in order to rule out holomorphic disks (without punctures) with boundary on Λ. That in turn is vital to proving $\partial\varphi_\Lambda = \varphi_\Lambda\partial$.

Computability?

It is virtually impossible to carry out any computation of symplectic field theory or contact homology (or a map induced by cobordism) based on the ideas presented so far.

The most classical case of the relative theory is that of Legendrian 1–submanifolds (knots and links) in the contact manifold $M = J^1\mathbf{R} \cong \mathbf{R}^3$.

In this case:

- A combinatorial version of relative contact homology was established by Chekanov.
- The existence and invariance of the map induced by an exact Lagrangian cobordism are known (folklore).

Definitions in the classical case

The *standard contact structure* on $\mathbf{R}^3_{xyz}$ is the tangent plane field given as the kernel of the one-form

$$\alpha = \mathrm{d}z - y\mathrm{d}x.$$

Its Reeb vector field is $R_\alpha = \frac{\partial}{\partial z}$.

A knot or link is Legendrian if it is everywhere tangent to the contact structure, that is, if

$$\frac{\mathrm{d}z(s)}{\mathrm{d}s} - y(s)\frac{\mathrm{d}x(s)}{\mathrm{d}s} \equiv 0$$

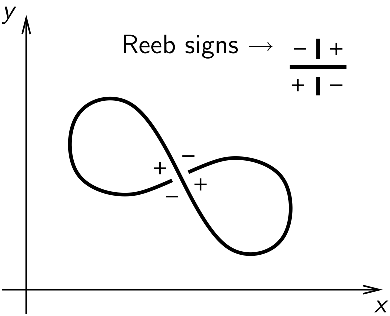
along the curve.

Lagrangian (xy) projection

Special case of $J^1N \to T^*N$. Reeb chords correspond to crossings. The length of the chord is also called the height of the crossing.

A Legendrian curve has an xy–projection so that

- it is immersed
- the area of any bounded complementary region equals the sum of the heights of its corners taken by the so-called *Reeb signs*.



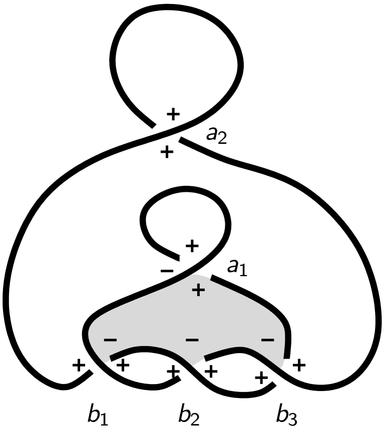
The latter implies that the signed area bounded by a Lagrangian projection is zero. (Check: If the projection γ bounds a 2–chain F, then $\text{area}(F) = \int_F \mathrm{d}x \wedge \mathrm{d}y = \int_\gamma -y\mathrm{d}x = \int_\gamma -\mathrm{d}z = -\Delta z = 0$.)

Linear constraints

This is the Lagrangian diagram of a certain Legendrian trefoil knot. The area shaded in the middle is expressed by the heights as

$$h(a_1) - h(b_1) - h(b_2) - h(b_3).$$

By *linear constraint* we mean that in particular, this quantity has to be positive.



Contractible crossings

A Reeb chord/crossing in a Lagrangian diagram is called *contractible* if it can be shrunk to length zero with a Legendrian isotopy while keeping all other chord lengths and areas positive.

In other words, if we substitute 0 for the length of a contractible crossing, the linear constraints can still be satisfied.

In our trefoil example, a_1 and a_2 are not contractible but b_1, b_2, and b_3 are.

Two easy Legendrian invariants

Lagrangian diagrams are always immersed, not just generically. In other words, Reidemeister I moves cannot occur. Thus, the following two quantities stay invariant throughout Legendrian isotopies:

- The *rotation number* is the total winding number (Whitney index) of a Lagrangian diagram (its sign depends on choice of orientation).
- The *Thurston–Bennequin number* is the writhe (algebraic crossing number) of a Lagrangian diagram.

Chekanov's observation

In order to compute the contact homology of a Legendrian in $\mathbf{R}^3$, instead of going one dimension higher to the symplectization, we may go one dimension lower to the Lagrangian projection.

In particular, the holomorphic disks of the definition remain holomorphic after projecting out t and z. A puncture asymptotic to a Reeb chord projects to a convex corner at the corresponding crossing. Positive punctures carry the Reeb sign $+$, and negative punctures carry a $-$.

Using the Riemann mapping theorem, holomorphic curves can be recovered from these planar immersions.

Admissible disks

More formally: a map $D^2 \to \mathbf{R}^2_{xy}$ is an *admissible disk* in the Lagrangian diagram γ if

- it takes ∂D^2 to the curve γ
- it is immersed away from finitely many convex corners on the boundary
- exactly one of those corners covers the Reeb sign $+$, all others cover a $-$.

Relative contact homology combinatorially I

As we said before, the chain complex has a product structure too, i.e. it is a differential graded algebra (DGA).

"A": $\mathcal{A}$ is non-commutative, associative, and unital, freely generated over $\mathbf{Z}/2\mathbf{Z}$ by the crossings of the xy–projection.

"G": $\mathcal{A}$ is graded modulo $2r$ by the following rule: Let a be a generator. Starting at the undercrossing at a, follow γ until we reach a again, this time on the upper strand. Denote this path by γ_a and define

$$|a| = -2r(\gamma_a) - \frac{1}{2},$$

where $r(\gamma_a)$ is the fractional number of rotations taken by γ_a (we pretend that the strands meeting at a make a right angle). This is extended to the rest of $\mathcal{A}$ multiplicatively.

Relative contact homology combinatorially II

"D": To compute the differential ∂a of a generator a, we enumerate all admissible disks in the diagram with positive corner at a, and sum the products of their negative corners. We extend to $\mathcal{A}$ by the Leibniz rule $\partial(ab) = \partial a \cdot b + a \cdot \partial b$.

Chekanov proved:

Theorem

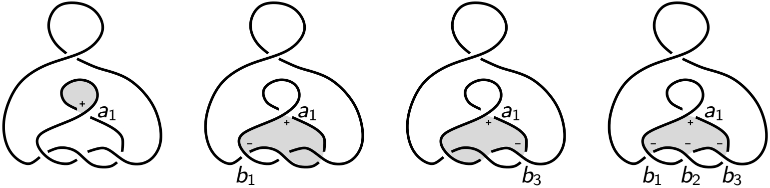
The differential ∂ lowers the grading by 1 and it satisfies $\partial^2 = 0$. The resulting so-called relative contact homology $CH(L)$ is itself a graded algebra, and it is invariant under Legendrian isotopies of the Legendrian knot L.

Example

For the trefoil knot shown previously we have $r = 0$, so the theory is $\mathbf{Z}$–graded. The indices (gradings) of the generators are

$$|a_1| = |a_2| = 1 \text{ and } |b_1| = |b_2| = |b_3| = 0.$$

Thus $\partial b_1 = \partial b_2 = \partial b_3 = 0$. Further, $\partial a_1 =$



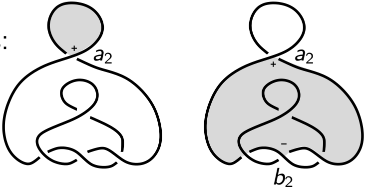
$$1 \quad + \quad b_1 \quad + \quad b_3 \quad + \quad b_1 b_2 b_3.$$

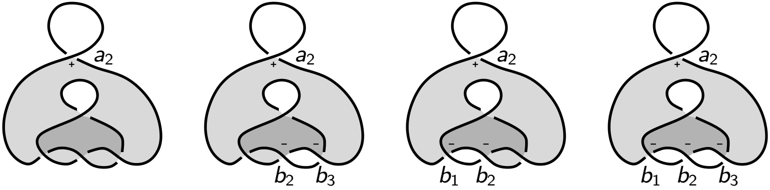
Example, continued

Finally,

$$\begin{aligned} \partial a_2 &= 1 + b_2 + 1 + b_2 b_3 + b_1 b_2 + b_2 b_3 b_1 b_2 \\ &= b_2 + b_2 b_3 + b_1 b_2 + b_2 b_3 b_1 b_2. \end{aligned}$$

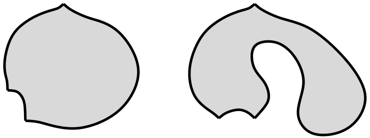
The six admissible disks are as follows:





Some immersed disks

The last four disks are not embedded. They can be visualized as follows (showing the one with three corners that contributes $b_2 b_3$):



Augmentations and linearization

We stopped the above computation after determining the differential. That is because as a vector space, both $\mathcal{A}$ and the quotient $CH(L)$ are infinite dimensional, so we cannot compute things like Betti numbers.

There is a way around this problem, namely a process called *linearization*. It requires (and depends on) the choice of a so-called *augmentation* of the DGA.

An augmentation is an algebra map $\varepsilon\colon \mathcal{A} \to \mathbf{Z}/2\mathbf{Z}$ supported in grading 0 which vanishes on the image of ∂. Thus it descends to an algebra homomorphism $E\colon CH \to \mathbf{Z}/2\mathbf{Z}$, too.

In other words, an augmentation (on the chain level) is a subset of the index 0 crossings of the Lagrangian diagram whose characteristic function solves the set of polynomial equations $\{\, \partial q = 0 \,|\, q \text{ is any crossing of the diagram} \,\}$.

Example

For the trefoil, we have two equations in three unknowns:

$$\begin{aligned} 1 + b_1 + b_3 + b_1 b_2 b_3 &= 0 \\ b_2 + b_2 b_3 + b_1 b_2 + b_2 b_3 b_1 b_2 &= 0, \end{aligned}$$

which is really only one equation if the unknowns commute.

Five of the eight possible sets of crossings are augmentations:

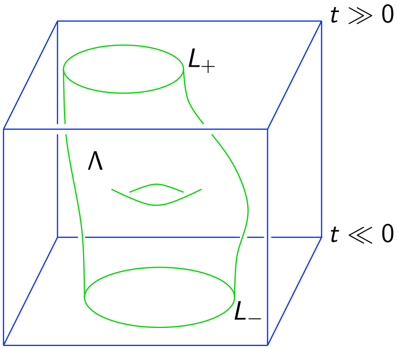
$$\{ b_1 \}, \{ b_3 \}, \{ b_1, b_2 \}, \{ b_2, b_3 \}, \text{ and } \{ b_1, b_2, b_3 \}.$$

These stay different on the homology level, too.

Our class of Lagrangians

We will consider Lagrangians Λ in the symplectization $\mathbf{R}^4_{xyzt}$ with the following properties:

- As $t \to \pm\infty$, Λ is asymptotic to Legendrians L_+ and L_-, respectively.
- Intersections of Λ with $|t| \leq$ const. sets are compact.
- Λ is exact, that is a function $F\colon \Lambda \to \mathbf{R}$ exists so that $\mathrm{d}F = e^t\alpha$ along Λ.



Stacking

It is just a technicality that we pushed the ends of Λ to infinity.

That does not stop us from stacking Λ_1 and Λ_2 together into a single Lagrangian, provided that the negative end of Λ_1 matches the positive end of Λ_2.

Theorem

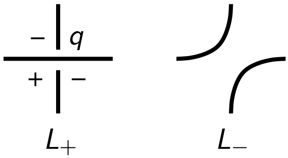
Let Λ be a generic exact Lagrangian as above. Then there is a well-defined map $\varphi_\Lambda \colon \mathcal{A}(L_+) \to \mathcal{A}(L_-)$ (given on generators as described before and extended as an algebra homomorphism), it is a chain map, and its induced map (degree preserving algebra homomorphism) $\Phi_\Lambda \colon CH(L_+) \to CH(L_-)$ is invariant under exact Lagrangian isotopies of Λ fixing $\partial\Lambda$. Moreover, Φ is functorial with respect to stacking.

In particular if $L_- = \varnothing$, we get an augmentation for L_+: $\Phi_\Lambda \colon CH(L_+) \to CH(\varnothing) = \mathbf{Z}/2\mathbf{Z}$.

Constructions

We introduce three basic building blocks.

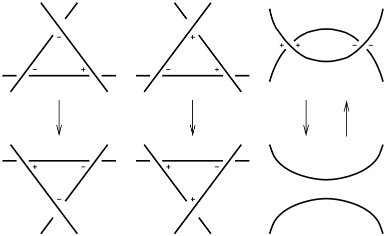
1. *Capping off unknots.* The standard Legendrian unknot (as L_+) bounds a Lagrangian disk (so that $L_- = \varnothing$).
2. *Isotopy* (Chantraine). An isotopy L_s, $0 \leq s \leq 1$ so that $L_s \subset \mathbf{R}^3_{xyz}$ is Legendrian for all s can be turned into a Lagrangian cobordism from L_0 to L_1.
3. *Zero-resolution.* If q is a contractible crossing in a Lagrangian diagram of the Legendrian L_+, then there exists a Lagrangian saddle from L_+ to L_-, where the xy-diagram of L_- differs from that of L_+ only as shown.



Main result

The maps induced by our three building blocks are as follows.

1. The cap on the unknot sends its only Reeb chord to zero.
2. For a generic Legendrian isotopy, the corresponding maps agree with those in Chekanov's proof that *CH* is invariant under such isotopies. In particular, if the *xy*-diagram does not undergo any Reidemeister moves in the isotopy, then the induced map is the identity.

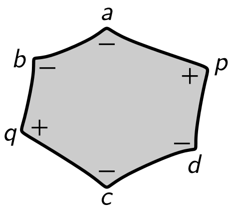


Main result, cont'd

3. The chain map φ induced by the zero-resolution at q sends q to

$$\varphi(q) = 1.$$

If p is another crossing, then $\varphi(p)$ is p plus the sum of the following contributions. Find all immersed disks in the xy-diagram of L_+ with convex corners so that exactly *two* of the corners are positive, and those are at p and q. Each such disk contributes the product of its negative corners.



$$\varphi(p) = p + abcd + \cdots$$

Method of proof

Exact Lagrangians Λ can be described in terms of front projections.

A *front* is a graph of the potential F as a multi-valued function of two of the four symplectic coordinates (q_1 and q_2) so that the other two (p_1 and p_2) can be recovered as partial derivatives.

A *flow tree* is a tree in the q_1q_2–plane built out of Morse flow lines of local differences of sheets of the front.

As we degenerate Λ by flattening out its front, holomorphic disks with boundary on Λ converge to flow trees. So holomorphic curve counts can be carried out in terms of flow tree counts.

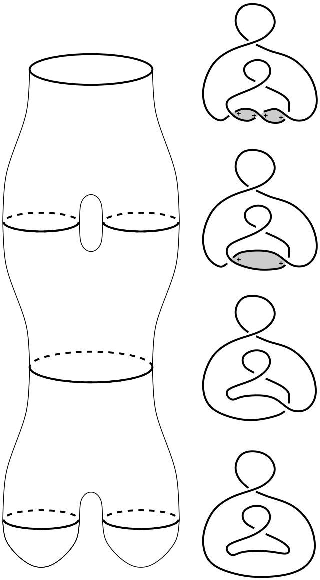
The trefoil once more

The three contractible crossings of our trefoil diagram can be resolved in any order. For example, using the order b_2, b_1, b_3 leads to the augmentation $\varphi(b_1) = 0$, $\varphi(b_2) = \varphi(b_3) = 1$.

The six orders produce all five augmentations. Therefore, up to isotopy through exact Lagrangians, the trefoil knot bounds at least five different exact Lagrangian punctured tori.

(So which two orders gave the same augmentation? As long as b_2 is present in the diagram, b_1 and b_3 are *simultaneously* contractible. Thus, the surface from b_1, b_3, b_2 is isotopic to that from b_3, b_1, b_2.)

Computing the induced map



b_1	b_2	b_3
$b_1 + 1$	1	$b_3 + 1$
$1 + 1$	1	$b_3 + 1 + 1$
0	1	$1 + 1 + 1 = 1$