Published at Asymptotic Analysis 23(2000), pp. 329-375.

SOME REFINEMENTS OF WIGNER'S SEMI-CIRCLE LAW FOR GAUSSIAN RANDOM MATRICES USING SUPERANALYSIS

ATSUSHI INOUE AND YUJI NOMURA

Dedicated to the memory of our friend Nobuhisa Iwasaki

Abstract. In Random Matrix Theory(=R.M.T.), Wigner's semi-circle law gives a well-known cornerstone. We report mathematical refinements of this law as an application of superanalysis. That is, using Efetov's idea, we rewrite the average of the empirical measure of the eigenvalue distribution of the Hermitian matrices in a compact form. Careful calculations give not only the precise convergence rate of that law, but also the precise rate of the edge mobility.

1. Introduction and Result

Let $\mathfrak{U}_N$ be a set of Hermitian $N \times N$ matrices, which is identified with $\mathbb{R}^{N^2}$ as a topological space. In this set, we introduce a probability measure $d\mu_N(H)$ by

$$\begin{gathered} d\mu_N(H) = \prod_{k=1}^N d(\Re H_{kk}) \prod_{j<k}^N d(\Re H_{jk}) d(\Im H_{jk}) P_{N,J}(H), \\ P_{N,J}(H) = Z_{N,J}^{-1} \exp\big[-\frac{N}{2J^2} \operatorname{tr} H^* H\big] \end{gathered} \tag{1.1}$$

where $H = (H_{jk})$, $H^* = (H^*_{jk}) = (\overline{H}_{kj}) = {}^t\overline{H}$, $\prod_{k=1}^N d(\Re H_{kk}) \prod_{j<k}^N d(\Re H_{jk}) d(\Im H_{jk})$ being the Lebesgue measure on $\mathbb{R}^{N^2}$, and $Z_{N,J}^{-1}$ is the normalizing constant given by $Z_{N,J} = 2^{N/2}(J^2\pi/N)^{3N/2}$.

Let $E_\alpha = E_\alpha(H)$ $(\alpha = 1, \cdots, N)$ be real eigenvalues of $H \in \mathfrak{U}_N$.

We put

$$\rho_N(\lambda) = \rho_N(\lambda; H) = N^{-1} \sum_{\alpha=1}^N \delta(\lambda - E_\alpha(H)), \tag{1.2}$$

where δ is the Dirac's delta. Denoting

$$\big\langle f \big\rangle_N = \big\langle f(\cdot) \big\rangle_N = \int_{\mathfrak{U}_N} d\mu_N(H)\, f(H),$$

for a function f on $\mathfrak{U}_N$, we get

Theorem 1.1 (Wigner's semi-circle law)**.**

$$\lim_{N\to\infty} \big\langle \rho_N(\lambda) \big\rangle_N = w_{sc}(\lambda) = \begin{cases} (2\pi J^2)^{-1}\sqrt{4J^2 - \lambda^2} & \text{for} \quad |\lambda| < 2J, \\ 0 & \text{for} \quad |\lambda| > 2J. \end{cases} \tag{1.3}$$

Remark. By definition, the limit (1.3) is interpreted as

$$\lim_{N\to\infty} \langle \phi, \int_{\mathfrak{U}_N} d\mu_N(H)\, N^{-1} \sum_{\alpha=1}^N \delta(\cdot - E_\alpha(H)) \rangle = \langle \phi, w_{sc} \rangle = \int_{\mathbb{R}} d\lambda\, \phi(\lambda) w_{sc}(\lambda)$$

Date: Department of Mathematics, Tokyo Institute of Technology.

The first author is partially supported by Monbusho Grand-in-aid No.08304010 and No.10640201.

for any $\phi \in C_0^\infty(\mathbb{R}) = \mathcal{D}(\mathbb{R})$. $\langle\cdot,\cdot\rangle$ stands for the duality between $\mathcal{D}(\mathbb{R})$ and $\mathcal{D}'(\mathbb{R})$. We need more interpretation to give the meaning to $\int_{\mathfrak{U}_N} d\mu_N(H)\, N^{-1}\sum_{\alpha=1}^N \delta(\cdot - E_\alpha(H))$, which will be given in §2.

Seemingly, there exist several methods to prove this fact. Here, we want to explain a new derivation of this fact using odd variables obtained by Efetov [6], mainly following from Fyodorov [8] and Brézin [1] (see also, Mello [15], Zirnbauer [21]).

Moreover, we get, as a byproduct of this new treatise,

Theorem 1.2 (A refined version of Wigner's semi-circle law)**.** *For each λ with $|\lambda| < 2J$, when $N \to \infty$, we have*

$$\big\langle\rho_N(\lambda)\big\rangle_N = \frac{\sqrt{4J^2-\lambda^2}}{2\pi J^2} - \frac{(-1)^N J}{\pi(4J^2-\lambda^2)}\cos(N[\frac{\lambda\sqrt{4J^2-\lambda^2}}{2J^2} + 2\arcsin(\frac{\lambda}{2J})])N^{-1} + O(N^{-2}). \tag{1.4}$$

When λ satisfies $|\lambda| > 2J$, there exist constants $C_\pm(\lambda) > 0$ and $k_\pm(\lambda) > 0$ such that

$$\Big|\big\langle\rho_N(\lambda)\big\rangle_N\Big| \le C_\pm(\lambda)\exp[-k_\pm(\lambda)N] \tag{1.5}$$

with $k_\pm(\lambda) \to 0$ and $C_\pm(\lambda) \to \infty$ for $\lambda \searrow 2J$ or $\lambda \nearrow -2J$, respectively.

Theorem 1.3 (The spectrum edge problem)**.** *Let $z \in [-1, 1]$. We have*

$$\begin{aligned} \big\langle\rho_N(2J - zN^{-2/3})\big\rangle_N &= N^{-1/3}f(z/J) + O(N^{-2/3}) \qquad \text{as } N \to \infty, \\ \big\langle\rho_N(-2J + zN^{-2/3})\big\rangle_N &= -N^{-1/3}f(z/J) + O(N^{-2/3}) \qquad \text{as } N \to \infty, \end{aligned} \tag{1.6}$$

where

$$f(w) = \frac{1}{4\pi^2 J}(\mathrm{Ai}'(w)^2 - \mathrm{Ai}''(w)\,\mathrm{Ai}(w)), \quad \mathrm{Ai}(w) = \int_{\mathbb{R}} dx\, \exp[-\frac{i}{3}x^3 + iwx].$$

(A) One of the key expression obtained by introducing new auxiliary variables, is

$$\big\langle\rho_N(\lambda)\big\rangle_N = \pi^{-1}\Im\int_{\mathfrak{Q}} dQ\,\big(\{(\lambda - i0)I_2 - Q\}^{-1}\big)_{bb}\exp[-N\mathcal{L}(\mathcal{Q})] \tag{1.7}$$

where I_n stands for $n\times n$-identity matrix and

$$\begin{gathered} \mathcal{L}(\mathcal{Q}) = \mathrm{str}\,[(2J^2)^{-1}Q^2 + \log((\lambda - i0)I_2 - Q)], \\ \mathfrak{Q} = \{Q = \begin{pmatrix} x_1 & \rho_1 \\ \rho_2 & ix_2\end{pmatrix} \mid x_1, x_2 \in \mathfrak{R}_{\mathrm{ev}},\ \rho_1,\rho_2 \in \mathfrak{R}_{\mathrm{od}}\} \cong \mathfrak{R}^{2|2},\ \ dQ = \frac{dx_1 dx_2}{2\pi}d\rho_1 d\rho_2, \\ \big(((\lambda - i0)I_2 - Q)^{-1}\big)_{bb} = \frac{(\lambda - i0 - x_1)(\lambda - i0 - ix_2) + \rho_1\rho_2}{(\lambda - i0 - x_1)^2(\lambda - i0 - ix_2)}. \end{gathered} \tag{1.8}$$

Here in (1.7), **the parameter N appears only in one place**. This formula is formidably charming but **not yet directly justified**, like Feynman's expression of certain quantum objects using his measure. (Unfamiliar notion like $\mathfrak{R}_{\mathrm{ev}}$, $\mathfrak{R}_{\mathrm{od}}$ and $\mathfrak{R}^{2|2}$ will be explained in §2.)

(B) In physics literatures, for example in [8],[21], they claim without proof that they may apply the method of steepest descent to (1.7) when $N \to \infty$. More precisely, as

$$\delta\mathcal{L}(Q)\tilde{Q} = \frac{d}{d\epsilon}\mathcal{L}(Q + \epsilon\tilde{Q})\Big|_{\epsilon=0},$$

they seek solutions of

$$\delta\mathcal{L}(Q) = \mathrm{str}\,(\frac{Q}{J^2} - \frac{1}{\lambda - Q}) = 0.$$

As a candidate of effective saddle points, they take

$$Q_c = (\frac{1}{2}\lambda + \frac{1}{2}\sqrt{\lambda^2 - 4J^2})I_2,$$

and they have

$$\lim_{N\to\infty}\big\langle\rho_N(\lambda)\big\rangle_N=\pi^{-1}\Im(\lambda-Q_c)^{-1}_{bb}=w_{sc}(\lambda).\quad\square$$

Remark. Not only the expression (1.7) nor the applicability of the saddle point method to it are not so clear. To get the mathematical rigour, we **dare to loose such a beautiful expression** like (1.7), but we have the two formulae (3.7) and (3.18) which lead to our results.

Remark. The set $(\mathfrak{U}_N,\ d\mu_N(\cdot))$ is called GUE=the Gaussian Unitary Ensemble. Other ensembles may be treated analogously as indicated in [21] but are not treated here.

Problem: Under what condition, do we have the following equality?

$$\lim_{N\to\infty}\lim_{\epsilon\to 0}\big\langle\frac{1}{N}\operatorname{tr}\frac{1}{(\lambda-i\epsilon)I_N-H}\big\rangle_N=\lim_{\epsilon\to 0}\lim_{N\to\infty}\big\langle\frac{1}{N}\operatorname{tr}\frac{1}{(\lambda-i\epsilon)I_N-H}\big\rangle_N.\tag{1.9}$$

If this assertion is true, may we justify the physicists argument of "saddle point method"?

Contents of this paper: We gather some notations from superanalysis=analysis on superspace but **without supersymmetry**, in §2. In §3, we introduce odd variables to reformulate $\big\langle N^{-1}\operatorname{tr}((\lambda-i\epsilon)I_N-H)^{-1}\big\rangle_N$ in a compact form. To make $\epsilon\to 0$ in that expression, we not only use Brézin's idea but also give our new expression using products of Hermite polynomials. Using Brézin's expression, in §4, we give the proof of Theorem 1.2 whose precise calculations are given in Appendix A. In §5, our expression of $\big\langle\rho_N(\lambda)\big\rangle_N$ is used to prove Theorem 1.3. To estimate the remainder terms, we use Lax's technique for the highly oscillatory integrals. In Appendix B, we clarify the ambiguity of Q-integration using the Rothstein measure instead of the Berezin measure as the volume form on the superspace.

Since not only authors main concern is to clarify the meaning of the formula (1.7) but also there exist so many papers on Wigner's semi-circle law, the references cited in this paper are so restricted to be complete. We apology to other persons whose works on this subject are not mentioned.

Acknowledgement: We would like to thank Prof. Kōhei Uchiyama for valuable discussion concerning the steepest descent method. We thanks also anonymous refree who pointed out Ziegler [20] to us.

2. Fundamentals of superanalysis

For symbols $\{\sigma_j\}_{j=1}^\infty$ satisfying the Grassmann relation

$$\sigma_j\sigma_k+\sigma_k\sigma_j=0,\quad j,k=1,2,\cdots,$$

we put

$$\mathfrak{C}=\{X=\sum_{I\in\mathcal{I}}X_I\sigma^I\ \big|\ X_I\in\mathbb{C}\}$$

where

$$\mathcal{I}=\{I=(i_k)\in\{0,1\}^{\mathbf{N}}\ \big|\ |I|=\sum_k i_k<\infty\},$$

$$\sigma^I=\sigma_1^{i_1}\sigma_2^{i_2}\cdots,\quad I=(i_1,i_2,\cdots),\quad \sigma^{\tilde 0}=1,\quad \tilde 0=(0,0,\cdots)\in\mathcal{I}.$$

Besides trivially defined linear operations of sums and scalar multiplications, we have a product operation in $\mathfrak{C}$: For

$$X=\sum_{J\in\mathcal{J}}X_J\sigma^J,\quad Y=\sum_{K\in\mathcal{I}}Y_K\sigma^K,$$

we put

$$XY=\sum_{I\in\mathcal{I}}(XY)_I\sigma^I\quad\text{with}\quad (XY)_I=\sum_{I=J+K}(-1)^{\tau(I;J,K)}X_JY_K.$$

Here, $\tau(I;J,K)$ is an integer defined by

$$\sigma^J\sigma^K = (-1)^{\tau(I;J,K)}\sigma^I, \quad I = J + K.$$

Proposition 2.1 ([12]). *$\mathfrak{C}$ forms an ∞-dimensional Fréchet-Grassmann algebra over $\mathbb{C}$, that is, an associative, distributive and non-commutative ring with degree, which is endowed with the Fréchet topology.*

Remark. (1) Degree in $\mathfrak{C}$ is defined by introducing subspaces

$$\mathfrak{C}^{[j]} = \{X = \sum_{I\in\mathcal{I},|\mathcal{I}|=|} X_I\sigma^I\} \quad \text{for} \quad j = 0, 1, \cdots$$

which satisfy

$$\mathfrak{C} = \oplus_{j=0}^{\infty}\mathfrak{C}^{[j]}, \quad \mathfrak{C}^{[j]}\cdot\mathfrak{C}^{[k]} \subset \mathfrak{C}^{[j+k]}.$$

(2) Define

$$\mathrm{proj}_I(X) = X_I \quad \text{for} \quad X = \sum_{I\in\mathcal{I}} X_I\sigma^I \in \mathfrak{C}.$$

The topology in $\mathfrak{C}$ is given by $X \to 0$ in $\mathfrak{C}$ if and only if $\mathrm{proj}_I(X) \to 0$ in $\mathbb{C}$, for any $I \in \mathcal{I}$.

This topology is equivalent to the one introduced by the metric $\mathrm{dist}(X,Y) = \mathrm{dist}(X - Y)$ where $\mathrm{dist}(X)$ is defined by

$$\mathrm{dist}(X) = \sum_{I\in\mathcal{I}} \frac{1}{2^{r(I)}}\frac{|\,\mathrm{proj}_I(X)|}{1 + |\,\mathrm{proj}_I(X)|} \quad \text{with} \quad r(I) = 1 + \frac{1}{2}\sum_{k=1}^{\infty} 2^k i_k \quad \text{for} \quad I \in \mathcal{I}.$$

(3) We introduce parity in $\mathfrak{C}$ by setting

$$p(X) = \begin{cases} 0 & \text{if } X = \sum_{I\in\mathcal{I},|\mathcal{I}|=\mathrm{ev}} X_I\sigma^I, \\ 1 & \text{if } X = \sum_{I\in\mathcal{I},|\mathcal{I}|=\mathrm{od}} X_I\sigma^I, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

We put

$$\begin{cases} \mathfrak{C}_{\mathrm{ev}} = \oplus_{j=0}^{\infty}\mathfrak{C}^{[2j]} = \{X \in \mathfrak{C} \,|\, p(X) = 0\}, \\ \mathfrak{C}_{\mathrm{od}} = \oplus_{j=0}^{\infty}\mathfrak{C}^{[2j+1]} = \{X \in \mathfrak{C} \,|\, p(X) = 1\}, \\ \mathfrak{C} \cong \mathfrak{C}_{\mathrm{ev}} \oplus \mathfrak{C}_{\mathrm{od}} \cong \mathfrak{C}_{\mathrm{ev}} \times \mathfrak{C}_{\mathrm{od}}. \end{cases}$$

Analogous to $\mathfrak{C}$, we define

$$\begin{cases} \mathfrak{R} = \{X \in \mathfrak{C} \,|\, \pi_{\mathrm{B}}X \in \mathbb{R}\},\ \mathfrak{R}^{[j]} = \mathfrak{R} \cap \mathfrak{C}^{[j]}, \\ \mathfrak{R}_{\mathrm{ev}} = \mathfrak{R} \cap \mathfrak{C}_{\mathrm{ev}}, \quad \mathfrak{R}_{\mathrm{od}} = \mathfrak{R} \cap \mathfrak{C}_{\mathrm{od}} = \mathfrak{C}_{\mathrm{od}}, \\ \mathfrak{R} \cong \mathfrak{R}_{\mathrm{ev}} \oplus \mathfrak{R}_{\mathrm{od}} \cong \mathfrak{R}_{\mathrm{ev}} \times \mathfrak{R}_{\mathrm{od}}. \end{cases}$$

We introduced the body (projection) map π_{B} by

$$\pi_{\mathrm{B}}X = \mathrm{proj}_{\tilde{0}}(X) = X_{\tilde{0}} = X^{[0]} = X_{\mathrm{B}} \quad \text{for any} \quad X \in \mathfrak{C},$$

and the soul part X_{S} of X as

$$X_{\mathrm{S}} = X - X_{\mathrm{B}} = \sum_{|I|\geq 1} X_I\sigma^I.$$

We define the (real) superspace $\mathfrak{R}^{m|n}$ by

$$\mathfrak{R}^{m|n} = \mathfrak{R}_{ev}^m \times \mathfrak{R}_{od}^n.$$

The distance between $X, Y \in \mathfrak{R}^{m|n}$ is defined by,

$$\mathrm{dist}_{m|n}(X,Y) = \mathrm{dist}_{m|n}(X - Y)$$

with

$$\mathrm{dist}_{m|n}(X)=\sum_{j=1}^{m}\left(\sum_{I\in\mathcal{I}}\frac{1}{2^{r(I)}}\frac{|\,\mathrm{proj}_I(x_j)|}{1+|\,\mathrm{proj}_I(x_j)|}\right)+\sum_{k=1}^{n}\left(\sum_{I\in\mathcal{I}}\frac{1}{2^{r(I)}}\frac{|\,\mathrm{proj}_I(\theta_k)|}{1+|\,\mathrm{proj}_I(\theta_k)|}\right).$$

We use the following notation:

$$X=(X_A)_{A=1}^{m+n}=(x,\theta)\in\mathfrak{R}^{m|n}\quad\text{with}$$
$$x=(X_A)_{A=1}^{m}=(x_j)_{j=1}^{m}\in\mathfrak{R}^{m|0},\quad \theta=(X_A)_{A=m+1}^{m+n}=(\theta_k)_{k=1}^{n}\in\mathfrak{R}^{0|n}.$$

We generalize the body map π_B from $\mathfrak{R}^{m|n}$ or $\mathfrak{R}^{m|0}$ to $\mathbb{R}^m$ by putting,

$$X=(x,\theta)\in\mathfrak{R}^{m|n}\longrightarrow\pi_\mathrm{B}X=X_\mathrm{B}=(x_\mathrm{B},0)\cong x_\mathrm{B}=\pi_\mathrm{B}x=(\pi_\mathrm{B}x_1,\cdots,\pi_\mathrm{B}x_m)\in\mathbb{R}^m.$$

We call $x_j\in\mathfrak{R}_\mathrm{ev}$ and $\theta_k\in\mathfrak{R}_\mathrm{od}$ as even and odd (alias bosonic and ferminionic) variable, respectively.

[**Linear Algebra**]

Definition 2.1. *A rectangular array M, whose cells are indexed by pairs consisting of a row number and a column number, is called a supermatrix and denoted by $M\in\mathrm{Mat}\,((m|n)\times(r|s):\mathfrak{C})$, if it satisfies the following:*

1. *A $(m+n)\times(r+s)$ matrix M is decomposed blockwisely as $M=\begin{bmatrix}A & C\\ D & B\end{bmatrix}$ where A, B, C and D are $m\times r$, $n\times s$, $m\times s$ and $n\times r$ matrices with elements in $\mathfrak{C}$, respectively.*
2. *One of the following conditions is satisfied: Either*
   - *$p(M)=0$, that is, $p(A_{jk})=0=p(B_{uv})$ and $p(C_{jv})=1=p(D_{uk})$ or*
   - *$p(M)=1$, that is, $p(A_{jk})=1=p(B_{uv})$ and $p(C_{jv})=0=p(D_{uk})$.*

We call M is even denoted by $\mathrm{Mat}_\mathrm{ev}((m|n)\times(r|s):\mathfrak{C})$ (resp. odd denoted by $\mathrm{Mat}_\mathrm{od}((m|n)\times(r|s):\mathfrak{C})$) if $p(M)=0$ (resp. $p(M)=1$). Therefore, we have

$$\mathrm{Mat}\,((m|n)\times(r|s):\mathfrak{C})=\mathrm{Mat}_\mathrm{ev}((m|n)\times(r|s):\mathfrak{C})\oplus\mathrm{Mat}_\mathrm{od}((m|n)\times(r|s):\mathfrak{C}).$$

Moreover, we may decompose M as $M=M_\mathrm{B}+M_\mathrm{S}$ where

$$M_\mathrm{B}=\begin{cases}\begin{bmatrix}A_\mathrm{B} & 0\\ 0 & B_\mathrm{B}\end{bmatrix} & \text{when}\quad p(M)=0,\\[2ex] \begin{bmatrix}0 & C_\mathrm{B}\\ D_\mathrm{B} & 0\end{bmatrix} & \text{when}\quad p(M)=1.\end{cases}$$

The summation of two matrices in $\mathrm{Mat}_\mathrm{ev}((m|n)\times(r|s):\mathfrak{C})$ or in $\mathrm{Mat}_\mathrm{od}((m|n)\times(r|s):\mathfrak{C})$ is defined as usual, but the sum of $\mathrm{Mat}_\mathrm{ev}((m|n)\times(r|s):\mathfrak{C})$ and $\mathrm{Mat}_\mathrm{od}((m|n)\times(r|s):\mathfrak{C})$ is not defined except zero matrix.

It is clear that if M is the $(m+n)\times(r+s)$ matrix and N is the $(r+s)\times(p+q)$ matrix, then we may define the product MN and its parity $p(MN)$ as

$$(MN)_{ij}=\sum_k M_{ik}N_{kj},\quad p(MN)=p(M)+p(N)\quad\mathrm{mod}\ 2.$$

Moreover, we define $\mathrm{Mat}\,[m|n:\mathfrak{C}]$ as the algebra of $(m+n)\times(m+n)$ supermatrices.

Definition 2.2. *Let $M=\begin{bmatrix}A & C\\ D & B\end{bmatrix}\in\mathrm{Mat}\,[m|n:\mathfrak{C}]$. We define the supertrace of M by*

$$\mathrm{str}\ M=\mathrm{tr}\ A-(-1)^{p(M)}\ \mathrm{tr}\ B.$$

We get

Proposition 2.2. *(a) Let* $M, N \in \mathrm{Mat}\,[m|n : \mathfrak{C}]$ *such that* $p(M) + p(N) \equiv 0 \bmod 2$. *Then, we have*

$$\mathrm{str}\,(M + N) = \mathrm{str}\ M + \mathrm{str}\ N.$$

(b) M *is a matrix of size* $(m + n) \times (r + s)$ *and* N *is a matrix of size* $(r + s) \times (m + n)$. *Then,*

$$\mathrm{str}\,(MN) = (-1)^{p(M)p(N)}\,\mathrm{str}\,(NM).$$

If $M \in \mathrm{Mat}\,[m|n : \mathfrak{C}]$ is even, denoted by $M \in \mathrm{Mat}_{\,\mathrm{ev}}[m|n : \mathfrak{C}]$, then M acts on $\mathfrak{R}^{m|n}$ linearly. Denoting this by T_M, we call it super linear transformation on $\mathfrak{R}^{m|n}$ and M is called the representative matrix of T_M.

Proposition 2.3. *Let* $M \in \mathrm{Mat}_{\,\mathrm{ev}}[m|n : \mathfrak{C}]$ *and assume* $\det M_{\mathrm{B}} \neq 0$. *Then, for given* $Y \in \mathfrak{R}^{m|n}$,

$$T_M X = Y$$

has the unique solution $X \in \mathfrak{R}^{m|n}$, *which is denoted by* $X = M^{-1}Y$.

Definition 2.3. $M \in \mathrm{Mat}_{\,\mathrm{ev}}[m|n : \mathfrak{C}]$ *is called invertible or non-singular if* M_{B} *is invertible, i.e.* $\det A_{\mathrm{B}} \det B_{\mathrm{B}} \neq 0$, *and denoted by* $M \in \mathrm{GL}_{\,\mathrm{ev}}[m|n : \mathfrak{C}]$.

Definition 2.4. *Let* $B = (B_{jk})$ *be* $(\ell \times \ell)$*-matrix with elements in* $\mathfrak{C}_{\mathrm{ev}}$, *denoted by,* $B \in \mathrm{Mat}\,[\ell : \mathfrak{C}_{\mathrm{ev}}]$. *As* $\mathfrak{C}_{\mathrm{ev}}$ *is a commutative ring, we may define* $\det B$ *as usual:*

$$\det B = \sum_{\rho \in \wp_\ell} \mathrm{sgn}\,(\rho) B_{1\,\rho(1)} \cdots B_{\ell\,\rho(\ell)}.$$

Definition 2.5. *Let* M *be an even supermatrix. When* $\det B_{\mathrm{B}} \neq 0$, *we put*

$$\mathrm{sdet}\ M = (\det(A - CB^{-1}D))(\det B)^{-1}$$

and call it superdeterminant or Berezinian of M.

Theorem 2.4. *Let* $M, N \in \mathrm{Mat}\,[m|n : \mathfrak{C}]$.

(1) If M *is invertible, then we have* $\mathrm{sdet}\ M \neq 0$. *Moreover, if* A *is nonsingular, then*

$$(\mathrm{sdet}\ M)^{-1} = (\det A)^{-1}(\det(B - DA^{-1}C)).$$

(2) Multiplicativity of sdet *on* $\mathrm{GL}_{\,\mathrm{ev}}[m|n : \mathfrak{C}]$*:*

$$\mathrm{sdet}\,(MN) = \mathrm{sdet}\ M\,\mathrm{sdet}\ N.$$

(3) str *and* sdet *are matrix invariants. That is,*

$$\begin{cases} \mathrm{str}\ M = (-1)^{p(M)+p(N)}\,\mathrm{str}\ NMN^{-1} & \text{if } N \text{ is invertible},\\ \mathrm{sdet}\ M = \mathrm{sdet}\ NMN^{-1} & \text{if } M \in \mathrm{GL}_{\,\mathrm{ev}}[m|n : \mathfrak{C}] \text{ and } N \text{ is invertible.} \end{cases}$$

(4) Moreover, we have

$$\exp(\mathrm{str}\ M) = \mathrm{sdet}\,(\exp M) \quad \text{for } M \in \mathrm{GL}_{\,\mathrm{ev}}[m|n : \mathfrak{C}].$$

[**Elementary analysis**]

Supersmooth functions: For any $f(q) \in C^\infty(\mathbb{R}^m : \mathbb{C})$, we put,

$$\tilde{f}(x) = \sum_{|\alpha|=0}^{\infty} \frac{1}{\alpha!} \partial_q^\alpha f(x_{\mathrm{B}}) x_{\mathrm{S}}^\alpha \quad \text{for } x = x_{\mathrm{B}} + x_{\mathrm{S}}$$

which is called the Grassmann continuation of $f(q)$, and denoted simply by $f(x)$. We define a function $u \in \mathcal{C}_{\rm SS,ev}(\mathfrak{R}^{m|n})$ by

$$u(X) = u(x,\theta) = \sum_{|a|\le n} \tilde{u}_a(x)\theta^a = \text{or simply } \sum_{|a|\le n} u_a(x)\theta^a,$$

called a supersmooth function on $\mathfrak{R}^{m|n}$.

Example. For $\xi = (\xi_1, \cdots, \xi_m) \in \mathfrak{R}^{m|0} = \mathfrak{R}^m_{\rm ev}$, we define $|\xi| \in \mathfrak{R}_{\rm ev}$ as follows: Putting

$$|\xi| = |\xi|_{\rm B} + |\xi|_{\rm S} \quad \text{with} \quad |\xi|_{\rm S} = \sum_{|I|=\text{even}\ge 2} |\xi|_I \sigma^I, \quad |\xi|_{\rm B} \ge 0,\ |\xi|_I \in \mathbb{R},$$

we should have

$$|\xi|^2 = \sum_{j=1}^m (\xi_{j,\rm B} + \xi_{j,\rm S})(\xi_{j,\rm B} + \overline{\xi_{j,\rm S}}) = \sum_{j=1}^m \xi^2_{j,\rm B} + \sum_{j=1}^m \xi_{j,\rm B}(\xi_{j,\rm S} + \overline{\xi_{j,\rm S}}) + \sum_{j=1}^m \xi_{j,\rm S}\overline{\xi_{j,\rm S}},$$

$$\xi_{j,\rm S} = \sum_{|I|=\text{even}\ge 2} \xi_{j,I}\sigma^I,\ \overline{\xi_{j,\rm S}} = \sum_{|I|=\text{even}\ge 2} \overline{\xi_{j,I}}\sigma^I \text{with } \overline{\xi_{j,I}} \text{ being the complex conjugate of } \xi_{j,I} \text{ in } \mathbb{C}.$$

Therefore,

$$|\xi|_{\rm B} = \{\sum_{j=1}^m \xi^2_{j,\rm B}\}^{1/2},$$

$$2|\xi|_K|\xi|_{\rm B} + \sum_{I+J=K} |\xi|_I \overline{|\xi|_J}(-1)^{\tau(K;I,J)} = \sum_{j=1}^m 2\xi_{j,\rm B}\,\Re\xi_{j,K} + \sum_{I+J=K}\sum_{j=1}^m \xi_{j,I}\overline{\xi_{j,J}}(-1)^{\tau(K;I,J)}$$

which are solved by induction with respect to the length $|K|$. For example, if $|K| = 2$, we have

$$|\xi|_K = |\xi|_{\rm B}^{-1}\sum_{j=1}^m \xi_{j,\rm B}\,\Re\xi_{j,K}.$$

If $|K| = 4$,

$$2|\xi|_K = |\xi|_{\rm B}^{-1}\big(2\sum_{j=1}^m \xi_{j,\rm B}\,\Re\xi_{j,K} + \sum_{I+J=K}\sum_{j=1}^m \xi_{j,I}\overline{\xi_{j,J}}(-1)^{\tau(K;I,J)} - \sum_{I+J=K}\sum_{j=1}^m |\xi|_I|\xi|_J(-1)^{\tau(K;I,J)}\big), \quad \text{etc.}$$

We define $\sin|\xi|$, $\cos|\xi|$ as

$$\sin|\xi| = \sum_{n=0}^\infty \frac{1}{n!}\sin\big(|\xi|_{\rm B} + \frac{n\pi}{2}\big)|\xi|^n_{\rm S}, \quad \cos|\xi| = \sum_{n=0}^\infty \frac{1}{n!}\cos\big(|\xi|_{\rm B} + \frac{n\pi}{2}\big)|\xi|^n_{\rm S}.$$

We may characterize this function as a solution of a certain Cauchy-Riemann type partial differential equation. See more exactly [12, 10].

Derivations: For a given supersmooth function $u(X)$ on $\mathfrak{R}^{m|n}$, we define its derivatives as follows: For $j = 1, 2, \cdots, m$ and $k = 1, 2, \cdots, n$, we put

$$\begin{cases} U_j(X) = \sum_{|a|\le n} \partial_{x_j} u_a(x)\theta^a, \\ U_{k+m}(X) = \sum_{|a|\le n} (-1)^{l_k(a)} u_a(x)\theta_1^{a_1}\cdots\theta_k^{a_k-1}\cdots\theta_n^{a_n} \end{cases}$$

where $l_k(a) = \sum_{j=1}^{k-1} a_j$ and $\theta_k^{-1} = 0$. $U_\kappa(X)$ are called the partial derivatives of u with respect to X_κ at $X = (x,\theta)$ and are denoted by

$$\begin{cases} U_j(X) = \dfrac{\partial}{\partial x_j}u(x,\theta) = \partial_{x_j}u(x,\theta) \quad \text{for} \quad j = 1, 2, \cdots, m, \\ U_{m+s}(X) = \dfrac{\partial}{\partial \theta_s}u(x,\theta) = \partial_{\theta_s}u(x,\theta) \quad \text{for} \quad s = 1, 2, \cdots, n \end{cases}$$

or simply by

$$U_\kappa(X) = \partial_{X_\kappa} u(X) \quad \text{for} \quad \kappa = 1, \cdots, m+n.$$

For

$$\mathfrak{a} = (\alpha, a),\ \alpha = (\alpha_1, \cdots, \alpha_m) \in \mathbf{N}^m,\ a = (a_1, \cdots, a_n) \in \{0,1\}^n,$$
$$|\alpha| = \sum_{j=1}^m \alpha_j,\ \ |a| = \sum_{k=1}^n a_k,\ \ |\mathfrak{a}| = |\alpha| + |a|,$$

we put

$$\partial_X^{\mathfrak{a}} = \partial_x^\alpha \partial_\theta^a \quad \text{with} \quad \partial_x^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_m}, \quad \partial_\theta^a = \partial_{\theta_1}^{a_1} \cdots \partial_{\theta_n}^{a_n}.$$

Example. $\partial_{\theta_2}\theta_1\theta_2\theta_3 = -\theta_1\theta_3, \quad \partial_{\theta_1}\partial_{\theta_3}\theta_1\theta_2\theta_3 = \theta_2 \neq -\theta_2 = \partial_{\theta_3}\partial_{\theta_1}\theta_1\theta_2\theta_3,$ *etc.*

Integration: We define

$$\begin{aligned}\int_{\mathfrak{R}^{m|n}} dx d\theta\, u(x,\theta) &= \int_{\mathfrak{R}^{m|0}} dx \left\{ \int_{\mathfrak{R}^{0|n}} d\theta\, u(x,\theta) \right\} \\ &= \int_{\mathbb{R}^m} dX_{\mathrm{B}} (\partial_{\theta_n} \cdots \partial_{\theta_1} u)(X_{\mathrm{B}}) \quad (\pi_{\mathrm{B}}(\mathfrak{R}^{m|0}) = \mathbb{R}^m) \\ &= \int_{\mathfrak{R}^{0|n}} d\theta \left\{ \int_{\mathfrak{R}^{m|0}} dx\, u(x,\theta) \right\} = \int_{\mathfrak{R}^{m|n}} d\theta dx\, u(x,\theta).\end{aligned}$$

Especially for odd integration, we have the following curious looking but well-known relations

$$\int_{\mathfrak{R}^{0|n}} d\theta_n \cdots d\theta_1\, \theta_1 \cdots \theta_n = 1 \quad \text{and} \quad \int_{\mathfrak{R}^{0|n}} d\theta_n \cdots d\theta_1\, 1 = 0 \quad \text{(Berezin integral)}.$$

Remarks for the need of ∞ number of Grassmann generators.

(i) Though $\mathfrak{C}$ does not form a field because $X^2 = 0$ for any $X \in \mathfrak{C}_{\mathrm{od}}$, but if $X, Y \in \mathfrak{C}$ satisfy $XY = 0$ for any $Y \in \mathfrak{C}_{\mathrm{od}}$, then $X = 0$. This property holds only when the number of generators is infinite. By this, we may determine the derivative $\partial_X^{\mathfrak{a}} u(X)$ uniquely.

(ii) In general, we need at least countable number of operations in doing analysis. If the number of Grassmann generators is finite, then the effect of odd variables may vanish after finitely many operations.

Remark. Though the differential calculus on Fréchet spaces has some difficulties in general, such calculus on Fréchet-Grassmann algebra holds safely in our case. For example, the implicit and inverse function theorems, and the chain rule for differentiation. See, Inoue and Maeda [12], Inoue [9, 11].

3. The derivation of (1.7) with $\lambda - i\epsilon$ ($\epsilon > 0$ fixed) and its consequences

It is well-known that

$$\delta(q) = \frac{1}{\pi} \lim_{\epsilon \to 0} \Im \frac{1}{q - i\epsilon} = \frac{1}{2\pi i} \lim_{\epsilon \to 0} \left[\frac{1}{q - i\epsilon} - \frac{1}{q + i\epsilon} \right] = \frac{1}{\pi} \lim_{\epsilon \to 0} \frac{\epsilon}{q^2 + \epsilon^2} \quad \text{in } \mathcal{D}'(\mathbb{R}),$$

that is, for any $\phi \in C_0^\infty(\mathbb{R})$,

$$\pi^{-1} \Im \int_{\mathbb{R}} dq\, \frac{\phi(q)}{q - i\epsilon} = \pi^{-1} \int_{\mathbb{R}} dq\, \frac{\epsilon \phi(q)}{q^2 + \epsilon^2} = \pi^{-1} \int_{\mathbb{R}} dq\, \frac{\phi(\epsilon q)}{1 + q^2} \to \phi(0) = \langle \phi, \delta \rangle \quad \text{as } \epsilon \to 0.$$

Therefore, for any fixed $\phi \in C_0^\infty(\mathbb{R})$, we have

$$\int_{\mathfrak{U}_N} d\mu_N(H)\langle\phi(\cdot), \frac{1}{N}\sum_{\alpha=1}^N \delta(\cdot - E_\alpha(H))\rangle \overset{\text{def}}{=} \int_{\mathfrak{U}_N} d\mu_N(H) \lim_{\epsilon\to 0}\int_{\mathbb{R}} d\lambda\,\phi(\lambda)\frac{1}{\pi N}\sum_{\alpha=1}^N \frac{\epsilon}{(\lambda - E_\alpha(H))^2+\epsilon^2}$$

$$= \lim_{\epsilon\to 0}\int_{\mathfrak{U}_N} d\mu_N(H)\int_{\mathbb{R}} d\lambda\,\phi(\lambda)\frac{1}{\pi N}\sum_{\alpha=1}^N \frac{\epsilon}{(\lambda - E_\alpha(H))^2+\epsilon^2} \quad \text{by Lebesgue's dom. conv. theorem}$$

$$= \lim_{\epsilon\to 0}\int_{\mathbb{R}} d\lambda\,\phi(\lambda)\int_{\mathfrak{U}_N} d\mu_N(H)\frac{1}{\pi N}\sum_{\alpha=1}^N \frac{\epsilon}{(\lambda - E_\alpha(H))^2+\epsilon^2} \quad \text{by Fubini's theorem.}$$

The second equality is garanteed by the fact that for any $\phi \in C_0^\infty(\mathbb{R})$, we have, for any $\epsilon > 0$ and $H \in \mathfrak{U}_N$,

$$\Big|\int_{\mathbb{R}} d\lambda\,\phi(\lambda)\frac{1}{\pi N}\sum_{\alpha=1}^N \frac{\epsilon}{(\lambda - E_\alpha(H))^2+\epsilon^2}\Big| \le \max|\phi(\lambda)|.$$

Here, we used the fact $\int_{\mathbb{R}} d\lambda\,\epsilon(\lambda^2+\epsilon^2)^{-1} = \pi$. The third equality holds because we have

$$\Big|\phi(\lambda)\frac{1}{\pi N}\sum_{\alpha=1}^N \frac{\epsilon}{(\lambda - E_\alpha(H))^2+\epsilon^2}\Big| \le \epsilon^{-1}|\phi(\lambda)|,$$

and the right hand side is integrable w.r.t. the product measure $d\lambda\, d\mu_N(H)$ for any fixed $\epsilon > 0$.

In order to check whether we may take the limit before integration w.r.t. $d\lambda$ in the last line above, we calculate the following quantity as explicitly as possible:

$$g(\lambda,\epsilon,N) = \int_{\mathfrak{U}_N} d\mu(H)\frac{1}{\pi N}\Im\sum_{\alpha=1}^N \frac{1}{\lambda - i\epsilon - E_\alpha(H)}. \tag{3.1}$$

We claim in this section that (i) $g(\lambda,\epsilon,N)$ exists as a function of λ for any $\epsilon > 0$ and $N \in \mathbb{N}$ and (ii) $\lim_{\epsilon\to 0} g(\cdot,\epsilon,N)$ exists in $\mathcal{D}'(\mathbb{R})$ for any $N \in \mathbb{N}$ and it is denoted by $\big\langle \rho_N(\lambda)\big\rangle_N$.

Now, we put

$$z_j = x_j + iy_j,\ \bar z_j = x_j - iy_j,\ x_j,\, y_j \in \mathfrak{R}_{\text{ev}};\ \theta_k, \bar\theta_k \in \mathfrak{R}_{\text{od}} = \mathfrak{C}_{\text{od}},$$
$$X = {}^t(z,\theta),\ z = {}^t(z_1,\cdots,z_N),\ \theta = {}^t(\theta_1,\cdots,\theta_N),$$
$$X^* = (z^*,\theta^*),\ z^* = (\bar z_1,\cdots,\bar z_N),\ \theta^* = (\bar\theta_1,\cdots,\bar\theta_N).$$

Here, θ_k and $\bar\theta_k$ are considered as two different odd variables.

The following is the key formula which is well known:

Lemma 3.1. *Put $\mu = \lambda - i\epsilon$ $(\epsilon > 0)$.*

$$\begin{aligned}\operatorname{tr}\frac{1}{\mu I_N - H} &= \sum_{\alpha=1}^N \frac{1}{\mu - E_\alpha(H)}\\ &= i\int_{\mathfrak{C}^{N|2N}}\prod_{j=1}^N \frac{d\bar z_j\, dz_j}{2\pi i}\prod_{k=1}^N d\bar\theta_k d\theta_k\,(z^*\!\cdot z)\exp\left[-iX^*(I_2\otimes(\mu I_N - H))X\right].\end{aligned} \tag{3.2}$$

To prove this lemma, we need the following lemma.

Lemma 3.2. *Let Γ=the diagonal matrix with diagonal given by $(\gamma_1,\cdots,\gamma_N)$ where $\gamma_j \in \mathbb{R}$. Putting $(z^*\!\cdot z) = \sum_{j=1}^N \bar z_j z_j = |z|^2$, we have*

$$i\int_{\mathfrak{C}^{N|2N}}\prod_{j=1}^N \frac{d\bar z_j\, dz_j}{2\pi i}\prod_{k=1}^N d\bar\theta_k d\theta_k\,(z^*\!\cdot z)\exp\left[-iX^*(I_2\otimes(\Gamma - i\epsilon I_N))X\right] = \sum_{j=1}^N \frac{1}{\gamma_j - i\epsilon}. \tag{3.3}$$

Proof. Remarking

$$i\int_{\mathfrak{C}^{N|0}}\prod_{j=1}^N \frac{d\overline{z}_j\,dz_j}{2\pi i}\,(z^*\!\cdot z)\exp\left[-iz^*(\Gamma-i\epsilon I_N)z\right]=\left(\sum_{j=1}^N\frac{1}{\gamma_j-i\epsilon}\right)\prod_{j=1}^N\frac{1}{\epsilon+i\gamma_j},$$

and

$$\int_{\mathfrak{C}^{0|2N}}\prod_{k=1}^N d\overline{\theta}_k d\theta_k\,\exp\left[-i\theta^*(\Gamma-i\epsilon I_N)\theta\right]=\prod_{k=1}^N(\epsilon+i\gamma_k),$$

we get the result (3.3) readily. □

Proof of Lemma 2.1. Considering Γ as the diagonalization of $\lambda I_N - H$ for any $N\times N$-Hermitian matrix H, we get (3.2) readily. □

Lemma 3.3. *For $\mu=\lambda-i\epsilon$ ($\epsilon>0$),*

$$\Big\langle \operatorname{tr}\frac{1}{\mu I_N-H}\Big\rangle_N = i\int\prod_{j=1}^N\frac{d\overline{z}_j\,dz_j}{2\pi i}\prod_{k=1}^N d\overline{\theta}_k d\theta_k\,(z^*\!\cdot z)\exp\left[-iX^*(I_2\otimes\mu I_N)X\right]$$
$$\times\exp\Big[-\frac{J^2}{2N}\sum_{j,k=1}^N(\overline{z}_j z_k+\overline{\theta}_j\theta_k)(\overline{z}_k z_j+\overline{\theta}_k\theta_j)\Big]. \tag{3.4}$$

Proof. By definition, we have

$$\langle\operatorname{tr}\frac{1}{\mu I_N-H}\rangle_N=\int_{\mathfrak{U}_N}d\mu_N(H)\Big[i\int\prod_{j=1}^N\frac{d\overline{z}_j\,dz_j}{2\pi i}\prod_{k=1}^N d\overline{\theta}_k d\theta_k$$
$$\times(z^*\!\cdot z)\exp\left[-iX^*(I_2\otimes(\mu I_N-H)X\right]\Big]. \tag{3.5}$$

As $X^*(I_2\otimes H)X=H_{jk}(\overline{z}_j z_k+\overline{\theta}_j\theta_k)$, we have

$$\Big\langle\exp\Big[\pm i\sum_{j,k=1}^N H_{jk}(\overline{z}_j z_k+\overline{\theta}_j\theta_k)\Big]\Big\rangle_N=\exp\Big[-\frac{J^2}{2N}\sum_{j,k=1}^N(\overline{z}_j z_k+\overline{\theta}_j\theta_k)(\overline{z}_k z_j+\overline{\theta}_k\theta_j)\Big]. \tag{3.6}$$

After changing the order of integration and substituting (3.6) into (3.5), we get (3.4) . □

There are at least two approach from (3.4) to Wigner's law: The method (I) permits us to make $\epsilon\to 0$ rather easily and leads us to a not so simple looking formula but which is calculable, the other one (II) yields the beautiful formula (1.7) formally, but in order to make $\epsilon\to 0$ in that formula rigorously, we reform it until it is represented by Hermite polynomials, and at that time the beauty of the formula (1.7) is lost.

(I) The following calculation is proposed by E. Brézin [1, 2]:

Using

$$\sum_{j,k=1}^N(\overline{z}_j z_k+\overline{\theta}_j\theta_k)(\overline{z}_k z_j+\overline{\theta}_k\theta_j)=(z^*\!\cdot z)^2+2(\theta^*\!\cdot z)(z^*\!\cdot\theta)-(\theta^*\!\cdot\theta)^2,$$

we have

$$\exp\Big[\frac{J^2}{2N}(\theta^*\!\cdot\theta)^2\Big]=\left(\frac{N}{2\pi J^2}\right)^{1/2}\int_{-\infty}^{\infty}d\tau\,\exp\Big[-\tau(\theta^*\!\cdot\theta)-\frac{N}{2J^2}\tau^2\Big].$$

Substituting this relation into (3.5), we get

$$\begin{aligned}\langle \operatorname{tr} \frac{1}{\mu I_N - H}\rangle_N =& i\int \prod_{j=1}^N \frac{d\bar{z}_j\, dz_j}{2\pi i} \prod_{k=1}^N d\bar\theta_k d\theta_k\,(z^*{\cdot}z) \\ &\times \exp[-iX^*(I_2\otimes \mu I_N)X - \frac{J^2}{2N}((z^*{\cdot}z)^2 + 2(\theta^*{\cdot}z)(z^*{\cdot}\theta))] \\ &\times \left(\frac{N}{2\pi J^2}\right)^{1/2} \int_{-\infty}^{\infty} d\tau\, \exp[-\tau(\theta^*{\cdot}\theta) - \frac{N}{2J^2}\tau^2].\end{aligned}$$

Proposition 3.4. *We have the following formula:*

$$\begin{aligned}\langle \operatorname{tr} \frac{1}{(\lambda - i0)I_N - H}\rangle_N &= i\frac{1}{(N-1)!}\left(\frac{N}{2\pi J^2}\right)^{1/2}\left(\frac{N}{J^2}\right)^{N+1}\int_0^\infty ds\, s^N \exp[-\frac{N}{2J^2}(2i\lambda s + s^2)] \\ &\quad\times \int_{-\infty}^{\infty} d\tau\,(\tau + i\lambda)^{N-1}(\tau + i\lambda + s)\exp[-\frac{N}{2J^2}\tau^2] \\ &= i\frac{1}{(N-1)!}\left(\frac{N}{2\pi J^2}\right)^{1/2}\left(\frac{N}{J^2}\right)^{N+1}\iint_{\mathbb{R}_+\times\mathbb{R}} ds d\tau\,(1 + (\tau + i\lambda)^{-1}s) \\ &\quad\times \exp[-N(\frac{1}{2J^2}(\tau^2 + 2i\lambda s + s^2) - \log s(\tau + i\lambda))].\end{aligned} \tag{3.7}$$

Proof. Since

$$-i\mu(\theta^*{\cdot}\theta) - \frac{J^2}{N}(\theta^*{\cdot}z)(z^*{\cdot}\theta) - \tau(\theta^*{\cdot}\theta) = -\sum_{a,b}\bar\theta_a((\tau + i\mu)\delta_{ab} + \frac{J^2}{N}z_a\bar{z}_b)\theta_b,$$

using Lemma 2.3 and Lemma 3.6 below, we get

$$\int \prod_{k=1}^N d\bar\theta_k d\theta_k\, \exp[-i\mu(\theta^*{\cdot}\theta) - \frac{J^2}{N}(\theta^*{\cdot}z)(z^*{\cdot}\theta) - \tau(\theta^*{\cdot}\theta)] = (\tau + i\mu)^{N-1}(\tau + i\mu + \frac{J^2}{N}(z^*{\cdot}z)). \tag{3.8}$$

Using the expression (3.8), we have

$$\begin{aligned}\langle \operatorname{tr} \frac{1}{\mu I_N - H}\rangle_N =& i\int \prod_{j=1}^N \frac{d\bar{z}_j\, dz_j}{2\pi i}(z^*{\cdot}z)\exp[-i\mu(z^*{\cdot}z) - \frac{J^2}{2N}(z^*{\cdot}z)^2] \\ &\times \left(\frac{N}{2\pi J^2}\right)^{1/2}\int_{-\infty}^{\infty} d\tau\,(\tau + i\mu)^{N-1}(\tau + i\mu + \frac{J^2}{N}(z^*{\cdot}z))\exp[-\frac{N}{2J^2}\tau^2].\end{aligned}$$

Identifying $\mathbb{C}^N = \mathbb{R}^{2N}$ by $z_j = x_j + iy_j$, $\bar{z}_j = x_j - iy_j$, $d\bar{z}_j \wedge dz_j = 2i dx_j \wedge dy_j$ and using the polar coordinate $(r,\omega) \in \mathbb{R}_+ \times S^{2N-1}$ with $\prod_{j=1}^N \frac{d\bar{z}_j\, dz_j}{2\pi i} = \prod_{j=1}^N \frac{dx_j dy_j}{\pi} = \pi^{-N} r^{2N-1} dr\, d\omega$, $\int_{S^{2N-1}} d\omega = \mathrm{vol}(S^{2N-1}) = 2\pi^N/(N-1)!$, we get,

$$\begin{aligned}\langle \operatorname{tr} \frac{1}{\mu I_N - H}\rangle_N &= i\frac{1}{(N-1)!}\int_0^\infty dr^2\, r^{2N}\exp[-i\mu r^2 - \frac{J^2}{2N}r^4] \\ &\quad\times \left(\frac{N}{2\pi J^2}\right)^{1/2}\int_{-\infty}^{\infty} d\tau\,(\tau + i\mu)^{N-1}(\tau + i\mu + \frac{J^2}{N}r^2)\exp[-\frac{N}{2J^2}\tau^2].\end{aligned}$$

Changing the independent variables as $r^2 = (N/J^2)\tilde{r}$ and making $\epsilon \to 0$, i.e. $\mu \to \lambda - i0$, we get the result. Here, this procedure of making $\epsilon \to 0$ under integral sign is admitted because of Lebesgue's dominated convergence theorem. □

Remark. The formula (3.7) equals to Brézin's one (2.16) of [1] where he takes $\lambda + i\epsilon$ instead of our choice $\lambda - i\epsilon$. On the other hand, seemingly, he miscopies his equality in (44) of [2], more precisely, the term $(1 + xy)$ in (44) should be $(1 + (x - iz)^{-1}y)$.

Corollary 3.5. *For $\lambda = 0$, we get readily*

$$\langle \rho_N(0) \rangle_N = \frac{1}{\pi J}[1 - (-1)^N \frac{1}{4} N^{-1} + \frac{1}{32} N^{-2} + (-1)^N \frac{5}{128} N^{-3} + O(N^{-4})]. \tag{3.9}$$

Proof. Using (A.6), (A.7) to have

$$\begin{aligned}
\langle \rho_N(0) \rangle_N &= \frac{1}{\pi N} \Im \langle \text{tr } \frac{1}{-i0 I_N - H} \rangle_N \\
&= \frac{1}{\pi N!} \left(\frac{N}{2\pi J^2}\right)^{1/2} \left(\frac{N}{J^2}\right)^{N+1} \int_0^\infty ds\, s^N \exp[-\frac{N}{2J^2} s^2] \int_{-\infty}^\infty d\tau\, \tau^{N-1} (\tau + s) \exp[-\frac{N}{2J^2} \tau^2] \\
&= \frac{1}{\pi N!} \left(\frac{N}{2\pi J^2}\right)^{1/2} \left(\frac{N}{J^2}\right)^{N+1} \times \begin{cases} \frac{1}{2} \left(\frac{N}{2J^2}\right)^{-N-1} \Gamma\left(\frac{N+1}{2}\right)^2, & N\text{=even}, \\ \frac{1}{2} \left(\frac{N}{2J^2}\right)^{-N-1} \Gamma\left(\frac{N}{2}\right) \Gamma\left(\frac{N+2}{2}\right), & N\text{=odd}, \end{cases}
\end{aligned}$$

we may calculate explicitly by the Stirling formula. □

In proving (3.8) above, we used

Lemma 3.6. *Let $M = (M_{ab})$ with $M_{ab} = \alpha \delta_{ab} + \beta z_a \overline{z}_b$. Then, we have*

$$\det M = \alpha^{N-1} (\alpha + \beta |z|^2), \quad |z|^2 = z^* \cdot z.$$

Proof. Let u satisfy $Mu = \gamma u$. Then, we have

$$\overline{z} \cdot Mu = \gamma \overline{z} \cdot u, \quad \overline{u} \cdot Mu = \gamma \overline{u} \cdot u. \tag{3.10}$$

From the first equation above, we get

$$(\gamma - \alpha - \beta |z|^2) \sum_{j=1}^N u_j \overline{z}_j = 0.$$

If $\sum_{j=1}^N u_j \overline{z}_j \neq 0$, $\gamma = \alpha + \beta |z|^2$. On the other hand, if $\sum_{j=1}^N u_j \overline{z}_j = 0$, the second one in (3.10) implies that $\gamma = \alpha$. Taking into account the multiplicity, we have the desired result. □

(II) In order to get the expression (1.7), we proceed as follows: Putting

$$A_X = \begin{pmatrix} \sum_{j=1}^N \overline{z}_j z_j & \sum_{j=1}^N \overline{\theta}_j z_j \\ \sum_{j=1}^N \theta_j \overline{z}_j & \sum_{j=1}^N \overline{\theta}_j \theta_j \end{pmatrix},$$

we have

$$\text{str } A_X^2 = \sum_{j,k=1}^N (\overline{z}_j z_k + \overline{\theta}_j \theta_k)(\overline{z}_k z_j + \overline{\theta}_k \theta_j).$$

On the other hand, the following is known as the Hubbard-Stratonovich formula:

Lemma 3.7. *Let A be any even 2×2 supermatrix. For $Q \in \mathfrak{Q}$ given in (1.8), we have*

$$\exp[-\frac{J^2}{2N} \text{str } A^2] = \int_{\mathfrak{Q}} dQ \exp[-\frac{N}{2J^2} \text{str } Q^2 \pm i\, \text{str}\,(QA)]. \tag{3.11}$$

Proof. Let $A = \begin{pmatrix} a & \theta_1 \\ \theta_2 & b \end{pmatrix}$ with $a, b \in \mathfrak{R}_{\text{ev}}$ and $\theta_1, \theta_2 \in \mathfrak{R}_{\text{od}}$. For any $\gamma > 0$, we claim

$$\int_{\mathfrak{Q}} dQ \exp[-\frac{1}{2} \text{str}\,(\gamma Q \pm i\gamma^{-1} A)^2] = 1.$$

As we have readily

$$\text{str}\,(\gamma Q \pm i\gamma^{-1} A)^2 = \gamma^2 (x_1^2 + x_2^2 + 2\rho_1 \rho_2) \pm 2i(x_1 a + \rho_1 \theta_2 - \rho_2 \theta_1 - i x_2 b) - \gamma^{-2}(a^2 - b^2 + 2\theta_1 \theta_2),$$

we get

$$\int d\rho_1 d\rho_2 \, \exp[-\gamma^2\rho_1\rho_2 \mp i(\rho_1\theta_2-\rho_2\theta_1)+\gamma^{-2}\theta_1\theta_2] = (\gamma^2-\theta_1\theta_2)(1+\gamma^{-2}\theta_1\theta_2)=\gamma^2,$$

$$\int \frac{dx_1dx_2}{2\pi} \, \exp[-\frac{\gamma^2}{2}(x_1^2+x_2^2)\mp i(x_1a-ix_2b)+\frac{\gamma^{-2}}{2}(a^2-b^2)]$$
$$= \int \frac{dx_1dx_2}{2\pi} \, \exp[-\frac{1}{2}(\gamma x_1 \pm i\gamma^{-1}a)^2-\frac{1}{2}(\gamma x_2\pm\gamma^{-1}b)^2]=\gamma^{-2}. \quad \square$$

Substituting (3.11) with $A=A_X$ into (3.4), noting $\mathrm{str}\,(QA_X)=X^*(Q\otimes I_N)X$, taking the part of integral and changing the order of integration, we have

$$i\int \prod_{j=1}^N \frac{d\bar z_j\,dz_j}{2\pi i}\prod_{k=1}^N d\bar\theta_k d\theta_k\,(z^*\cdot z)\exp[-iX^*\big((\mu I_2-Q)\otimes I_N\big)X]$$
$$=\sum_{j=1}^N(\{(\mu I_2-Q)\otimes I_N\}^{-1})_{bb,jj}\,\mathrm{sdet}^{-1}(i(\mu I_2-Q)\otimes I_N).$$

Therefore, we have

Lemma 3.8. *For $\mu=\lambda-i\epsilon$ ($\epsilon>0$),*

$$\big\langle \mathrm{tr}\,\frac{1}{\mu I_N-H}\big\rangle_N=\int_{\mathfrak{Q}} dQ\sum_{j=1}^N(\{(\mu I_2-Q)\otimes I_N\}^{-1})_{bb,jj}\,\mathrm{sdet}^{-1}(i(\mu I_2-Q)\otimes I_N). \tag{3.12}$$

Here, $(C)_{bb,jj}$ is the j-th diagonal element of the boson-boson block of the (even) supermatrix C.

Remarking

$$(\{\mu I_2-Q)\otimes I_N\}^{-1})_{bb,jj}=(\{\mu I_2-Q\}^{-1})_{bb} \quad \text{for any} \quad j=1,2,\cdots,N,$$
$$\mathrm{sdet}^{-1}(i(\mu I_2-Q)\otimes I_N)=\mathrm{sdet}^{-N}(\mu I_2-Q),$$
$$\mathrm{str}\,(AB)=\mathrm{str}\,(BA),\quad \mathrm{str}\,(A+B)=\mathrm{str}\,A+\mathrm{str}\,B,\quad \log(\mathrm{sdet}^{\ell}A)=\ell\,\mathrm{str}\,(\log A)\quad\text{for}\quad \ell\in\mathbb{Z},$$

we have

$$\big\langle \mathrm{tr}\,\frac{1}{\mu I_N-H}\big\rangle_N=\int_{\mathfrak{Q}} dQ\,N\big(\{\mu I_2-Q\}^{-1}\big)_{bb}\exp[-N\mathcal{L}(\mu;Q)] \tag{3.13}$$

with

$$\mathcal{L}(\mu;Q)=\mathrm{str}\,[(2J^2)^{-1}Q^2+\log(\mu I_2-Q)],$$
$$\big(\{\mu I_2-Q)\}^{-1}\big)_{bb,11}=\frac{\mu-ix_2}{(\mu-x_1)(\mu-ix_2)-\rho_1\rho_2}=\frac{(\mu-x_1)(\mu-ix_2)+\rho_1\rho_2}{(\mu-x_1)^2(\mu-ix_2)},$$
$$\mathfrak{Q}=\big\{Q=\begin{pmatrix}x_1&\rho_1\\ \rho_2&ix_2\end{pmatrix}\,\big|\,x_1,x_2\in\mathfrak{R}_{\mathrm{ev}},\ \rho_1,\rho_2\in\mathfrak{R}_{\mathrm{od}}\big\}.$$

Remark. If we could make directly $\epsilon\to0$ in (3.13), we had the formula (1.7). We claim at least symbolically we do that.

Lemma 3.9. *For $\mu=\lambda-i\epsilon$ ($\epsilon>0$),*

$$\frac{1}{N}\langle\mathrm{tr}\,\frac{1}{\mu I_N-H}\rangle_N=\int_{\mathbb{R}^2}\frac{dx_1dx_2}{2\pi}\frac{N(\mu-x_1-ix_2)}{J^2(\mu-x_1)(\mu-ix_2)}\exp[-N\Phi(x_1,x_2;\mu)], \tag{3.14}$$

where

$$\Phi(x_1,x_2;\mu)=\frac{x_1^2+x_2^2}{2J^2}+\log\frac{\mu-x_1}{\mu-ix_2}.$$

Proof. As the integrand in (3.13) is represented by

$$\frac{(\mu-x_1)(\mu-ix_2)+\rho_1\rho_2}{(\mu-x_1)^2(\mu-ix_2)}\exp\,[-N\{\frac{1}{2J^2}(x_1^2+x_2^2+2\rho_1\rho_2)+\log\frac{\mu-x_1}{\mu-ix_2}-\frac{\rho_1\rho_2}{(\mu-x_1)(\mu-ix_2)}\}],$$

we have

$$\begin{aligned}\int d\rho_1 d\rho_2\,\frac{(\mu-x_1)(\mu-ix_2)+\rho_1\rho_2}{(\mu-x_1)^2(\mu-ix_2)}\exp\,[-N\{\frac{1}{J^2}-\frac{1}{(\mu-x_1)(\mu-ix_2)}\}\rho_1\rho_2]\\ =\frac{-1}{(\mu-x_1)^2(\mu-ix_2)}+\frac{N}{\mu-x_1}\{\frac{1}{J^2}-\frac{1}{(\mu-x_1)(\mu-ix_2)}\}.\end{aligned}\tag{3.15}$$

Remarking $(\mu-x_1)^{-2}=\partial_{x_1}(\mu-x_1)^{-1}$, by integration by parts, we have

$$\begin{aligned}\int\frac{dx_1dx_2}{2\pi}\frac{-1}{(\mu-x_1)^2(\mu-ix_2)}\exp\,[-N\Phi(x_1,x_2;\mu)]\\ =\int\frac{dx_1dx_2}{2\pi}\frac{-N}{(\mu-x_1)(\mu-ix_2)}\{\frac{x_1}{J^2}-\frac{1}{\mu-x_1}\}\exp\,[-N\Phi(x_1,x_2;\mu)],\end{aligned}$$

which yields (3.14). □

Remark. As the right-hand side of (3.14) is rewritten

$$\frac{1}{N}\langle\mathrm{tr}\,\frac{1}{\mu I_N-H}\rangle_N=\frac{N}{2\pi J^2}\int_{\mathbb{R}^2}dx_1dx_2\frac{(\mu-x_1-ix_2)(\mu-ix_2)^{N-1}}{(\mu-x_1)^{N+1}}\exp\,[-N\frac{x_1^2+x_2^2}{2J^2}],\tag{3.16}$$

there is no singularity in the integrand when $\Im\mu\neq 0$. □

Using the fact that for any real smooth integrable function f,

$$\lim_{\epsilon\to 0}\Im\int_{\mathbb{R}}dx\,(\lambda-i\epsilon-x)^{-1}f(x)=\lim_{\epsilon\to 0}\int_{\mathbb{R}}dx\frac{\epsilon}{(\lambda-x)^2+\epsilon^2}f(x)=\pi f(\lambda),$$

and integrating by parts based on $\partial_{x_1}^{\ell}(\mu-x_1)^{-1}=\ell!(\mu-x_1)^{-\ell-1}$, we have,

$$\lim_{\epsilon\to 0}\Im\int_{\mathbb{R}}dx_1\,(\lambda-i\epsilon-x_1)^{-\ell-1}\exp[-N\frac{x_1^2}{2J^2}]=\pi\frac{(-1)^{\ell}}{\ell!}\partial_{\lambda}^{\ell}\exp[-N\frac{\lambda^2}{2J^2}].$$

Using the Hermite polynomial $H_\ell(x)$ defined by

$$H_\ell(x)=(-1)^{\ell}e^{x^2/2}\partial_x^{\ell}e^{-x^2/2}=\sum_{k=0}^{[\ell/2]}\frac{(-1)^k\ell!x^{\ell-2k}}{2^k k!(\ell-2k)!}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dt\,e^{-t^2/2}(x\mp it)^{\ell},$$

with

$$H_\ell(\gamma x)=\frac{\gamma^{\ell+1}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dt\,e^{-\gamma^2t^2/2}(x\mp it)^{\ell},$$

we have

Lemma 3.10.

$$\pi\frac{(-1)^{\ell}}{\ell!}\partial_{\lambda}^{\ell}\exp[-N\frac{\lambda^2}{2J^2}]=\frac{\pi}{\ell!}\left(\frac{N}{J^2}\right)^{(2\ell+1)/2}\exp[-N\frac{\lambda^2}{2J^2}]\int_{\mathbb{R}}dt\,(\lambda\mp it)^{\ell}\exp[-N\frac{t^2}{2J^2}].\tag{3.17}$$

Proof. Using Bell's polynomial, we have

$$\begin{aligned}\pi\frac{(-1)^{\ell}}{\ell!}\partial_{\lambda}^{\ell}\exp[-N\frac{\lambda^2}{2J^2}]&=\pi\sum_{k=0}^{[\ell/2]}\frac{(-1)^k2^{-k}}{k!(\ell-2k)!}\left(\frac{N}{J^2}\right)^{\ell-k}\lambda^{\ell-2k}\exp[-N\frac{\lambda^2}{2J^2}]\\ &=\frac{\pi}{\ell!}\left(\frac{N}{J^2}\right)^{\ell/2}H_\ell\left(\left(\frac{N}{J^2}\right)^{1/2}\lambda\right)\exp[-N\frac{\lambda^2}{2J^2}].\quad\Box\end{aligned}$$

On the other hand,

$$\int_{\mathbb{R}}dx_2\,(\mu-ix_2)^{\ell}\exp\,[-N\frac{x_2^2}{2J^2}]=\sqrt{2\pi}\left(\frac{N}{J^2}\right)^{-(\ell+1)/2}H_\ell\left(\left(\frac{N}{J^2}\right)^{1/2}\mu\right),$$

and

$$\int_{\mathbb{R}} dx_2\,(-ix_2)(\mu-ix_2)^\ell \exp[-N\frac{x_2^2}{2J^2}] = -\ell\frac{J^2}{N}\int_{\mathbb{R}} dx_2\,(\mu-ix_2)^{\ell-1}\exp[-N\frac{x_2^2}{2J^2}]$$
$$= -\ell\sqrt{2\pi}\Big(\frac{N}{J^2}\Big)^{-(\ell+2)/2} H_{\ell-1}\Big(\Big(\frac{N}{J^2}\Big)^{1/2}\mu\Big).$$

Therefore, we have

$$\big\langle \rho_N(\lambda)\big\rangle_N = \frac{N}{2\pi^2 J^2}\lim_{\epsilon\to 0}(\Im K_1 + \Im K_2),$$

where

$$K_1 = \int_{\mathbb{R}^2} dx_1 dx_2\,(\mu-x_1)^{-N}(\mu-ix_2)^{N-1}\exp[-N\frac{x_1^2+x_2^2}{2J^2}],$$
$$K_2 = \int_{\mathbb{R}^2} dx_1 dx_2\,(\mu-x_1)^{-N-1}(-ix_2)(\mu-ix_2)^{N-1}\exp[-N\frac{x_1^2+x_2^2}{2J^2}].$$

Moreover, we get

$$\lim_{\epsilon\to 0}\Im K_1 = \frac{\pi}{(N-1)!}\Big(\frac{N}{J^2}\Big)^{(N-1)/2} H_{N-1}\Big(\Big(\frac{N}{J^2}\Big)^{1/2}\lambda\Big)\exp[-N\frac{\lambda^2}{2J^2}]$$
$$\times\sqrt{2\pi}\Big(\frac{N}{J^2}\Big)^{-N/2} H_{N-1}\Big(\Big(\frac{N}{J^2}\Big)^{1/2}\lambda\Big),$$
$$\lim_{\epsilon\to 0}\Im K_2 = \frac{\pi}{N!}\Big(\frac{N}{J^2}\Big)^{N/2} H_N\Big(\Big(\frac{N}{J^2}\Big)^{1/2}\lambda\Big)\exp[-N\frac{\lambda^2}{2J^2}]$$
$$\times(-1)(N-1)\sqrt{2\pi}\Big(\frac{N}{J^2}\Big)^{-(N+1)/2} H_{N-2}\Big(\Big(\frac{N}{J^2}\Big)^{1/2}\lambda\Big).$$

Combining these, we have proved

Proposition 3.11. *For any $\lambda\in\mathbb{R}$, we have*

$$\begin{aligned}\big\langle \rho_N(\lambda)\big\rangle_N &= \frac{1}{\sqrt{2\pi}J(N-1)!}\exp[-\frac{N}{2J^2}\lambda^2]\Big[\sqrt{N}H_{N-1}^2\Big(\Big(\frac{N}{J^2}\Big)^{1/2}\lambda\Big)\\ &\qquad -\frac{N-1}{\sqrt{N}}H_N\Big(\Big(\frac{N}{J^2}\Big)^{1/2}\lambda\Big)H_{N-2}\Big(\Big(\frac{N}{J^2}\Big)^{1/2}\lambda\Big)\Big]\\ &= \Big(\frac{N}{2\pi J^2}\Big)^{1/2}\frac{1}{2\pi(N-1)!}\Big(\frac{N}{J^2}\Big)^N\iint_{\mathbb{R}^2} dt\,ds\,\exp[-N\phi_\pm(t,s,\lambda)]a_\pm(t,s,\lambda;N),\end{aligned}\tag{3.18}$$

where

$$\begin{aligned}\phi_\pm(t,s,\lambda) &= \frac{1}{2J^2}(t^2+s^2+\lambda^2)-\log(\lambda\mp it)(\lambda\mp is),\\ a_\pm(t,s,\lambda:N) &= \frac{1}{(\lambda\mp it)(\lambda\mp is)}-\frac{1}{2}(1-N^{-1})\Big[\frac{1}{(\lambda\mp it)^2}+\frac{1}{(\lambda\mp is)^2}\Big].\end{aligned}\tag{3.19}$$

4. The proof of semi-circle law and beyond that

Now, we study the asymptotic behavior of the following integral w.r.t. N:

$$\begin{aligned}\Big\langle \operatorname{tr}\frac{1}{(\lambda-i0)I_N-H}\Big\rangle_N &= i\frac{1}{(N-1)!}\Big(\frac{N}{2\pi J^2}\Big)^{1/2}\Big(\frac{N}{J^2}\Big)^{N+1}(I_1+I_2),\\ I_1 &= \int_0^\infty ds\,s^N\exp[-\frac{N}{2J^2}(2i\lambda s+s^2)]\int_{-\infty}^\infty d\tau\,(\tau+i\lambda)^N\exp[-\frac{N}{2J^2}\tau^2],\\ I_2 &= \int_0^\infty ds\,s^{N+1}\exp[-\frac{N}{2J^2}(2i\lambda s+s^2)]\int_{-\infty}^\infty d\tau\,(\tau+i\lambda)^{N-1}\exp[-\frac{N}{2J^2}\tau^2].\end{aligned}\tag{4.1}$$

Theorem 4.1. *Let $|\lambda| < 2J$. Putting $\theta = -\arg\tau_+$, $\tau_+ = 2^{-1}(-i\lambda + \sqrt{4J^2-\lambda^2})$, we have*

$$I_1 + I_2 = 2\pi e^{-N} J^{2(N+1)} \left[e^{-i\theta}N^{-1} + \frac{1}{12}\left(e^{-i\theta} - (-1)^N \frac{3e^{-iN(\sin 2\theta + 2\theta)}}{\cos^2\theta} \right) N^{-2} + O(N^{-3}) \right]. \tag{4.2}$$

For the proof, see Appendix A.4.

Remarking the Stirling formula

$$(N-1)! = e^{-N}N^{N-1/2}\sqrt{2\pi}(1 + \frac{1}{12}N^{-1} + \frac{1}{288}N^{-2} - \frac{139}{51840}N^{-3} - \frac{571}{2488320}N^{-4} + O(N^{-5})),$$

that is,

$$\frac{1}{(N-1)!} = \frac{e^N N^{-N+1/2}}{\sqrt{2\pi}}(1 - \frac{1}{12}N^{-1} - \frac{1}{96}N^{-2} + O(N^{-3})), \tag{4.3}$$

we get

$$\begin{aligned}
\Big\langle \mathrm{tr}\,\frac{1}{(\lambda - i0)I_N - H}\Big\rangle_N &= i\frac{1}{(N-1)!}\left(\frac{N}{2\pi J^2}\right)^{1/2}\left(\frac{N}{J^2}\right)^{N+1}(I_1+I_2),\\
&= i\frac{N^2}{J}\left(e^{-i\theta}N^{-1} + \frac{1}{12}[e^{-i\theta} - (-1)^N\frac{3e^{-iN(\sin 2\theta+2\theta)}}{\cos^2\theta}]N^{-2} + O(N^{-3})\right)(1 - \frac{1}{12}N^{-1} + O(N^{-2}))\\
&= i\frac{N}{J}\left(e^{-i\theta} - \frac{(-1)^N}{4}\frac{e^{-iN(\sin 2\theta+2\theta)}}{\cos^2\theta}N^{-1} + O(N^{-2})\right).
\end{aligned}$$

Therefore, we proved the first part of Theorem 1.2. □

The relation (1.5) for $|\lambda| \geq 2J$ is proved analogously: That is, we have

Theorem 4.2. *Let $\lambda > 2J$. There exists constsnt $k(\lambda) > 0$ and $C(\lambda) > 0$ such that*

$$I_1 + I_2 = J^{2N}e^{-N}K(N) + \textit{pure imaginary part} \quad \textit{with} \quad \big|K(N)\big| \leq C(\lambda)N^{-\frac{1}{2}}e^{-k(\lambda)N}.$$

See, Appendix A.4, for the proof.

Substituting this estimate into the definition of $\big\langle \rho_N(\lambda)\big\rangle_N$, we get

$$\begin{aligned}
\big\langle \rho_N(\lambda)\big\rangle_N &= \Im\frac{1}{\pi N}\Big\langle \mathrm{tr}\,\frac{1}{(\lambda - i0)I_N - H}\Big\rangle_N\\
&= \Im i\frac{1}{\pi N!}\left(\frac{N}{2\pi J^2}\right)^{1/2}\left(\frac{N}{J^2}\right)^{N+1}(I_1+I_2),\\
&= \Im i\frac{1}{\pi N!}\left(\frac{N}{2\pi J^2}\right)^{1/2}\left(\frac{N}{J^2}\right)^{N+1}J^{2N}e^{-N}\,(K(N) + \text{pure imaginary part})\\
&= \frac{1}{\pi N!}\left(\frac{N}{2\pi J^2}\right)^{1/2}\left(\frac{N}{J^2}\right)^{N+1}J^{2N}e^{-N}K(N).
\end{aligned}$$

Applying the Stirling formula to the last line of the above, we get the estimate (1.5). □

5. Edge mobility

To study the asymptotic behavior of $\big\langle \rho_N(2J - zN^{-2/3})\big\rangle_N$, or $\big\langle \rho_N(-2J + zN^{-2/3})\big\rangle_N$ for $|z| \leq 1$ as $N \to \infty$, we use in this section the formula (3.18):

$$\begin{aligned}
\big\langle \rho_N(2J - zN^{-2/3})\big\rangle_N = \frac{N^{N+1/2}}{(2\pi)^{3/2}(N-1)!J^{2N+1}} &\iint_{\mathbb{R}^2} dt\,ds\,\exp\left[-N\phi_+(t,s,2J - zN^{-2/3})\right]\\
&\times a_+(t,s,2J-zN^{-2/3};N),
\end{aligned} \tag{5.1}$$

$$\langle \rho_N(-2J+zN^{-2/3})\rangle_N = \frac{N^{N+1/2}}{(2\pi)^{3/2}(N-1)!J^{2N+1}} \iint_{\mathbb{R}^2} dt\,ds\, \exp\,[-N\phi_-(t,s,-2J+zN^{-2/3})] \times a_-(t,s,-2J+zN^{-2/3};N), \tag{5.2}$$

where

$$\begin{aligned} \phi_\pm(t,s,\lambda) &= \frac{1}{2J^2}(t^2+s^2+\lambda^2) - \log(\lambda\mp it)(\lambda\mp is), \\ a_\pm(t,s,\lambda;N) &= \frac{2(\lambda\mp it)(\lambda\mp is)-(1-N^{-1})[(\lambda\mp it)^2+(\lambda\mp is)^2]}{2(\lambda\mp it)^2(\lambda\mp is)^2}. \end{aligned} \tag{5.3}$$

Proposition 5.1. *For $|z|\le 1$, we have*

$$\langle \rho_N(2J-zN^{-2/3})\rangle_N = \frac{N^{-1/3}}{8\pi^2 J^5}\iint_{\mathbb{R}^2} dx\,dy\,(x-y)^2 \exp\,[-\frac{i}{3J^3}(x^3+y^3-3zJ(x+y))] + O(N^{-2/3}). \tag{5.4}$$

The right-hand integral above, should be interpreted as the oscillatory one.

Proof. In this proof, we abbreviate the subscript $+$ of ϕ_+ and a_+.

Put $u=N^{-1/3}$. For $\lambda=2J-zu^2$, using the change of variables $s=-iJ+yu$, $t=-iJ+xu$, we have

$$\varphi(x,y,z;u)=\phi(-iJ+xu,-iJ+yu,2J-zu^2)=h(u)-\log g(u),$$

where

$$\begin{aligned} h(u) &= \frac{1}{2J^2}((-iJ+xu)^2+(-iJ+yu)^2+(2J-zu^2)^2) \\ &= 1-\frac{i(x+y)}{J}u+\frac{x^2+y^2-4zJ}{2J^2}u^2+\frac{z^2}{2J^2}u^4, \\ g(u) &= (2J-zu^2-i(-iJ+xu))(2J-zu^2-i(-iJ+yu)) \\ &= J^2-iJ(x+y)u-(xy+2zJ)u^2+iz(x+y)u^3+z^2u^4. \end{aligned}$$

Analogously, we put

$$\begin{aligned} \alpha(x,y,z;u) &= a(-iJ+xu,-iJ+yu,2J-zu^2;u^{-3}) \\ &= \frac{(x-y)^2u^2+u^3[2J^2-2iJ(x+y)u-(x^2+y^2+4zJ)u^2+2i(x+y)zu^3+2z^2u^4]}{2g^2(u)}. \end{aligned}$$

Using Taylor's expansion of $\varphi(x,y,z;u)$ w.r.t. u at $u=0$, we get

$$\varphi(x,y,z;u)=1-\log J^2+\frac{i}{3J^3}[x^3+y^3-3zJ(x+y)]u^3+R(u),$$

with

$$\begin{aligned} R(u) &= \frac{u^4}{3!}\int_0^1 d\tau\,(1-\tau)^3\varphi^{(4)}(x,y,z;\tau u), \\ \varphi^{(4)}(x,y,z;u) &= \frac{4!z^2}{g(u)}-\frac{3(g''(u)^2+g'(u)g^{(3)}(u))}{g(u)^2}+\frac{12g'(u)^2g''(u)}{g(u)^3}-\frac{6g'(u)^4}{g(u)^4}, \\ |\partial_x^k\partial_y^\ell R(u)| &\le C_{k,\ell}u^4 \quad \text{for } u\ge 0,\ x,\ y\in\mathbb{R},\ |z|\le 1,\ k+\ell\le 2. \end{aligned}$$

Moreover,

$$e^{-NR(u)}=1+S(u),\quad S(u)=-u^{-2}\int_0^1 d\tau\,R'(\tau u),$$

$$R'(u)=\frac{2u^3}{3}\int_0^1 d\tau\,(1-\tau)^3\varphi^{(4)}(x,y,z;\tau u)+\frac{u^5}{6}\int_0^1 d\tau\,(1-\tau)^3\varphi^{(5)}(x,y,z;\tau u).$$

Therefore, we have

$$\exp\,[-N\varphi(x,y,z;N^{-1/3})]=e^{-N}J^{2N}\exp\,[-\frac{i}{3J^3}(x^3+y^3-3zJ(x+y))]e^{-NR(N^{-1/3})}.$$

On the other hand, as we have

$$g(u)^{-2} = J^{-4} - 2u \int_0^1 d\tau\, g'(\tau u) g(\tau u)^{-3},$$

we get

$$\alpha(x,y,z;u) = \frac{(x-y)^2}{2J^4} u^2 + A(u), \quad A(u) = -u^3 (x-y)^2 \int_0^1 d\tau\, g'(\tau u) g(\tau u)^{-3}.$$

with

$$|\partial_x^k \partial_y^\ell A(u)| \le C_{k,\ell} u^3 \quad \text{for } u \ge 0,\ x,\ y \in \mathbb{R},\ |z| \le 1,\ k+\ell \le 2.$$

Combining these, we get

$$\begin{aligned}
&\iint_{\mathbb{R}^2} dt\, ds\, \exp[-N\phi(t,s,2J - zN^{-2/3})]\, a(t,s,2J - zN^{-2/3}; N) \\
&\quad = N^{-2/3} \iint_{\mathbb{R}^2} dx\, dy\, \exp[-N\varphi(x,y,z;N^{-1/3})] \alpha(x,y,z;N^{-1/3}) \\
&\quad = e^{-N} J^{2N} N^{-2/3} \iint_{\mathbb{R}^2} dx\, dy\, \exp[-\frac{i}{3J^3}(x^3 + y^3 - 3zJ(x+y))] \\
&\qquad\qquad \times (1 + S(N^{-1/3})) \Big(\frac{(x-y)^2}{2J^4} N^{-2/3} + A(N^{-1/3}) \Big) \\
&\quad = e^{-N} J^{2N} \Big[N^{-4/3} \iint_{\mathbb{R}^2} dx\, dy\, \frac{(x-y)^2}{2J^4} \exp[-\frac{i}{3J^3}(x^3 + y^3 - 3zJ(x+y))] + O(N^{-5/3}) \Big].
\end{aligned} \tag{5.5}$$

Here, we applied the lemma below to

$$f(x,y) = A(u),\ A(u)S(u),\ S(u)\frac{(x-y)^2}{2J^2} u^2,$$

for getting the last term $O(N^{-5/3})$.

Moreover, we may rewrite the above (5.5) using the Stirling formula to get

$$\begin{aligned}
\big\langle \rho_N(2J - zN^{-2/3}) \big\rangle_N = \frac{N^{-1/3}}{8\pi^2 J^5} \iint_{\mathbb{R}^2} dx\, dy\, \times \exp[-\frac{i}{3J^3}(x^3 + y^3 - 3zJ(x+y))](x-y)^2 \\
+ O(N^{-2/3}). \quad \square
\end{aligned}$$

Lemma 5.2. *If f satisfies*

$$|\partial_x^k \partial_y^\ell f(x,y)| \le C_{k,\ell},$$

we have

$$\Big| \iint_{\mathbb{R}^2} dxdy\, f(x,y) \exp[-\frac{i}{3J^3}(x^3 + y^3 - 3zJ(x+y))] \Big| \le C < \infty.$$

Proof. We use Lax's technique to estimate the oscillatory integrals noting

$$(1 - \partial_x^2 - \partial_y^2) \exp[-\frac{i}{3J^3}(x^3 + y^3 - 3zJ(x+y))] = \Phi(x,y) \exp[-\frac{i}{3J^3}(x^3 + y^3 - 3zJ(x+y))]$$

with

$$\Phi(x,y) = 1 + 18z^2 J^2 + \frac{2i(x+y)}{J^3} + \frac{6z(x^2+y^2)}{J^2} + \frac{x^4 + y^4}{J^6}.$$

Therefore, we have

$$\begin{aligned}
&\iint_{\mathbb{R}^2} dxdy\, f(x,y) \exp[-\frac{i}{3J^3}(x^3 + y^3 - 3zJ(x+y))] \\
&\quad = \iint_{\mathbb{R}^2} dxdy\, (1 - \partial_x^2 - \partial_y^2) \frac{f(x,y)}{\Phi(x,y)} \exp[-\frac{i}{3J^3}(x^3 + y^3 - 3zJ(x+y))].
\end{aligned}$$

By the assumption, $(1 - \partial_x^2 - \partial_y^2)(f(x,y)/\Phi(x,y))$ is integrable w.r.t. $dxdy$, we get the desired result. $\square$

Using the Airy function defined by

$$\mathrm{Ai}(z) = \int_{\mathbb{R}} dx\, \exp[-\frac{i}{3}x^3 + izx] = \int_{\mathbb{R}} dx\, \exp[\frac{i}{3}x^3 - izx] = \overline{\mathrm{Ai}(z)} \quad \text{for } z \in \mathbb{R},$$

we have

$$\int_{\mathbb{R}} dx\, \exp[-\frac{ix^3}{3J^3}+\frac{izx}{J^2}] = J\,\mathrm{Ai}(\frac{z}{J}),$$
$$\int_{\mathbb{R}} dx\, x\, \exp[-\frac{ix^3}{3J^3}+\frac{izx}{J^2}] = -iJ^2\,\mathrm{Ai}'(\frac{z}{J}),$$
$$\int_{\mathbb{R}} dx\, x^2\, \exp[-\frac{ix^3}{3J^3}+\frac{izx}{J^2}] = -J^3\,\mathrm{Ai}''(\frac{z}{J}).$$

And we get

$$\big\langle \rho_N(2J-zN^{-2/3})\big\rangle_N = N^{-1/3}f(zJ^{-1})+O(N^{-2/3}),$$

where

$$f(z)=\frac{1}{4\pi^2 J}(\mathrm{Ai}'(z)\,\mathrm{Ai}'(z)-\mathrm{Ai}''(z)\,\mathrm{Ai}(z)). \quad \square$$

Corollary 5.3. *For $|z|\le 1$, we have*

$$\big\langle \rho_N(-2J+zN^{-2/3})\big\rangle_N = -N^{-1/3}f(zJ^{-1})+O(N^{-2/3}).$$

Remark. Though Brézin and Kazakov applied the Brézin formula (2.7) to obtain the analogous statement, but we can't follow their proof (48) of [2].

Appendix A. A typical application of the saddle points method

It seems rather rare to find "a complete prescription" of the saddle point method for a concrete example, see the old book of de Bruijn [5], pp.77-78 :

> The saddle point method, due to B. Riemann and P. Debye, is one of the most important and most powerful methods in asymptotics. ⋯ (omission) ⋯
>
> Any special application of the saddle point method consists of two stages.
>
> (i) The stage of exploring, conjecturing and scheming, which is usually the most difficult one. It results in choosing a new integration path, made ready for application of (ii).
>
> (ii) The stage of carrying out the method. Once the path has been suitably chosen, this second stage is, as a rule, rather a matter of routine, although it may be complicated. It essentially depends on the Laplace method of Ch.4.
>
> ⋯ (omission) ⋯ Most authors dealing with special applications do not go into the trouble of explaining what arguments led to their choice of path. The main reason is that it is always very difficult to say why a certain possibility is tried and others are discarded, especially since this depends on personal imagination and experience.

Therefore, for the future use, we describe an application of this method to the problem in hand as complete as possible.

A.1. **A simple example: the case when $|\lambda|<2J$.** In this subsection, we consider the asymptotic behaviour of

$$\int_{\mathbb{R}} d\tau\, e^{-N\phi(\tau)} \quad \text{as} \quad N\to\infty \quad \text{where} \quad \phi(\tau)=\frac{\tau^2}{2J^2}-\log(\tau+i\lambda). \tag{A.1}$$

For $|\lambda|<2J$, we put

$$\alpha=\sqrt{4J^2-\lambda^2}>0 \quad \text{and} \quad \tau_\pm=\frac{-i\lambda\pm\alpha}{2}$$

where $\tau_\pm$ are the critical points of ϕ, namely the roots of $\phi'(\tau) = \tau/J^2 - 1/(\tau + i\lambda) = 0$.

Assuming that no confusion occurs for abusing the symbol θ, we put

$$\begin{aligned} &\theta = -\arg \tau_+ \text{ (this implies } \alpha = 2J\cos\theta,\ \lambda = 2J\sin\theta), \\ &a_2 = \frac{\cos\theta}{J^2},\ \beta = \frac{e^{-i\theta/2}}{J},\ t_1 = J\frac{\cos\theta}{\cos(\theta/2)} \quad \text{and} \quad t_0 = 2J\cos\frac{\theta}{2}. \end{aligned} \tag{A.2}$$

Therefore, we have

$$\tau_\pm = \begin{cases} Je^{-i\theta}, \\ -Je^{i\theta}, \end{cases} \qquad \phi(\tau_\pm) = \begin{cases} 2^{-1}e^{-2i\theta} - \log(Je^{i\theta}), \\ 2^{-1}e^{2i\theta} - \log(-Je^{-i\theta}), \end{cases} \qquad \phi''(\tau_\pm) = \begin{cases} J^{-2}(1+e^{-2i\theta}), \\ J^{-2}(1+e^{2i\theta}). \end{cases}$$

Moreover, we may show that the critical points $\tau_\pm$ are saddle points: If we complexify τ and put $\tilde\phi(t,\omega) = \phi(\tau_+ + te^{i\omega})$, then we have $\tilde\phi_t(0,\omega) = 0$ and $\tilde\phi_{tt}(0,\omega) = e^{2i\omega - i\theta}|\phi''(\tau_+)|$. Therefore, we have $\tilde\phi_{tt}(0,\theta/2) > 0$ and $\tilde\phi_{tt}(0,\theta/2+\pi/2) < 0$.

As ϕ is holomorphically extended near $\tau_\pm$, we may deform the path of integration such that $\tau_\pm$ are crossed by this path. We parameterize the pathes l_1 and l_2 by

$$l_1; \tau_1(t) = \tau_+ + te^{i\theta/2}, \quad t \in I_1 = [-t_1, t_0], \tag{A.3}$$

$$l_2; \tau_2(t) = \tau_- + te^{-i\theta/2}. \quad t \in I_2 = [-t_0, t_1], \tag{A.4}$$

Putting $\tau_0 \equiv \tau_1(t_0) = -\tau_2(-t_0) = J(2\cos\theta + 1)$, we have

$$\int_{\mathbb{R}} d\tau\, e^{-N\phi(\tau)} = \int_{-\infty}^{-\tau_0} d\tau\, e^{-N\phi(\tau)} + \int_{l_2} d\tau\, e^{-N\phi(\tau)} + \int_{l_1} d\tau\, e^{-N\phi(\tau)} + \int_{\tau_0}^{\infty} d\tau\, e^{-N\phi(\tau)}. \tag{A.5}$$

Before calculating the each term in (A.5), we prepare a technical lemma:

Lemma A.1. *For any $\ell > 0$ and $n = 0, 1, 2, 3, \cdots$,*

$$\int_{-\infty}^{\infty} dt\, e^{-\ell t^2} t^{2n+1} = 0, \quad \int_0^{\infty} dt\, e^{-\ell t^2} t^{2n+1} = \frac{n!}{2\ell^{n+1}}, \tag{A.6}$$

$$\int_{-\infty}^{\infty} dt\, e^{-\ell t^2} t^{2n} = \ell^{-n-\frac12}\Gamma\left(\frac{2n+1}{2}\right) = \ell^{-n-\frac12}\frac{(2n)!}{n! 2^{2n}}\pi^{\frac12}. \tag{A.7}$$

Let $\delta_0 > 0,\ d_0 > 0$. For $\tau_N = d_0 N^{-\gamma}$ such that $0 \le \gamma \le 1/2$, we have

$$\int_{\tau_N}^{\infty} dt\, e^{-\delta_0 N t^2} < (2\delta_0 d_0 N^{1-\gamma})^{-1} e^{-\delta_0 d_0^2 N^{1-2\gamma}}. \tag{A.8}$$

Proof. The first two are well known. As $t > \tau_N$, we have

$$\delta N(t^2 - \tau_N^2) = \delta N(t - \tau_N)(t + \tau_N) > 2\delta N \tau_N (t - \tau_N).$$

Therefore, we get

$$\int_{\tau_N}^{\infty} dt\, e^{-\delta_0 N t^2} < e^{-\delta_0 N \tau_N^2} \int_{\tau_N}^{\infty} dt\, e^{-2\delta_0 d_0 N^{1-\gamma}(t-\tau_N)} = (2\delta_0 d_0 N^{1-\gamma})^{-1} e^{-\delta_0 d_0^2 N^{1-2\gamma}}. \quad \square$$

Now, we consider the integral $\int_{l_1} e^{-N\phi(\tau)} d\tau$.

Proposition A.2. *We have*

$$\int_{l_1} d\tau\, e^{-N\phi(\tau)} = e^{-N\phi(\tau_+)}\left(A_{\frac12} N^{-\frac12} + A_{\frac32} N^{-\frac32} + A_{\frac52} N^{-\frac52} + O(N^{-\frac72})\right), \quad \text{as} \quad N \to \infty \tag{A.9}$$

where

$$A_{\frac{1}{2}} = \left(\frac{2\pi}{\phi''(\tau_+)}\right)^{\frac{1}{2}} = e^{i\theta/2}\left(\frac{\pi}{a_2}\right)^{\frac{1}{2}}, \quad A_{\frac{3}{2}} = e^{i\theta/2}\left(-\frac{1}{4}\beta^4 a_2^{-\frac{5}{2}}\Gamma\left(\frac{5}{2}\right) + \frac{1}{18}\beta^6 a_2^{-\frac{7}{2}}\Gamma\left(\frac{7}{2}\right)\right),$$

$$A_{\frac{5}{2}} = e^{i\theta/2}\left(-\frac{1}{6}\beta^6 a_2^{-\frac{7}{2}}\Gamma\left(\frac{7}{2}\right) + \frac{47}{480}\beta^8 a_2^{-\frac{9}{2}}\Gamma\left(\frac{9}{2}\right) - \frac{1}{72}\beta^{10} a_2^{-\frac{11}{2}}\Gamma\left(\frac{11}{2}\right) + \frac{1}{1944}\beta^{12} a_2^{-\frac{13}{2}}\Gamma\left(\frac{13}{2}\right)\right).$$

Proof. Substituting the relations (A.3) into $\phi(\tau)$ to have

$$\phi(\tau_1(t)) = \frac{1}{2J^2}\left(e^{i\theta}t^2 + (\alpha - i\lambda)e^{i\theta/2}t + \frac{\alpha^2-\lambda^2}{4} - \frac{i\alpha\lambda}{2}\right) - \log\left(\frac{\alpha+i\lambda}{2} + e^{i\theta/2}t\right), \tag{A.10}$$

we put

$$\begin{aligned} F_1(t) &\equiv \phi(\tau_1(t)) - \phi(\tau_1(0)) \\ &= \frac{1}{2J^2}e^{i\theta}t^2 + \frac{1}{J}e^{-i\theta/2}t - \log(\frac{\alpha+i\lambda}{2} + e^{i\theta/2}t) + \log\frac{\alpha+i\lambda}{2} \\ &= \frac{t^2}{2J^2}\cos\theta + \frac{t}{J}\cos\frac{\theta}{2} - \frac{1}{2}\log(J^2+t^2+2Jt\cos\frac{\theta}{2}) + \log J \\ &\qquad + i[\frac{t^2}{2J^2}\sin\theta - \frac{t}{J}\sin\frac{\theta}{2} - \arctan\frac{J\sin\theta + t\sin(\theta/2)}{J\cos\theta + t\cos(\theta/2)} + \theta] \\ &= \frac{\cos\theta}{J^2}t^2 + \sum_{n=3}^{\infty}\frac{(-1)^n}{n}\left(\frac{e^{-i\theta/2}}{J}\right)^n t^n \quad \text{for} \quad (|J^{-1}t|<1). \end{aligned} \tag{A.11}$$

To proceed further, we prepare

Lemma A.3. *For any $0 < \eta_1 < a_2$, there exist constants $t_2 > 0$ and $m_1 > 0$ such that $(-t_2, t_2) \subset I_1$, $\Re F_1(t) \geq \eta_1 t^2$ for $t \in (-t_2, t_2)$ and $\Re F_1(t) > m_1$ for $t \in I_1 \setminus (-t_2, t_2)$. Moreover, we have, as $N \to \infty$,*

$$\int_{-t_1}^{-t_2} dt\, e^{-NF_1(t)} + \int_{t_2}^{t_0} dt\, e^{-NF_1(t)} = O(e^{-m_1 N}). \tag{A.12}$$

Putting $\tau_N = N^{-1/3}$ and letting $N \to \infty$, we have

$$\int_{\tau_N}^{t_2} dt\, e^{-NF_1(t)} + \int_{-t_2}^{-\tau_N} dt\, e^{-NF_1(t)} = O(e^{-\eta_1 N^{1/3}}). \tag{A.13}$$

Proof. For $|J^{-1}t| < 1$, we have

$$\begin{aligned} f_1(t) = \Re F_1(t) &= \frac{\cos\theta}{J^2}t^2 + \sum_{n=3}^{\infty}\frac{(-1)^n}{n}J^{-n}t^n\cos\left(\frac{n\theta}{2}\right) \\ &= \frac{t^2}{2J^2}\cos\theta + \frac{t}{J}\cos\frac{\theta}{2} - \frac{1}{2}\log(J^2+t^2+2Jt\cos\frac{\theta}{2}) + \log J. \end{aligned}$$

Therefore, if $\eta_1 < a_2$, we take t_2 sufficiently small such that we have $\Re F_1(t) \geq \eta_1 t^2$ for $t \in (-t_2, t_2)$. Moreover, since

$$f_1'(t) = \frac{\cos\theta}{J^2}t + \frac{\cos(\theta/2)}{J} - \frac{t + J\cos(\theta/2)}{J^2+t^2+2Jt\cos(\theta/2)} \begin{cases} > 0 & \text{for} \quad t > 0, \\ < 0 & \text{for} \quad t < 0, \end{cases}$$

we have $\Re F_1(t) \geq m_1$ for $t \in I_1 \setminus (-t_2, t_2)$ where $m_1 = \min_{t\in I_1\setminus(-t_2,t_2)} f_1(t) > 0$.

Because $\Re F_1(t) \geq m_1$ on $I_1 \setminus (-t_2, t_2)$, we get readily

$$\int_{-t_1}^{-t_2} dt\, e^{-N\Re F_1(t)} < e^{-Nm_1}(t_1 - t_2) \quad \text{and} \quad \int_{t_2}^{t_0} dt\, e^{-N\Re_1 F(t)} < e^{-Nm_1}(t_0 - t_2),$$

which yields (A.12).

Taking $\gamma = 1/3,\ d_0 = 1,\ \delta_0 = \eta_1$ in (A.8) and $N > {t_2}^{-3}$, we have

$$\int_{\tau_N}^{t_2} dt\, e^{-N\Re F_1(t)} + \int_{-t_2}^{-\tau_N} dt\, e^{-N\Re F_1(t)} < \eta_1^{-1} e^{-\eta_1 N^{\frac{1}{3}}}$$

which gives (A.13). The bounds in (A.12) and (A.13) are locally bounded in w.r.t. λ in $(-2J, 2J)$. $\square$

Now, we continue the proof of Proposition A.2: We consider $e^{-N(F_1(t)-a_2t^2)}$. Since we have

$$F_1(t) - a_2t^2 = t^3\left(\frac{-1}{3}\beta^3 + \frac{1}{4}\beta^4 t + \frac{-1}{5}\beta^5 t^2 + \frac{1}{6}\beta^6 t^3 + O(t^4)\right) \quad \text{as } t \to 0$$

by (A.11) ($\beta = J^{-1}e^{-i\theta/2}$), we have for $-\tau_N \le t \le \tau_N$

$$\begin{aligned} e^{-N(F_1(t)-a_2t^2)} =& 1 - N(I_0^1 + I_1^1) + \frac{1}{2!}N^2(I_0^2 + I_1^2) - \frac{1}{3!}N^3(I_0^3 + I_1^3) \\ &+ \frac{1}{4!}N^4(I_0^4 + I_1^4) + N^4 O(t^{14}) \\ =& P_0(t) + P_1(t) + R(t,N). \end{aligned} \tag{A.14}$$

Here

$$\begin{aligned} I_0^1 &= \frac{1}{4}\beta^4 t^4 + \frac{1}{6}\beta^6 t^6, \quad I_1^1 = \frac{-1}{3}\beta^3 t^3 + \frac{-1}{5}\beta^5 t^5, \\ I_0^2 &= \frac{1}{9}\beta^6 t^6 + \frac{47}{240}\beta^8 t^8, \quad I_1^2 = \frac{-1}{6}\beta^7 t^7 + \frac{-19}{90}\beta^9 t^9, \\ I_0^3 &= \frac{1}{12}\beta^{10} t^{10}, \quad I_1^3 = \frac{-1}{27}\beta^9 t^9 + \frac{-31}{240}\beta^{11} t^{11}, \\ I_0^4 &= \frac{1}{81}\beta^{12} + O(t^{14}), \quad I_1^4 = O(t^{15}), \end{aligned}$$

and

$$\begin{aligned} P_0(t) &= 1 - \frac{1}{4}\beta^4 N t^4 - \frac{1}{6}\beta^6 N t^6 + \frac{1}{18}\beta^6 N^2 t^6 + \frac{47}{480}\beta^8 N^2 t^8 - \frac{1}{72}\beta^{10} N^3 t^{10} + \frac{1}{1944}\beta^{12} N^4 t^{12}, \\ P_1(t) &= N\left(\frac{1}{3}\beta^3 t^3 + \frac{1}{5}\beta^5 t^5\right) + N^2\left(\frac{-1}{12}\beta^7 t^7 + \frac{-19}{180}\beta^9 t^9\right) + N^3\left(\frac{1}{182}\beta^9 t^9 + \frac{1}{1440}\beta^{11} t^{11}\right), \\ |R(t,N)| &\le \sum_{\ell=1}^{4} c_\ell N^\ell t^{2(\ell+3)} \quad \text{with suitable constants } c_\ell. \end{aligned}$$

Note that the term $\int dt\, P_1(t) e^{-Na_2t^2} = 0$ on $(-\tau_N, \tau_N)$ or $(-\infty, \infty)$ because of $P_1(-t) = -P_1(t)$.

Since for any $n \in \mathbb{N}$ there exists $d_n = \left(\frac{n}{a_2}\right)^{n/2} e^{-n/2} > 0$ such that $|t|^n \le d_n e^{Na_2t^2/2}$ on $\mathbb{R}$, it follows that

$$\int_{\mathbb{R}\setminus(-\tau_N, \tau_N)} e^{-Na_2t^2} |t|^n dt \le 2d_n a_2^{-1} e^{-\frac{1}{2}a_2 N^{\frac{1}{3}}}$$

by (A.8) with $\gamma = 1/3,\ d_0 = 1,\ \delta_0 = a_2$. Then we have

$$\left| \int_{\mathbb{R}\setminus(-\tau_N, \tau_N)} dt\, e^{-Na_2t^2} P_0(t) \right| \le c_2 N^4 e^{-\frac{1}{2}a_2 N^{\frac{1}{3}}}. \tag{A.15}$$

By (A.6) and (A.7), we have

$$\begin{aligned} &\int_{-\infty}^{\infty} dt\, e^{-Na_2t^2} P_0(t) \\ &= \left(\frac{\pi}{a_2}\right)^{\frac{1}{2}} N^{-\frac{1}{2}} + \left(-\frac{1}{4}\beta^4 a_2^{-\frac{5}{2}}\Gamma\left(\frac{5}{2}\right) + \frac{1}{18}\beta^6 a_2^{-\frac{7}{2}}\Gamma\left(\frac{7}{2}\right)\right) N^{-\frac{3}{2}} \\ &\quad + \left(-\frac{1}{6}\beta^6 a_2^{-\frac{7}{2}}\Gamma\left(\frac{7}{2}\right) + \frac{47}{480}\beta^8 a_2^{-\frac{9}{2}}\Gamma\left(\frac{9}{2}\right) - \frac{1}{72}\beta^{10} a_2^{-\frac{11}{2}}\Gamma\left(\frac{11}{2}\right)\right) \\ &\qquad + \frac{1}{1944}\beta^{12} a_2^{-\frac{13}{2}}\Gamma\left(\frac{13}{2}\right) N^{-\frac{5}{2}} \\ &\equiv A'_{\frac{1}{2}} N^{-\frac{1}{2}} + A'_{\frac{3}{2}} N^{-\frac{3}{2}} + A'_{\frac{5}{2}} N^{-\frac{5}{2}}. \end{aligned} \tag{A.16}$$

By (A.15) and (A.16), we have

$$\int_{-\tau_N}^{\tau_N} dt\, e^{-Na_2t^2} P_0(t) = A'_{\frac{1}{2}} N^{-\frac{1}{2}} + A'_{\frac{3}{2}} N^{-\frac{3}{2}} + A'_{\frac{5}{2}} N^{-\frac{5}{2}} + O\left(e^{-\frac{1}{3}a_2 N^{\frac{1}{3}}}\right) \tag{A.17}$$

as $N \to \infty$. By (A.14), (A.6), (A.7) and the definition of $P_0(t)$, we have

$$\left| \int_{-\tau_N}^{\tau_N} dt\, e^{-Na_2t^2} \left(e^{-N(F_1(t)-a_2t^2)} - P_0(t) \right) \right| \le c_3 N^{-\frac{7}{2}}. \tag{A.18}$$

By (A.17) and (A.18), we get

$$\int_{-\tau_N}^{\tau_N} dt\, e^{-NF_1(t)} = A'_{\frac{1}{2}} N^{-\frac{1}{2}} + A'_{\frac{3}{2}} N^{-\frac{3}{2}} + A'_{\frac{5}{2}} N^{-\frac{5}{2}} + O\left(N^{-\frac{7}{2}}\right). \tag{A.19}$$

By (A.3), (A.12), (A.13) and (A.19), we have

$$\begin{aligned} \int_{l_1} d\tau\, e^{-N\phi(\tau)} &= e^{i\theta/2} e^{-N\phi(\tau_1(0))} \int_{-t_1}^{t_0} e^{-NF_1(t)} dt \\ &= e^{-N\phi(\tau_+)} \left(e^{i\theta/2} A'_{\frac{1}{2}} N^{-\frac{1}{2}} + e^{i\theta/2} A'_{\frac{3}{2}} N^{-\frac{3}{2}} + e^{i\theta/2} A'_{\frac{5}{2}} N^{-\frac{5}{2}} + O(N^{-\frac{7}{2}}) \right). \end{aligned} \tag{A.20}$$

Remarking $\phi''(\tau_+) = 2\alpha/(J^2(\alpha + i\lambda)) = 2\cos\theta/(J^2 e^{i\theta})$, we have

$$A_{\frac{1}{2}} \equiv e^{i\theta/2} A'_{\frac{1}{2}} = e^{i\theta/2} \left(\frac{\pi}{a_2} \right)^{\frac{1}{2}} = \left(\frac{2\pi}{\phi''(\tau_+)} \right)^{\frac{1}{2}}, \quad A_{\frac{3}{2}} \equiv e^{i\theta/2} A'_{\frac{3}{2}}, \quad A_{\frac{5}{2}} \equiv e^{i\theta/2} A'_{\frac{5}{2}}.$$

Thus Proposition A.2 is proved. □

Next we consider the integral $\int_{l_2} d\tau\, e^{-N\phi(\tau)}$.

Proposition A.4.

$$\int_{l_2} d\tau\, e^{-N\phi(\tau)} = e^{-N\phi(\tau_-)} \left(B_{\frac{1}{2}} N^{-\frac{1}{2}} + B_{\frac{3}{2}} N^{-\frac{3}{2}} + B_{\frac{5}{2}} N^{-\frac{5}{2}} + O(N^{-\frac{7}{2}}) \right) \quad \text{as } N \to \infty, \tag{A.21}$$

where

$$B_{\frac{1}{2}} = \left(\frac{2\pi}{\phi''(\tau_-)} \right)^{\frac{1}{2}} = \overline{A_{\frac{1}{2}}}, \quad B_{\frac{3}{2}} = \overline{A_{\frac{3}{2}}}, \quad B_{\frac{5}{2}} = \overline{A_{\frac{5}{2}}}.$$

Proof. We have

$$\phi(\tau_2(t)) = \frac{1}{2J^2} \left(e^{-i\theta} t^2 - (\alpha + i\lambda) e^{-i\theta/2} t + \frac{\alpha^2 - \lambda^2}{4} + \frac{i\alpha\lambda}{2} \right) - \log\left(\frac{-\alpha + i\lambda}{2} + e^{-i\theta/2} t \right).$$

Putting

$$\begin{aligned} F_2(t) &= \phi(\tau_2(t)) - \phi(\tau_2(0)) \\ &= \frac{1}{2J^2} e^{-i\theta} t^2 - \frac{1}{J} e^{i\theta/2} t - \log\left(\frac{-\alpha + i\lambda}{2} + e^{-i\theta/2} t\right) + \log\frac{-\alpha + i\lambda}{2} \\ &= \frac{\cos\theta}{J^2} t^2 + \sum_{n=3}^{\infty} \frac{1}{n} \left(\frac{e^{i\theta/2}}{J} \right)^n t^n, \end{aligned}$$

and

$$\begin{aligned} f_2(t) = \Re F_2(t) &= \frac{\cos\theta}{J^2} t^2 + \sum_{n=3}^{\infty} \frac{1}{n} J^{-n} t^n \cos\left(\frac{n\theta}{2}\right) \\ &= \frac{t^2}{2J^2} \cos\theta - \frac{t}{J} \cos\frac{\theta}{2} - \frac{1}{2} \log\left(J^2 + t^2 - 2Jt\cos\frac{\theta}{2}\right) + \log J, \end{aligned}$$

we may proceeed in a similar fashion to that used in the proof of Proposition A.2. □

By Proposition A.2 and Proposition A.4, we have

Proposition A.5. *When $N \to \infty$, we have*

$$\int_{\mathbb{R}} d\tau\, e^{-N\phi(\tau)} = e^{-N\phi(\tau_+)} \left(A_{\frac{1}{2}} N^{-\frac{1}{2}} + A_{\frac{3}{2}} N^{-\frac{3}{2}} + A_{\frac{5}{2}} N^{-\frac{5}{2}} + O(N^{-\frac{7}{2}}) \right) + e^{-N\phi(\tau_-)} \left(B_{\frac{1}{2}} N^{-\frac{1}{2}} + B_{\frac{3}{2}} N^{-\frac{3}{2}} + B_{\frac{5}{2}} N^{-\frac{5}{2}} + O(N^{-\frac{7}{2}}) \right). \tag{A.22}$$

Proof. Since it follows that there exists $\varepsilon_1 > 0$ s.t. $\inf_{(-\infty,-\tau_0)} \Re\phi(\tau) > \Re\phi(\tau_-) + \varepsilon_1$ and $e^{-\phi(\tau)}$ is integrable on $(-\infty, -\tau_0)$, we have,

$$\left| \int_{-\infty}^{-\tau_0} d\tau\, e^{-N\phi(\tau)} \right| < \left[e^{\Re\phi(\tau_-)} \int_{-\infty}^{-\tau_0} d\tau\, e^{-\phi(\tau)} \right] e^{-N\Re\phi(\tau_-) - \varepsilon_1 N} \leq C e^{-N\Re\phi(\tau_-)} e^{-\varepsilon_1 N}.$$

This term is negligible compared with $N^{-\frac{7}{2}}$ when $N \to \infty$. Similarly, $\int_{\tau_0}^{\infty} d\tau\, e^{-N\phi(\tau)}$ is negligible when $N \to \infty$. Therefore from Proposition A.2 and Proposition A.4, we get (A.22). □

Now, we consider the case where the non-trivial amplitude exists.

Proposition A.6. *When $N \to \infty$, we get*

$$\int_{l_1} d\tau\, \frac{1}{\tau + i\lambda} e^{-N\phi(\tau)} = \frac{1}{\tau_+ + i\lambda} e^{-N\phi(\tau_+)} \left(A_{\frac{1}{2}} N^{-\frac{1}{2}} + C_{\frac{3}{2}} N^{-\frac{3}{2}} + C_{\frac{5}{2}} N^{-\frac{5}{2}} + O(N^{-\frac{7}{2}}) \right) \tag{A.23}$$

and

$$\int_{l_2} d\tau\, \frac{1}{\tau + i\lambda} e^{-N\phi(\tau)} = \frac{1}{\tau_- + i\lambda} e^{-N\phi(\tau_-)} \left(B_{\frac{1}{2}} N^{-\frac{1}{2}} + \overline{C_{\frac{3}{2}}} N^{-\frac{3}{2}} + \overline{C_{\frac{5}{2}}} N^{-\frac{5}{2}} + O(N^{-\frac{7}{2}}) \right). \tag{A.24}$$

Here

$$C_{\frac{3}{2}} = e^{i\theta/2} \left(\beta^2 a_2^{-\frac{3}{2}} \Gamma\left(\frac{3}{2}\right) - \frac{7}{12} \beta^4 a_2^{-\frac{5}{2}} \Gamma\left(\frac{5}{2}\right) + \frac{1}{18} \beta^6 a_2^{-\frac{7}{2}} \Gamma\left(\frac{7}{2}\right) \right),$$

$$C_{\frac{5}{2}} = e^{i\theta/2} \left(\beta^4 a_2^{-\frac{5}{2}} \Gamma\left(\frac{5}{2}\right) - \frac{19}{20} \beta^6 a_2^{-\frac{7}{2}} \Gamma\left(\frac{7}{2}\right) + \frac{341}{1440} \beta^8 a_2^{-\frac{9}{2}} \Gamma\left(\frac{9}{2}\right) - \frac{13}{648} \beta^{10} a_2^{-\frac{11}{2}} \Gamma\left(\frac{11}{2}\right) + \frac{1}{1944} \beta^{12} a_2^{-\frac{13}{2}} \Gamma\left(\frac{13}{2}\right) \right).$$

Proof. Putting

$$a(t) = \frac{1}{\tau_1(t) + i\lambda} = \frac{1}{Je^{i\theta}} \sum_{n=0}^{\infty} (-1)^n \beta^n t^n,$$

we have

$$\begin{aligned} &Je^{i\theta} a(t) e^{-N(F(t) - a_2 t^2)} \\ &= 1 - \beta t + \beta^2 t^2 - \beta^3 t^3 + \beta^4 t^4 \\ &\quad - N t^3 \left(\frac{-1}{3} \beta^3 + \frac{7}{12} \beta^4 t + \frac{-47}{60} \beta^5 t^2 + \frac{19}{20} \beta^6 t^3 \right) \\ &\quad + \frac{1}{2!} N^2 t^6 \left(\frac{1}{9} \beta^6 - \frac{5}{18} \beta^7 t + \frac{341}{720} \beta^8 t^2 \right) \\ &\quad - \frac{1}{3!} N^3 t^9 \left(\frac{-1}{27} \beta^9 + \frac{13}{108} \beta^{10} t \right) + \frac{1}{4!} N^4 t^{12} \frac{1}{81} \beta^{12}) + N^4 0(t^{14}). \end{aligned}$$

Calculating as before, we get the result. □

Combining above, we get

Proposition A.7. *Making $N \to \infty$, we have*

$$\int_{\mathbb{R}} d\tau\, \frac{1}{\tau + i\lambda} e^{-N\phi(\tau)} = \frac{1}{\tau_+ + i\lambda} e^{-N\phi(\tau_+)} \left(A_{\frac{1}{2}} N^{-\frac{1}{2}} + C_{\frac{3}{2}} N^{-\frac{3}{2}} + C_{\frac{5}{2}} N^{-\frac{5}{2}} + O(N^{-\frac{7}{2}}) \right) + \frac{1}{\tau_- + i\lambda} e^{-N\phi(\tau_-)} \left(B_{\frac{1}{2}} N^{-\frac{1}{2}} + \overline{C_{\frac{3}{2}}} N^{-\frac{3}{2}} + \overline{C_{\frac{5}{2}}} N^{-\frac{5}{2}} + O(N^{-\frac{7}{2}}) \right). \tag{A.25}$$

Analogously, we have

Proposition A.8. *When $N \to \infty$, we have*

$$\int_{\mathbb{R}} d\tau \, \frac{1}{(\tau + i\lambda)^2} e^{-N\phi(\tau)} = \frac{1}{(\tau_+ + i\lambda)^2} e^{-N\phi(\tau_+)} \left(A_{\frac{1}{2}} N^{-\frac{1}{2}} + D_{\frac{3}{2}} N^{-\frac{3}{2}} + D_{\frac{5}{2}} N^{-\frac{5}{2}} + O(N^{-\frac{7}{2}}) \right) + \frac{1}{(\tau_- + i\lambda)^2} e^{-N\phi(\tau_-)} \left(B_{\frac{1}{2}} N^{-\frac{1}{2}} + \overline{D_{\frac{3}{2}}} N^{-\frac{3}{2}} + \overline{D_{\frac{5}{2}}} N^{-\frac{5}{2}} + O(N^{-\frac{7}{2}}) \right). \tag{A.26}$$

Here

$$D_{\frac{3}{2}} = e^{i\theta/2} \left(3\beta^2 a_2^{-\frac{3}{2}} \Gamma\left(\frac{3}{2}\right) - \frac{11}{12} \beta^4 a_2^{-\frac{5}{2}} \Gamma\left(\frac{5}{2}\right) + \frac{1}{18} \beta^6 a_2^{-\frac{7}{2}} \Gamma\left(\frac{7}{2}\right) \right),$$

$$D_{\frac{5}{2}} = e^{i\theta/2} \left(5\beta^4 a_2^{-\frac{5}{2}} \Gamma\left(\frac{5}{2}\right) - \frac{159}{60} \beta^6 a_2^{-\frac{7}{2}} \Gamma\left(\frac{7}{2}\right) + \frac{207}{480} \beta^8 a_2^{-\frac{9}{2}} \Gamma\left(\frac{9}{2}\right) - \frac{17}{648} \beta^{10} a_2^{-\frac{11}{2}} \Gamma\left(\frac{11}{2}\right) + \frac{1}{1944} \beta^{12} a_2^{-\frac{13}{2}} \Gamma\left(\frac{13}{2}\right) \right).$$

Remark. This estimate is not used directly in this paper. But instead of Brézin's expression (3.7), if we use the expression (3.18), we need this estimate.

A.2. **A simple example: the case when $|\lambda| \le 2J$ on the half line.**

Proposition A.9. *For λ satisfying $|\lambda| \le 2J$, we have*

$$\int_0^\infty ds \, s^N e^{-\frac{N}{2J^2}(2i\lambda s + s^2)} = e^{-N\varphi(s_+)} \left(E_{\frac{1}{2}} N^{-\frac{1}{2}} + E_{\frac{3}{2}} N^{-\frac{3}{2}} + E_{\frac{5}{2}} N^{-\frac{5}{2}} + O(N^{-\frac{7}{2}}) \right) \tag{A.27}$$

as $N \to \infty$. Here

$$\varphi(s) = \frac{s^2 + 2i\lambda s}{2J^2} - \log s, \quad s_+ = \frac{-i\lambda + \sqrt{4J^2 - \lambda^2}}{2} = \tau_+,$$

$$E_{\frac{1}{2}} = \left(\frac{2\pi}{\varphi''(s_+)} \right)^{\frac{1}{2}} = \overline{A_{\frac{1}{2}}}, \quad E_{\frac{3}{2}} = \overline{A_{\frac{3}{2}}}, \quad E_{\frac{5}{2}} = \overline{A_{\frac{5}{2}}}.$$

Proof. We deform the integration contour from $[0, \infty)$ to the path $l_3 \cup l_4 \cup l_5$ where

$$l_3; s_3(t) = -it, \qquad 0 \le t \le t_3 \quad \text{with} \quad t_3 = \frac{J \sin(\theta/2)}{\cos(\theta/2)},$$

$$l_4; s_4(t) = s_+ + te^{-i\theta/2}, \qquad -t_1 \le t \le t_1,$$

$$l_5; s_5(t) = t + \frac{J(\cos(\theta/2) + \cos(3\theta/2))}{\cos(\theta/2)} - i\frac{J \sin(3\theta/2)}{\cos(\theta/2)}, \qquad 0 \le t < \infty,$$

$$l_6^M; s_6(t) = M + it, \qquad -t_4 \le t \le 0 \quad \text{with} \quad t_4 = \frac{J \sin(3\theta/2)}{\cos(\theta/2)}.$$

Since for large M, we have

$$\left| i \int_{-t_4}^0 dt \, e^{-N\varphi(s_6(t))} \right| = o(M^{-1}),$$

we get

$$\int_0^\infty ds \, e^{-N\varphi(s)} = \left(\int_{l_3} + \int_{l_4} + \int_{l_5} \right) ds \, e^{-N\varphi(s)}.$$

(i) Putting

$$\begin{aligned}G_4(t) &\equiv \varphi(s_4(t)) - \varphi(s_4(0)) \\ &= \frac{1}{2J^2}(t^2e^{-i\theta} + 2Jte^{i\theta/2}) - \log(1+\frac{te^{i\theta/2}}{J}) \\ &= \frac{t^2}{2J^2}\cos\theta + \frac{t}{J}\cos\frac{\theta}{2} - \frac{1}{2}\log(J^2+t^2+2Jt\cos\frac{\theta}{2}) + \log J \\ &\qquad + i\Big[-\frac{t^2}{2J^2}\sin\theta + \frac{t}{J}\cos\frac{\theta}{2} - \arctan\frac{t\sin(\theta/2)}{J+t\cos(\theta/2)}\Big] \\ &= \frac{t^2}{2}\beta^2 + \sum_{n=3}^{\infty}\frac{(-1)^n}{n}(t\bar\beta)^n,\end{aligned}$$

we have

$$\Re G_4(t) = \frac{t^2}{2J^2}\cos\theta + \frac{t}{J}\cos\frac{\theta}{2} - \frac{1}{2}\log(J^2+t^2+2Jt\cos\frac{\theta}{2}) + \log J = f_1(t),$$

where $f_1(t)$ is same as the one in Lemma A.3. Therefore, we remark $\Re\varphi(s_+ - t_1e^{-i\theta/2}) \geq \Re\varphi(s_+)$ and we may apply the procedure in proving Lemma A.3 to estimate

$$\int_{l_4} dt\, e^{-N\varphi(s_4(t))} = e^{i\theta/2}\int_{-t_1}^{t_1} dt\, e^{-N\varphi(s_4(t))}. = e^{-N\varphi(s_+)}e^{i\theta/2}\int_{-t_1}^{t_1} dt\, e^{-NG_4(t)}.$$

(ii) As $f_3(t) \equiv \Re\,\varphi(s_3(t))$ is decreasing on $[0,t_3]$ and by the above remark, there exists $\varepsilon_2 > 0$ such that $\inf_{t\in l_3}\Re\,\varphi(s_3(t)) > \Re\varphi(s_+) + \varepsilon_2$. Therefore, we have

$$\Big| -i\int_{l_3} dt\, e^{-N\varphi(s_3(t))}\Big| = \int_0^{t_3} dt\, e^{-N\Re\varphi(s_3(t))} \leq t_3e^{-N\Re\varphi(s_+)}e^{-\varepsilon_2 N}.$$

(iii) There exists $\varepsilon_3 > 0$ such that $\inf_{t\in l_5}\Re\varphi(s_5(t)) > \Re\varphi(s_+) + \varepsilon_3$.

$$\Big|\int_{l_5} dt\, e^{-N\varphi(s_5(t))}\Big| = \int_0^{\infty} dt\, e^{-N\Re\varphi(s_5(t))} \leq e^{-(N-1)\Re\varphi(s_+)}e^{-\varepsilon_3 N}\int_0^{\infty} dt\, e^{-\Re\varphi(s_5(t))}.$$

Combining these, we get the desired result. □

Analogously, we have

Proposition A.10. *Making $N\to\infty$, we get*

$$\int_0^{\infty} ds\, s^{N+1}e^{-\frac{N}{2J^2}(2i\lambda s+s^2)} = s_+e^{-N\varphi(s_+)}\left(E'_{\frac12}N^{-\frac12} + E'_{\frac32}N^{-\frac32} + E'_{\frac52}N^{-\frac52} + O(N^{-\frac72})\right)$$

where

$$E'_{\frac12} = \left(\frac{2\pi}{\eta''(s_+)}\right)^{\frac12} = \overline{A_{\frac12}},\quad E'_{\frac32} = e^{-i\theta/2}\left(\frac{1}{12}\overline{\beta}^{\,4}a_2^{-\frac52}\Gamma\left(\frac52\right) + \frac{1}{18}\overline{\beta}^{\,6}a_2^{-\frac72}\Gamma\left(\frac72\right)\right),$$

$$E'_{\frac52} = e^{-i\theta/2}\left(\frac{1}{30}\overline{\beta}^{\,6}a_2^{-\frac72}\Gamma\left(\frac72\right) + \frac{7}{480}\overline{\beta}^{\,8}a_2^{-\frac92}\Gamma\left(\frac92\right) - \frac{5}{648}\overline{\beta}^{\,10}a_2^{-\frac{11}{2}}\Gamma\left(\frac{11}{2}\right) + \frac{1}{1944}\overline{\beta}^{\,12}a_2^{-\frac{13}{2}}\Gamma\left(\frac{13}{2}\right)\right).$$

A.3. **Proof of Theorem 3.1.** By Proposition A.5 and Proposition A.9, when $N\to\infty$, we have

$$\begin{aligned}I_1 &= e^{-N\left(\phi(\tau_+)+\varphi(s_+)+\frac{\lambda^2}{2J^2}\right)}\left(|A_{\frac12}|^2N^{-1} + 2\Re(A_{\frac12}\overline{A_{\frac32}})N^{-2} + (2\Re(A_{\frac12}\overline{A_{\frac52}}) + |A_{\frac32}|^2)N^{-3} + O(N^{-4})\right) \\ &\quad + e^{-N\left(\phi(\tau_-)+\varphi(s_+)+\frac{\lambda^2}{2J^2}\right)}\left(\overline{A_{\frac12}}^2N^{-1} + 2\overline{A_{\frac12}A_{\frac32}}N^{-2} + (2\overline{A_{\frac12}A_{\frac52}} + \overline{A_{\frac32}}^2)N^{-3} + O(N^{-4})\right).\end{aligned}$$

By Proposition A.7 and Proposition A.10, we have, when $N\to\infty$,

$$\begin{aligned}I_2 &= \frac{s_+}{\tau_+ + i\lambda}e^{-N\left(\phi(\tau_+)+\varphi(s_+)+\frac{\lambda^2}{2J^2}\right)}\left(A_{\frac12}E'_{\frac12}N^{-1} + (A_{\frac12}E'_{\frac32} + C_{\frac32}E'_{\frac12})N^{-2} + O(N^{-3})\right) \\ &\quad + \frac{s_+}{\tau_- + i\lambda}e^{-N\left(\phi(\tau_-)+\varphi(s_+)+\frac{\lambda^2}{2J^2}\right)}\left(B_{\frac12}E'_{\frac12}N^{-1} + (B_{\frac12}E'_{\frac32} + \overline{C_{\frac32}}E'_{\frac12})N^{-2} + O(N^{-3})\right).\end{aligned}$$

On the other hand, we have

$$\phi(\tau_+)+\varphi(s_+)+\frac{\lambda^2}{2J^2}=1-\log J^2,$$
$$e^{-N\left(\phi(\tau_-)+\varphi(s_+)+\frac{\lambda^2}{2J^2}\right)}=e^{-N}J^{2N}e^{-iN(\sin 2\theta+2\theta)}(-1)^N,$$
$$\frac{s_+}{\tau_++i\lambda}=e^{-2i\theta},\quad \frac{s_+}{\tau_-+i\lambda}=-1.$$

Because of

$$|A_{\frac12}|^2=A_{\frac12}E'_{\frac12}=\frac{\pi}{a_2}=\frac{J^2}{\cos\theta}\pi,\quad \overline{A_{\frac12}}^2=B_{\frac12}E'_{\frac12},$$

we have

$$\text{the coefficients of } N^{-1} \text{ in } I_1+I_2=e^{-N}J^{2N}(1+e^{-2i\theta})\frac{J^2}{\cos\theta}\pi=2\pi e^{-N}J^{2(N+1)}e^{-i\theta}.$$

By simple but lengthy calculations, we have

$$2\Re(A_{\frac12}\overline{A_{\frac32}})+e^{-2i\theta}(A_{\frac12}E'_{\frac32}+C_{\frac32}E'_{\frac12})=\frac{\pi}{6}J^2e^{-i\theta},$$
$$\overline{A_{\frac12}A_{\frac32}}-(B_{\frac12}E'_{\frac32}+\overline{C_{\frac32}}E'_{\frac12})=-\frac{\pi J^2}{2\cos^2\theta},$$

which yields

$$\text{the coefficients of } N^{-2} \text{ in } I_1+I_2=2\pi e^{-N}J^{2(N+1)}\frac{1}{12}\left(e^{-i\theta}-(-1)^N\frac{3e^{-iN(\sin 2\theta+2\theta)}}{\cos^2\theta}\right).$$

Thus Theorem 3.1 is proved. □

A.4. **The case when $|\lambda|>2J$.** Here, we consider the asymptotic behavior of (3.1) in the case of $\lambda>2J$. The case when $\lambda<-2J$ is treated analogously, so omitted.

A.4.1. *The integrals* $\int_{\mathbb{R}}d\tau\, e^{-N\phi(\tau)}$ *and* $\int_{\mathbb{R}}d\tau\,\frac{1}{\tau+i\lambda}e^{-N\phi(\tau)}$. We put

$$\rho=\sqrt{\lambda^2-4J^2}>0,\quad\text{and}\quad \sigma_\pm=i\frac{-\lambda\pm\rho}{2}$$

where $\sigma_\pm$ are the critical points of ϕ. As ϕ is holomorphically extended to the domain $D_+=\{z\in\mathbb{C}\,|\,\Im F>-\lambda\}$, we may deform the path of integration to the one which passes σ_+. We parametrize this path l_7 by

$$l_7;\tau_7(t)=\sigma_++t,\qquad t\in\mathbb{R}. \tag{A.28}$$

Since ϕ is holomorphic in D_+, we have easily

$$\int_{\mathbb{R}}d\tau\, e^{-N\phi(\tau)}=\int_{l_7}d\tau\, e^{-N\phi(\tau)}\qquad\text{and}\qquad \int_{\mathbb{R}}d\tau\,\frac{1}{\tau+i\lambda}e^{-N\phi(\tau)}=\int_{l_7}d\tau\,\frac{1}{\tau+i\lambda}e^{-N\phi(\tau)}.$$

Proposition A.11. *We have the following decomposition*

$$\int_{l_7}d\tau\, e^{-N\phi(\tau)}=e^{-N\phi(\sigma_+)}(K_1(N)+K_2(N)),$$

which satisfy the estimates below: There exist constants $k_0(\lambda)>0$ and $C_0(\lambda)>0$ such that

$$|K_1(N)|\le C_0(\lambda)e^{-k_0(\lambda)N}\quad\textit{and}\quad |K_2(N)|\le C_0(\lambda)N^{-1/2},$$

with $k_0(\lambda)\to 0$ and $C_0(\lambda)\to\infty$ for $\lambda\searrow 2J$ respectively and $K_2(N)$ is a real valued function.

Proof. Substituting the relation (A.28) into $\phi(\tau)$ to have

$$\phi(\tau_7(t))=\frac{1}{2J^2}\big(t^2-i(\lambda-\rho)t-\frac{(\lambda-\rho)^2}{4}\big)-\log\big(t+i\frac{\lambda+\rho}{2}\big), \tag{A.29}$$

we put

$$\begin{aligned} F_7(t) &\equiv \phi(\tau_7(t)) - \phi(\tau_7(0)) \\ &= \frac{1}{2J^2}\left(t^2 - i(\lambda-\rho)t\right) - \log\left(t + i\frac{\lambda+\rho}{2}\right) + \log\left(i\frac{\lambda+\rho}{2}\right) \\ &= b_2 t^2 + \sum_{n=3}^{\infty}\frac{(-1)^n}{n}\eta^n t^n \qquad \text{for} \qquad (|\eta t| < 1), \end{aligned} \tag{A.30}$$

where

$$b_2 \equiv \frac{\rho(\lambda-\rho)}{4J^4} = \frac{1}{2}\phi''(\sigma_+) > 0 \qquad \text{and} \qquad \eta \equiv \frac{\sigma_+}{J^2} = i\frac{-\lambda+\rho}{2J^2} \in i\mathbb{R}. \tag{A.31}$$

To proceed further, we prepare two lemmas.

Lemma A.12. *There exists $t_5 > 0$, $m_2 > 0$ and $C_0 > 0$ such that*

$$\left|\int_{t_5/2}^{t_5} dt\, e^{-NF_7(t)}\right| + \left|\int_{-t_5}^{-t_5/2} dt\, e^{-NF_7(t)}\right| \le \frac{4}{b_2 t_5 N} e^{-8^{-1} b_2 t_5^2 N} \tag{A.32}$$

and

$$\left|\int_{-\infty}^{-t_5} dt e^{-NF_7(t)}\right| + \left|\int_{t_5}^{\infty} dt e^{-NF_7(t)}\right| \le C_0 e^{-m_2 N}. \tag{A.33}$$

Proof. By (A.30), we have

$$f_7(t) \equiv \Re F_7(t) = b_2 t^2 + \sum_{n=3}^{\infty}\frac{(-1)^n}{n}\Re(\eta^n)t^n = \frac{t^2}{2J^2} - \frac{1}{2}\log(1 + \frac{4t^2}{(\lambda+\rho)^2})$$

for $|\eta t| < 1$. As $b_2 > 0$, there exists $t_5 > 0$ such that $\Re F_7(t) \ge (b_2/2)t^2$ for $t \in (-t_5, t_5)$. By putting $\tau_N = t_5/2$ in (A.8), we get (A.32)

$$\int_{\tau_N}^{t_5} dt\, e^{-N\Re F_7(t)} + \int_{-t_5}^{-\tau_N} dt\, e^{-N\Re F_7(t)} \le \frac{4}{b_2 t_5 N} e^{-8^{-1} b_2 t_5^2 N}.$$

On the other hand, since

$$f_7'(t) = \frac{t}{J^2} - \frac{4t}{4t^2 + (\lambda+\rho)^2} \begin{cases} > 0 & \text{for } t > 0, \\ < 0 & \text{for } t < 0, \end{cases}$$

we have $f_7(t) \ge m_2 = f_7(t_5) > 0$ for $t \in \mathbb{R} \setminus (-\approx_{\ntrianglelefteq}, \approx_{\ntrianglelefteq})$. Hence,

$$\left|\int_{t_5}^{\infty} dt\, e^{-NF_7(t)}\right| = \int_{t_5}^{\infty} dt\, e^{-(N-1)f_7(t) - f_7(t)} \le e^{-m_2(N-1)} \int_0^{\infty} dt \sqrt{1 + \frac{4t^2}{(\lambda+\rho)^2}} e^{-t^2/(2J^2)}.$$

Putting $C_0 = e^{m_2}(\int_0^\infty dt\,(1+t^2)e^{-t^2/(2J^2)})^{1/2}(\int_0^\infty dt\, e^{-t^2/(2J^2)})^{1/2} = e^{m_2}(\pi(J^2+J^4)/2)^{1/2}$, we have (A.33). □

Lemma A.13. *We have, with $t_5 > 0$ and $b_2 > 0$ defined above,*

$$\int_{-t_5/2}^{t_5/2} dt\, e^{-NF_7(t)} \in \mathbb{R} \quad \text{and} \quad \left|\int_{-t_5/2}^{t_5/2} dt\, e^{-NF_7(t)}\right| \le \sqrt{\frac{2\pi}{b_2}} N^{-1/2}.$$

Proof. By (A.30) we have

$$F_7(t) = (b_2 - \frac{1}{2}\eta^2)t^2 + \eta t - \log(1 + \eta t). \tag{A.34}$$

Putting $\eta = i\nu \in i\mathbb{R}$ we have

$$\begin{aligned} e^{-NF_7(t)} &= (1+\eta t)^N e^{(-b_2+\eta^2/2)Nt^2-N\eta t} \\ &= e^{(-b_2-\nu^2/2)Nt^2}(\cos(N\nu t) - i\sin(N\nu t))\sum_{n=0}^{N}\binom{N}{n}(i\nu t)^n \\ &= e^{(-b_2-\nu^2/2)Nt^2}(H_1(t)+H_2(t)) \end{aligned} \tag{A.35}$$

where

$$H_1(t) = \cos(N\nu t)\sum_{n=\text{even}}\binom{N}{n}(i\nu t)^n - i\sin(N\nu t)\sum_{n=\text{odd}}\binom{N}{n}(i\nu t)^n \tag{A.36}$$

and

$$H_2(t) = \cos(N\nu t)\sum_{n=\text{odd}}\binom{N}{n}(i\nu t)^n - i\sin(N\nu t)\sum_{n=\text{even}}\binom{N}{n}(i\nu t)^n. \tag{A.37}$$

Because $0 \neq \eta \in i\mathbb{R}$, $H_1(t)$ is a real valued and even function and $H_2(t)$ is a pure imaginary valued and odd function. Hence from (A.35) we have

$$\int_{-\tau_N}^{\tau_N} dt\, e^{-NF_7(t)} = \int_{-\tau_N}^{\tau_N} dt\, e^{-(b_2+\nu^2/2)t^2} H_1(t) \in \mathbb{R}.$$

Moreover, as $\Re F_7(t) \geq 2^{-1}b_2t^2$ for $t \in (-t_5/2, t_5/2) \subset (-t_5, t_5)$, we have

$$\left|\int_{-t_5/2}^{t_5/2} dt\, e^{-NF_7(t)}\right| = \int_{-t_5/2}^{t_5/2} dt\, e^{-N\Re F_7(t)} \leq \int_{-t_5/2}^{t_5/2} dt\, e^{-N2^{-1}b_2t^2} \leq \sqrt{\frac{2\pi}{b_2}}N^{-1/2}.$$

Thus Lemma A.13 is proved. □

Putting $K_1(N) = \int_{\mathbb{R}\setminus(-\tau_\mathbb{N},\tau_\mathbb{N})} dt\, e^{-NF_7(t)}$, $K_2(t) = \int_{-\tau_N}^{\tau_N} dt\, e^{-NF_7(t)}$, $k_0(\lambda) = \min(8^{-1}b_2t_5^2,\ m_2)$ and $C_0(\lambda) = \max(C_0, \sqrt{2\pi/b_2}, 4/(b_2t_5))$, we have proved Proposition A.11. □

Proposition A.14. *We have the following decomposition*

$$\int_{l_7} d\tau\, \frac{1}{\tau + i\lambda} e^{-N\phi(\tau)} = e^{-N\phi(\sigma_+)}(K_1'(N) + K_2'(N)),$$

with the estimates below: There exist constants $k_0'(\lambda) > 0$ and $C_0'(\lambda) > 0$ such that

$$\left|K_1'(N)\right| \leq C_0'(\lambda)e^{-k_0'(\lambda)N} \quad \text{and} \quad \left|K_2'(N)\right| \leq C_0'(\lambda)N^{-1/2}$$

with $k_0'(\lambda) \to 0$ and $C_0'(\lambda) \to \infty$ for $\lambda \searrow 2J$ respectively and $K_2'(N)$ is a purely imaginary valued function.

Proof. We use the same notation $t_5,\ b_2,\ m_2,\ \tau_N = t_5/2$ as in Lemma A.12. Since $(t+\sigma_+ + i\lambda)^{-1} = \eta(1+\eta t)^{-1}$, we have

$$\int_{-\tau_N}^{\tau_N} dt\, \frac{1}{t+\sigma_+ + i\lambda} e^{-NF_7(t)} = \eta \int_{-\tau_N}^{\tau_N} dt\, (1+\eta t)^{N-1} e^{(-b_2+\frac{1}{2}\eta^2)Nt^2 - N\eta t} \equiv K_2'(N).$$

By the similar fashion used in the proof of Lemma A.13, it follows that

$$\int_{-\tau_N}^{\tau_N} dt\, (1+\eta t)^{N-1} e^{(-b_2+\frac{1}{2}\eta^2)Nt^2 - N\eta t} \in \mathbb{R}.$$

Since $\eta \in i\mathbb{R}$, $K_2'(N)$ is pure imaginary. We put

$$K_1'(N) \equiv \int_{\mathbb{R}\setminus(-\tau_\mathbb{N},\tau_\mathbb{N})} dt\, \frac{1}{t+\sigma_+ + i\lambda} e^{-NF_7(t)}.$$

Noting that $|(t+\sigma_+ + i\lambda)^{-1}| \leq J^{-1}$ on $\mathbb{R}$ when $\lambda \geq 2J$, proceeding as the proofs of Lemma A.12 and Lemma A.13, we get the desired result. □

A.4.2. *The integrals* $\int_{[0,\infty)} ds\, e^{-N\varphi(s)}$ *and* $\int_{[0,\infty)} ds\, se^{-N\varphi(s)}$. We deform the integration contour from $[0,\infty)$ to the path $l_8 \cup l_9$ where

$$\begin{aligned} &l_8; s_8(t) = \sigma_+ - it, \qquad -\frac{\lambda-\rho}{2} \le t \le \rho, \\ &l_9; s_9(t) = \sigma_- + t \qquad 0 \le t. \end{aligned}$$

Proposition A.15. *We have the following decomposition*

$$\int_{l_8\cup l_9} ds\, e^{-N\varphi(s)} = -ie^{-N\varphi(\sigma_+)}(L_1(N) + L_2(N)),$$

which satusfy the estimates below: There exist constants $\ell_0(\lambda) > 0$ *and* $D_0(\lambda) > 0$ *such that*

$$\big|L_1(N)\big| \le D_0(\lambda)e^{-\ell_0(\lambda)N} \quad \text{and} \quad \big|L_2(N)\big| \le D_0(\lambda)N^{-1/2}$$

with $\ell_0(\lambda) \to 0$ *and* $D_0(\lambda) \to \infty$ *for* $\lambda \searrow 2J$ *respectively and* $L_2(N)$ *is real valued.*

Proof. On $\int_{l_8} ds\, e^{-N\varphi(s)}$: As

$$\varphi(s_8(t)) = \frac{1}{2J^2}\big(-t^2 + (\lambda+\rho)t + \frac{(\lambda-\rho)(3\lambda+\rho)}{4}\big) - \log\big(i\frac{\rho-\lambda}{2} - it\big),$$

we have

$$\begin{aligned} F_8(t) = \varphi(s_8(t)) - \varphi(s_8(0)) &= \frac{-t^2 + (\lambda+\rho)t}{2J^2} - \log\big(1 + \frac{2t}{\lambda-\rho}\big) \\ &= c_2t^2 + \sum_{n=3}^{\infty} \frac{(-1)^n}{n}\zeta^n t^n \quad \text{for } (|\zeta t| < 1) \end{aligned}$$

with

$$c_2 = \frac{\rho(\lambda+\rho)}{4J^4}, \quad \zeta = \frac{\lambda+\rho}{2J^2}.$$

Noting $F_8(t)$ is real, we have that (i) there exists $t_6 > 0$ such that $F_8(t) \ge (c_2/2)t^2$ on $[-t_6, t_6]$ and (ii) as $F_8'(t) \ge 0$ for $0 \le t \le \rho$ and $F_8'(t) \le 0$ for $-(\lambda-\rho)/2 \le t \le 0$, there exists $m_3 = \min_{[-(\lambda-\rho)/2,\rho]\setminus[-t_6,t_6]} F_8(t) > 0$.

Then, we have

$$\int_{l_8} ds\, e^{-N\varphi(s)} = -i\left(\int_{-t_6}^{t_6} dt\, e^{-N\varphi(s_8(t))} + \int_{[-(\lambda-\rho)/2,\rho]\setminus[-t_6,t_6]} dt\, e^{-N\varphi(s_8(t))}\right).$$

Since

$$\int_{-t_6}^{t_6} dt\, e^{-N\varphi(s_8(t))} = e^{-N\varphi(\sigma_+)} \int_{-t_6}^{t_6} dt\, e^{-NF_8(t)},$$

calculating as in Lemma A.13, we get readily

$$\mathbb{R} \ni \int_{-t_6}^{t_6} dt\, e^{-NF_8(t)}, \quad \left|\int_{-t_6}^{t_6} dt\, e^{-NF_8(t)}\right| \le \sqrt{\frac{2\pi}{c_2}}N^{-1/2}.$$

On the other hand, we get

$$\int_{[-(\lambda-\rho)/2,\rho]\setminus[-t_6,t_6]} dt\, e^{-N\varphi(s_8(t))} = e^{-N\varphi(\sigma_+)} \int_{[-(\lambda-\rho)/2,\rho]\setminus[-t_6,t_6]} dt\, e^{-NF_8(t)},$$

and $\varphi(s_8(t)) \ge m_3$ on $[-(\lambda-\rho)/2, \rho] \setminus [-t_6, t_6]$, we have with some constant $D_0 > 0$,

$$\left|\int_{[-(\lambda-\rho)/2,\rho]\setminus[-t_6,t_6]} dt\, e^{-NF_8(t)}\right| \le D_0e^{-m_3N}. \tag{A.38}$$

On $\int_{l_9} ds\, e^{-N\varphi(s)}$: Putting

$$f_9(t) = \Re\varphi(s_9(t)) = \frac{1}{2J^2}\big(t^2 + \lambda(\lambda+\rho) - \frac{(\lambda+\rho)^2}{4}\big) - \frac{1}{2}\log\big(t^2 + \frac{(\lambda+\rho)^2}{4}\big),$$

and remarking $f_9'(t) \geq 0$ for $t \geq 0$, there exists $\varepsilon_4 > 0$ such that

$$\inf \Re\varphi(s_9(t)) > \Re\varphi(\sigma_+) + \varepsilon_4, \quad \varepsilon_4 = \Re\varphi(\sigma_-) - \Re\varphi(\sigma_+) = \frac{\lambda\rho}{2j^2} + \log\frac{\lambda-\rho}{\lambda+\rho}.$$

As $e^{-N\varphi(s_9(t))}$ is integrable, we get

$$\left|\int_{l_9} ds\, e^{-N\varphi(s)}\right| \leq Ce^{-N\Re\varphi(\sigma_+)-\varepsilon_4 N}, \tag{A.39}$$

with some constant $C > 0$. Thus, we may rewrite

$$\int_{l_9} ds\, e^{-N\varphi(s)} = -ie^{-N\varphi(\sigma_+)} L_3(N)$$

where

$$\left|L_3(N)\right| \leq Ce^{-\varepsilon_4 N}.$$

Putting

$$L_1(N) \equiv \int_{[-(\lambda-\rho)/2,\rho]\setminus[-t_6,t_6]} dt\, e^{-NF_8(t)} + L_3(N), \quad L_2(N) \equiv \int_{-t_6}^{t_6} dt\, e^{-NF_8(t)},$$

and choosing $\ell_0(\lambda)$ and $D_0(\lambda)$ suitably as before, we have the desired result. The end of the proof of Proposition A.15. □

Proposition A.16. *We have the following decomposition*

$$\int_{l_8\cup l_9} ds\, se^{-N\varphi(s)} = e^{-N\varphi(\sigma_+)}(L_1'(N) + L_2'(N)),$$

which satisfy the estimates below: There exist constants $\ell_0'(\lambda) > 0$ and $D_0'(\lambda) > 0$ such that

$$\left|L_1'(N)\right| \leq D_0'(\lambda)e^{-\ell_0'(\lambda)N} \quad \textit{and} \quad \left|L_2'(N)\right| \leq D_0'(\lambda)N^{-1/2}$$

with $\ell_0'(\lambda) \to 0$ and $D_0'(\lambda) \to \infty$ for $\lambda \searrow 2J$ respectively and $L_2'(N)$ is real valued function.

Proof. By the definition of l_8 we have

$$\begin{aligned}\int_{l_8} ds\, se^{-N\varphi(s)} &= -i\left(\int_{-t_6}^{t_6} dt\,(\sigma_+ - it)e^{-N\varphi(s_8(t))} + \left(\int_{-\frac{\lambda-\rho}{2}}^{-t_6} + \int_{t_6}^{\frac{\lambda-\rho}{2}}\right) dt\,(\sigma_+ - it)e^{-N\varphi(s_8(t))}\right)\\ &= e^{-N\varphi(\sigma_+)}\left(\int_{-t_6}^{t_6} dt\,(-i\sigma_+ - t)e^{-NF_8(t)} + \left(\int_{-\frac{\lambda-\rho}{2}}^{-t_6} + \int_{t_6}^{\frac{\lambda-\rho}{2}}\right) dt\,(-i\sigma_+ - t)e^{-NF_8(t)}\right)\end{aligned}$$

Because $-i\sigma_+ = -(\lambda-\rho)/2 \in \mathbb{R}$ and F_8 is a real valued function, the integral

$$\int_{-t_6}^{t_6} dt\,(-i\sigma_+ - t)e^{-NF_8(t)}$$

is real valued. Since $|-i\sigma_+ - t| \leq 2^{-1}(\lambda+\rho)$ on $[-(\lambda-\rho)/2, \rho]$, we have

$$\left|\int_{-t_6}^{t_6} dt\,(-i\sigma_+ - t)e^{-NF_8(t)}\right| \leq \frac{\lambda+\rho}{2}\sqrt{\frac{2\pi}{c_2}}N^{-\frac{1}{2}}.$$

Moreover, we have

$$\left|\left(\int_{-(\lambda-\rho)/2}^{-t_6} + \int_{t_6}^{\rho}\right) dt\,(-i\sigma_+ - t)e^{-NF_8(t)}\right| \leq \frac{\lambda+\rho}{2}D_0 e^{-m_3 N}$$

where m_3 is given in Proposition A.15.

Next we consider

$$\int_{l_9} ds\, se^{-N\varphi(s)} = \int_0^\infty dt\,(\sigma_- + t)e^{-N\varphi(s_9(t))}.$$

Since there exist $\varepsilon_5 > 0$ such that

$$\inf_{t\geq 0} \Re\varphi(s_9(t)) > \Re\varphi(\sigma_+) + \varepsilon_5,$$

we have

$$\left|\int_0^\infty dt\,(\sigma_- + t)e^{-N\varphi(s_9(t))}\right| < \left|e^{\Re\varphi(\sigma_+)+\varepsilon_5}\int_0^\infty dt\,(\sigma_- + t)e^{-\varphi(s_9(t))}\right| e^{-N\Re\varphi(\sigma_+)}e^{-\varepsilon_5 N}$$

by the integrability of $(\sigma_- + t)e^{-\varphi(s_9(t))}$ on $[0,\infty)$. Hence we have

$$\int_0^\infty dt\,(\sigma_- + t)e^{-N\varphi(s_9(t))} = e^{-N\varphi(\sigma_+)}L_3'(N),$$

where with some constant $C > 0$

$$|L_3'(N)| \le Ce^{-\varepsilon_5 N}.$$

Therefore putting

$$L_1'(N) \equiv \left(\int_{-\frac{\lambda-\rho}{2}}^{-t_6} + \int_{t_6}^{\rho}\right) dt\,(-i\sigma_+ - t)e^{-NF_8(t)} + L_3'(N),$$
$$L_2'(N) \equiv \int_{-\tau}^{\tau} dt\,(-i\sigma_+ - t)e^{-NF_8(t)},$$

we get the result. □

The Proof of Theorem 3.2: By Proposition A.11, A.14, A.15, A.16 and using

$$\phi(\sigma_+) + \varphi(\sigma_-) = 1 - \log J^2$$

we have

$$\int_{l_7} d\tau\, e^{-N\phi(\tau)}\cdot\int_{l_8\cup l_9} ds\, e^{-N\varphi(s)} + \int_{l_7} d\tau\,\frac{1}{\tau + i\lambda}e^{-N\phi(\tau)}\cdot\int_{l_8\cup l_9} ds\, se^{-N\varphi(s)}$$
$$= e^{-N(1-\log J^2)}(-i(K_1L_1 + K_1L_2 + K_2L_1 + K_2L_2) + (K_1'L_1' + K_1'L_2' + K_2'L_1' + K_2'L_2')).$$

From Proposition A.11, A.14, A.15, A.16, we have

$$\left| -i(K_1L_1 + K_1L_2 + K_2L_1) + K_1'L_1' + K_1'L_2' + K_2'L_1'\right| \le C(\lambda)N^{-k(\lambda)N}$$

where $C(\lambda) = \max(C_0(\lambda), C_0'(\lambda), D_0(\lambda), D_0'(\lambda))$ and $k(\lambda) = \min(k_0(\lambda), k_0'(\lambda), \ell_0(\lambda), \ell_0'(\lambda))$ and $-iK_2L_2$ and $K_2'L_2'$ are pure imaginary. Therefore Theorem 3.2 is proved. □

Appendix B. A resolution of an ambiguity of Q-integration

B.1. **Diagonalization.** It is known that

$$\int_{\mathfrak{Q}} dQ\, e^{-\operatorname{str} Q^2} = \int_{\mathfrak{R}^{2|2}} \frac{dx_1dx_2}{2\pi}d\theta_1 d\theta_2\, e^{-(x_1^2+x_2^2+2\theta_1\theta_2)} = 1. \tag{B.1}$$

We may diagonalize the matrix Q by using the change of variables

$$\begin{cases} y_1 = x_1 + \dfrac{\theta_1\theta_2}{x_1 - ix_2},\ y_2 = x_2 - \dfrac{i\theta_1\theta_2}{x_1 - ix_2}, \\ \rho_1 = \dfrac{\theta_1}{x_1 - ix_2},\ \rho_2 = -\dfrac{\theta_2}{x_1 - ix_2}, \end{cases} \tag{B.2}$$

or

$$\begin{cases} x_1 = y_1 + \rho_1\rho_2(y_1 - iy_2),\ x_2 = y_2 - i\rho_1\rho_2(y_1 - iy_2), \\ \theta_1 = \rho_1(y_1 - iy_2),\ \theta_2 = -\rho_2(y_1 - iy_2), \end{cases} \tag{B.3}$$

such that

$$GQG^{-1} = \begin{pmatrix} y_1 & 0 \\ 0 & iy_2 \end{pmatrix},\quad GQ^2G^{-1} = \begin{pmatrix} y_1^2 & 0 \\ 0 & -y_2^2 \end{pmatrix} \tag{B.4}$$

where

$$G = \begin{pmatrix} 1 + 2^{-1}\rho_1\rho_2 & \rho_1 \\ \rho_2 & 1 - 2^{-1}\rho_1\rho_2 \end{pmatrix},\quad G^{-1} = \begin{pmatrix} 1 + 2^{-1}\rho_1\rho_2 & -\rho_1 \\ -\rho_2 & 1 - 2^{-1}\rho_1\rho_2 \end{pmatrix}.$$

It is clear that

$$x_1 - ix_2 = y_1 - iy_2, \quad \text{and} \quad \text{str } Q^2 = x_1^2 + x_2^2 + 2\theta_1\theta_2 = y_1^2 + y_2^2.$$

On the other hand, the so-called Berezinian is given by

$$dQ = \frac{dx_1 dx_2}{2\pi} d\theta_1 d\theta_2 = -\frac{dy_1 dy_2}{2\pi} d\rho_1 d\rho_2 \, (y_1 - iy_2)^{-2}.$$

"An ambiguity" related to the formula for the integration under the change of variables, occurs because

$$-\int \frac{dy_1 dy_2}{2\pi} d\rho_1 d\rho_2 \, (y_1 - iy_2)^{-2} e^{-(y_1^2+y_2^2)} = 0.$$

An explanation given in Fyodorov [8] (pp. 501-502) or F. Constantinescu and H.F. de Groote [4] (p.991) is outside of our comprehension.

We use the prescription of Rothstein [17] (see also M.Martellini and P. Teofilatto [14], M.R. Zirnbauer [21]) to remedy this ambiguity by altering the notion of the volume form on the superspace.

The change of variables (B.3), which is called degree increasing in [17] or non-splitting in [14], may be generated by a vector field $Y(y,\rho)$ with $(x,\theta) = e^{Y(y,\rho)}(y,\rho)$, given by

$$e^{Y(y,\rho)} = 1 + \rho_1\rho_2(y_1 - iy_2)\frac{\partial}{\partial y_1} - i\rho_1\rho_2(y_1 - iy_2)\frac{\partial}{\partial y_2} + (y_1 - iy_2 - 1)\rho_1\frac{\partial}{\partial \rho_1} - (y_1 - iy_2 + 1)\rho_2\frac{\partial}{\partial \rho_2}.$$

By the prescription given in [17] and more precisely in [14], we should have

$$\frac{dx_1 dx_2}{2\pi} d\theta_1 d\theta_2 = -\frac{dy_1 dy_2}{2\pi} d\rho_1 d\rho_2 [1 - \big(\rho_1\rho_2(y_1 - iy_2)\frac{\partial}{\partial y_1} - i\rho_1\rho_2(y_1 - iy_2)\frac{\partial}{\partial y_2}\big)](y_1 - iy_2)^{-2}.$$

Using this, we have

$$\begin{aligned}\int \frac{dx_1 dx_2}{2\pi} d\theta_1 d\theta_2 \, e^{-(x_1^2+x_2^2+2\theta_1\theta_2)} &= -\int \frac{dy_1 dy_2}{2\pi} d\rho_1 d\rho_2 \, (y_1 - iy_2)^{-2} e^{-(y_1^2+y_2^2)} \\ &\quad + \int \frac{dy_1 dy_2}{2\pi} d\rho_1 d\rho_2 \, \rho_1\rho_2 (y_1 - iy_2)(\frac{\partial}{\partial y_1} - i\frac{\partial}{\partial y_2})(y_1 - iy_2)^{-2} e^{-(y_1^2+y_2^2)}.\end{aligned}$$

Though the singularity $(y_1 - iy_2)^{-2}$ must be treated by deleting $|y| < \epsilon$ from the integration domain and making $\epsilon \to 0$, after calculating w.r.t. odd variables, we have

$$-\int \frac{dy_1 dy_2}{2\pi} (y_1 - iy_2)(\frac{\partial}{\partial y_1} - i\frac{\partial}{\partial y_2})(y_1 - iy_2)^{-2} e^{-(y_1^2+y_2^2)} = 1.$$

B.2. **An application of the diagonalization** (B.3) **to** (3.13)**.** We apply the above diagonalization (B.3) to the expression below (3.13):

$$\big\langle \text{tr } \frac{1}{\mu I_N - H} \big\rangle_N = \int_{\mathfrak{Q}} dQ \, N\big(\{\mu I_2 - Q\}^{-1}\big)_{bb} \exp\,[-N\mathcal{L}(\mu; Q)].$$

By the argument in the previous subsection, we have

$$G(\mu - Q)_{bb}^{-1} G^{-1} = (\mu - y_1)^{-1}, \text{ sdet}\,(\mu - Q) = (\mu - y_1)(\mu - iy_2)^{-1},$$

$$\mathcal{L}(\mu; GQG^{-1}) = \frac{1}{2J^2}(y_1^2 + y_2^2) + \log\frac{\mu - y_1}{\mu - iy_2}.$$

Then, we have

$$\begin{aligned}\int_{\mathfrak{Q}} dQ \, N\big(\{\mu I_2 - Q\}^{-1}\big)_{bb} \exp\,[-N\mathcal{L}(\mu; Q)] &= -\int \frac{dy_1 dy_2}{2\pi} d\rho_1 d\rho_2 \, \Phi(y_1, y_2; \mu) \\ &\quad + \int \frac{dy_1 dy_2}{2\pi} d\rho_1 d\rho_2 \, \rho_1\rho_2 \, (y_1 - iy_2)(\frac{\partial}{\partial y_1} - i\frac{\partial}{\partial y_2})\Phi(y_1, y_2; \mu) \qquad \text{(B.5)}\end{aligned}$$

where

$$\Phi(y_1, y_2; \mu) = (\mu - y_1)^{-1}(y_1 - iy_2)^{-2} \exp\,[-N(\frac{y_1^2 + y_2^2}{2J^2} + \log\frac{\mu - y_1}{\mu - iy_2})].$$

As the second term of the right hand side of (B.5) yields the same form of (3.15), we have

$$\frac{1}{N}\langle \mathrm{tr}\ \frac{1}{\mu I_N - H}\rangle_N = \int_{\mathbb{R}^2} \frac{dy_1 dy_2}{2\pi} \frac{N(\mu - y_1 - iy_2)}{J^2(\mu - y_1)(\mu - iy_2)} \exp\left[-N\Phi(y_1, y_2; \mu)\right]. \tag{B.6}$$

References

[1] E. Brézin, *Grassmann variables and supersymmetry in the theory of disordered systems*, in Applications of field theory to statistical mechanics, ed. L. Garrido, Lecture Notes in Physics 216 (1985) pp 115-123.

[2] E. Brézin and V.A. Kazakov, *Exactly solvable field theories of closed strings*, Physics Letters B, 236(1990) pp 144-150.

[3] E. Brézin and A. Zee, *Universality of the correlations between eigenvalues of large random matrices*, Nucl.Phys. B402(1993) pp 613-627.

[4] F. Constantinescu and H.F. de Groote, *The integral theorem for supersymmetric invariants*, J.Math.Phys. 30(1989) pp. 981-992.

[5] N.G. de Bruijn, *Asymptotic Methods in Analysis*, North-Holland Publ. 1961.

[6] K.B. Efetov, *Supersymmetry and theory of disordered metals*, Advances in Physics 32(1983) pp. 53-127.

[7] P.J. Forrester, *The spectrum edge of random matrix ensembles*, Nucl.Phys. B402[FS](1993) pp. 709-728.

[8] Y.V. Fyodorov, *Basic features of Efetov's supersymmetry approach*, Les Houches, Session LXI, Physique Quantique Mésoscopique (eds. E. Akkermans, G. Montambaux, J.L. Pichand and J. Zinn-Justin), 1994 Elsevier pp. 493-532.

[9] A. Inoue *Foundations of real analysis on the superspace $\mathfrak{R}^{m|n}$ over ∞-dimensional Fréchet-Grassmann algebra*, J.Fac.Sci.Univ.Tokyo, 39(1992) pp. 419-474.

[10] A. Inoue, *On a construction of the fundamental solution for the free Weyl equation by Hamiltonian path-integral method – An exactly solvable case with "odd variables"*, Tôhoku Math.J. 50(1998), pp. 91-118.

[11] A. Inoue, *An introduction to superanalysis and its applications*, in preparation,

[12] A. Inoue and Y. Maeda, *Foundations of calculus on super Euclidean space $\mathfrak{R}^{m|n}$ based on a Fréchet-Grassmann algebra*, Kodai Math.J.14(1991) pp. 72-112.

[13] M.L. Mehta, *Random Matrices*, Academic Press, 2nd ed 1991 Boston.

[14] M.Martellini and P. Teofilatto, *Global structire of the superstring partition function and resolution of the supermoduli measure ambiguity*, Phys.Lett. B, 211 (1988) pp. 293-300.

[15] P.A. Mello, *Theory of Random Matrices: spectral statistics and scattering problems*, Les Houches, Session LXI, Physique Quantique Mésoscopique (eds. E. Akkermans, G. Montambaux, J.L. Pichand and J. Zinn-Justin), Elsevier 1994 pp. 435-491.

[16] M.J. Rothstein, *The axioms of supermanifolds and a new structure arising from them*, Trans.Amer.Math.Soc. 297(1986) pp. 159-180.

[17] ———, *Integration on non-compact supermanifolds*, Trans.Amer.Math.Soc. 299(1987) pp. 387-396.

[18] H. Sakamoto, *Master Thesis presented to Nihon University in 1997*, March 1998.

[19] J.J.M. Verbaarschot, H.A. Weidenmüller and M.R. Zirnbauer, *Grassmann integration in stochastic quantum physics: The case of compound-nucleus scattering*, Phys. Reports 129(1985) pp. 367-438.

[20] K. Ziegler, *Disorder system with n orbitals per site: Lagrange formulation without replica trick, and scaling law for the density of states*, Z.Phys.B-Condensed Matter48(1982) pp. 293-304.

[21] M.R. Zirnbauer, *Riemannian symmetric superspaces and their origin in random-matrix theory*, J.Math.Phys.37(1996) pp. 4986-5018.

Oh-okayama, Meguro-ku, Tokyo, 152, Japan

E-mail address: inoue@math.titech.ac.jp, nomura@math.titech.ac.jp