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On double points of immersed surfaces

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The purpose of this paper is to give a new proof of the following proposition, which is a sort of "folklore".

Proposition 1. *Let F be a closed surface, N an oriented 4-manifold and $f : F \to N$ a stable map (therefore an immersion). Then*

$$e + 2D(f) \equiv \left\langle \widetilde{\mathrm{P}}\big(\mathcal{D}_N f_*[F]_2\big), [N]_4 \right\rangle + 2\chi(F) \pmod 4, \tag{1}$$

where e is the normal Euler number of f, $D(f)$ denotes the number of double points of f, $\widetilde{\mathrm{P}} : H^2(N; \mathbb{Z}_2) \to H^4(N; \mathbb{Z}_4)$ is the Pontrjagin square and $[N]_4 \in H_4(N; \mathbb{Z}_2)$ is the modulo 4 fundamental class of N.

The only properties of the Pontrjagin square we will use are its naturality and that the Pontrjagin square of the modulo 2 reduction of a class $a \in H^2(N; \mathbb{Z})$ is the modulo 4 reduction of $a^2 \in H^4(N; \mathbb{Z})$. Recall that the normal Euler number of f can be defined even if F is nonorientable as the algebraic number of the zeros of a generic section of the normal bundle ν_f.

In [1] a more general proposition (Lemma 2.5.2) is stated but no formal proof is given; only an indication is made that it can be verified by the method of Lannes [3]. This method is based on the investigation of some cohomology operations in configuration spaces. Li [4, Theorem 2 (1)] proves the same result for embeddings $M^{2k} \to N^{4k}$, where M is closed, N is oriented, by a formula of Massey [5].

We will need the next lemma of the second author [6]; for the convenience of the reader we give here a short proof.

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Let $g: M^n \to \mathbb{R}^{2n-1}$ be a self-transverse immersion of a closed manifold and let $\theta_2(g) \subset \mathbb{R}^{2n-1}$ denote the double point set of g. This is the union of immersed circles; for each such circle C a double covering $\widetilde{C} \to C$ arises naturally. We call C *nontrivial*, if this is nontrivial, i.e., $\widetilde{C}$ is connected; otherwise C is *trivial*. Let $\delta(g) \in \mathbb{Z}_2$ denote the parity of the number of nontrivial circles.

Lemma 1. *For n even, $\delta(g) = \langle w_1(M) \cup \overline{w}_{n-1}(M), [M]_2\rangle$.*

Proof. Let ν denote the normal bundle of $\Delta_2(g) := g^{-1}(\theta_2(g))$ in M and let ε^1 be the trivial line bundle over $\Delta_2(g)$; then obviously $\nu \oplus \varepsilon^1 = TM\big|_{\Delta_2(g)}$ and hence $w_1(\nu) = w_1(M)\big|_{\Delta_2(g)}$. Then we have

$$\bigl\langle w_1(\nu), \bigl[\Delta_2(g)\bigr]_2\bigr\rangle = \bigl\langle w_1(M) \cup \overline{w}_{n-1}(M), [M]_2\bigr\rangle$$

because the homology class represented by $\Delta_2(g)$ in $H_1(M;\mathbb{Z}_2)$ is the Poincaré dual of $\overline{w}_{n-1}(M)$ (see [2]). The left hand side of this equation is precisely the parity of the number of those components of $\Delta_2(g)$ which have nonorientable normal bundles in M and for n even this parity coincides with $\delta(g)$ (there are two components of $\Delta_2(g)$ belonging to each trivial double point circle and either both or none of them has nonorientable normal bundle; to a nontrivial circle belongs one component of $\Delta_2(g)$ and in the case of n being even this must have a nonorientable normal bundle). □

To prove our proposition, we will need several steps.

Proof of Proposition 1. First we consider the case $N = \mathbb{R}^4$, along with the additional condition of the existence of a nowhere vanishing section for ν_f. In this case $e = 0$ and obviously the congruence $D(f) \equiv \chi(F) \pmod 2$ is to be proven. Using Hirsch's theory for immersions, we find that f is regularly homotopic to a self-transverse immersion $g: F \to \mathbb{R}^3$ ($\mathbb{R}^3 \subset \mathbb{R}^4$ being an affine subspace). It is easy to see that $D(f) \equiv \delta(g) \pmod 2$. Then, by Lemma 1, $D(f) \equiv \langle w_1(F) \cup \overline{w}_1(F), [F]_2\rangle \equiv \langle w_1^2(F), [F]_2\rangle \equiv \langle w_2(F), [F]_2\rangle \equiv \chi(F) \pmod 2$.

Another, almost trivial special case is that of F being orientable. Now $\langle \widetilde{\mathrm{P}}(\mathcal{D}_N f_*[F]_2), [N]_4\rangle \equiv f_*[F] \cdot f_*[F] \pmod 4$ and of course $2\chi(F) \equiv 0 \pmod 4$ and the proof is easy using the definition of the normal Euler number. If s is a generic section of ν_f and $i: \nu_f \to N$ is an immersion onto a tubular neighbourhood of $f(F)$, then both f and $i \circ s$ represent $f_*[F]$, and their intersection points are the following: one for each zero of s and two for each double point of f. Below, we will reduce our statement to this case.

First we reduce it to the case $\chi(F) \equiv 0 \pmod 2$. Fix an immersion $\mathbb{R}P^2 \to \mathbb{R}^3$ (e.g., Boy's surface), and by a regular homotopy move it to a self-transverse immersion $\alpha: \mathbb{R}P^2 \to \mathbb{R}^4$. If $\chi(F)$ is odd then let us choose a 'small' set in N, diffeomorphic to $\mathbb{R}^4$, 'close' to $\mathrm{im}(f)$, but disjoint from it and copy $\alpha(\mathbb{R}P^2)$ into it. Constructing the connected sum of $f(F)$ and $\alpha(\mathbb{R}P^2)$ we obtain a surface immersed in N with even Euler characteristic. By the construction of α, $e(\nu_\alpha) = 0$ and it is easy to see that our operation does not change the normal Euler number e of f. It does not change the summand in

(1) containing the Pontrjagin square either, because the original and modified surfaces are homological. The only one thing left to prove is that we increased the number of double points by an odd number, i.e., α has an odd number of those, but this is obvious by the first paragraph of the proof.

We can assume now that $\chi(F)$ is even. In this case $w_1(F)$ can be represented by a single circle $C \subset F$, disjoint from the double points of f and embedded with trivial normal bundle ν_C. The circle $f(C)$ is embedded in N and its normal bundle is trivial because of the orientability of N; ν_C can be identified with a trivial subbundle of this trivial bundle. Let K denote the Klein bottle and let $\beta : K \to S^4$ be an immersion obtained by lifting an immersion $K \to \mathbb{R}^3$ into $\mathbb{R}^4$ and then embedding the latter into S^4. This map β and a circle $C' \subset K$ representing $w_1(K)$ have the same properties as above, thus we can identify closed tubular neighbourhoods U and V of $f(C)$ and $\beta(C')$, respectively, in such a way that $U \cap f(F)$ and $V \cap \beta(K)$ correspond to each other. Now cut out $\operatorname{int} U$ and $\operatorname{int} V$ from N and S^4, respectively, and attach the resulting two 4-manifolds along their boundaries using the above identification. As a result we get an orientable 4-manifold N' ($\partial U \cong \partial V \cong S^1 \times S^2$ are connected) and an *orientable* surface F' immersed in it. Trivially $\chi(F') = \chi(F)$. Orient N' by extending the orientation of (the remainder of) N. Since β can be chosen to be an embedding and $e(\nu_\beta) = 0$, the summands on the left hand side of (1) do not change and neither does the second term on the right hand side. Again by our first paragraph, $\langle \widetilde{\mathrm{P}}(\mathcal{D}_{S^4}\beta_*[K]_2), [S^4]_4\rangle = 0$. Using the fact that N' is cobordant with the disjoint union of N and S^4, the proof will be completed by the next lemma. $\square$

Lemma 2. *Let A and B be oriented closed* 4*-manifolds and $f_1 : F_1 \to A$ and $f_2 : F_2 \to B$ be generic immersions of closed surfaces into them. Assume that there is a cobordism between the pairs of spaces $(A, f_1(F_1))$ and $(B, f_2(F_2))$, meaning the following. Let W be an oriented compact* 5*-manifold with boundary and $g : X \to W$ an immersion of a compact* 3*-manifold X with boundary. Assume that ∂W can be identified by the disjoint union $A \cup B$* (*in such a way that the orientation of W induces that of A but the reverse of that of B by some fixed convention*) *and ∂X by the disjoint union $F_1 \cup F_2$ and under these identifications $g|_{\partial X}$ coincides with $f_1 \cup f_2$. Then*

$$\big\langle \widetilde{\mathrm{P}}\big(\mathcal{D}_A f_{1*}[F_1]_2\big), [A]_4\big\rangle = \big\langle \widetilde{\mathrm{P}}\big(\mathcal{D}_B f_{2*}[F_2]_2\big), [B]_4\big\rangle.$$

Proof. It is sufficient to show that $\langle \widetilde{\mathrm{P}}(\mathcal{D}_{\partial W} g_*[\partial X]_2), [\partial W]_4\rangle = 0$. Denoting the identical embedding $\partial W \to W$ by j we have

$$\begin{aligned}\big\langle \widetilde{\mathrm{P}}\big(\mathcal{D}_{\partial W} g_*[\partial X]_2\big), [\partial W]_4\big\rangle &= \big\langle j^*\widetilde{\mathrm{P}}\big(\mathcal{D}_W g_*[X]_2\big), [\partial W]_4\big\rangle \\ &= \big\langle \widetilde{\mathrm{P}}\big(\mathcal{D}_W g_*[X]_2\big), j_*[\partial W]_4\big\rangle = \big\langle \widetilde{\mathrm{P}}\big(\mathcal{D}_W g_*[X]_2\big), 0\big\rangle = 0,\end{aligned}$$

as we wanted. $\square$

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