A memo
on a 2×2-matrix version
of Hamilton-Jacobi and continuity equations*

Atsushi INOUE†

In memory of Michihiro NAGASE

Abstract

The Hamilton-Jacobi equation for a scalar symbol $H(t,q,p)$ is well known. Here, we extend this notion to 2×2-matrix valued symbol $\mathbb{H}(t,q,p)$, which forces us to prepare the new treatize of matrix structure. This new treatize allows us to recover the dependence of full components of matrix as it is, which should be compared with the incompleteness only using eigenvalues of matrix after diagonalization.

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1 Hamilton-Jacobi equation

1.1 The scalar case—a well-known procedure

For $H(t,q,p)\in C^\infty(\mathbb{R}\times T^*\mathbb{R}^d:\mathbb{R})$, we may construct at least locally in time, a unique solution $S(t,\underline{t}\,;q,p)$ of the Hamilton-Jacobi equation

$$\frac{\partial S(t,\underline{t}\,;q,p)}{\partial t}+H\Big(t,q,\frac{\partial S(t,\underline{t}\,;q,p)}{\partial q}\Big)=0 \quad\text{with}\quad S(\underline{t},\underline{t}\,;q,p)=q\cdot p=\langle q|p\rangle=\sum_{j=1}^d q_jp_j. \tag{1}$$

Jacobi's Method: (i) Solve Hamilton's canonical equation whose solution is denoted by $(q(t,\underline{t}),p(t,\underline{t}))$ or more precisely, $(q(t,\underline{t}\,;\underline{q},\underline{p}),p(t,\underline{t}\,;\underline{q},\underline{p}))$,
(ii) Calculate inverse map of Hamilton flow, i.e. applying the inverse function theorem to $\bar{q}=q(\bar{t},\underline{t},\underline{q},\underline{p})$ for given $(\bar{q},\underline{p})$, we get $\underline{q}=y(\bar{t},\underline{t},\bar{q},\underline{p})=y(\bar{t},\underline{t})$,
(iii) Construct a solution of Hamilton-Jacobi equation by putting

$$S(\bar{t},\underline{t}\,;\bar{q},\underline{p})=\phi(\bar{t},\underline{t}\,;y(\bar{t},\underline{t}\,;\bar{q},\underline{p}),\underline{p}),$$

where

$$\phi(t,\underline{t})=\phi(t,\underline{t}\,;\underline{q},\underline{p})=\underline{q}\cdot\underline{p}+\int_{\underline{t}}^t(p\nabla_pH-H)(\tau,q(\tau,\underline{t}),p(\tau,\underline{t}))d\tau.$$

Important point: (iv) Since the dependence on $(\underline{q},\underline{p})$ for $(q(t,\underline{t}\,;\underline{q},\underline{p}),p(t,\underline{t}\,;\underline{q},\underline{p}))$ is calculable, that on $(\bar{q},\underline{p})$ for $S(\bar{t},\underline{t}\,;\bar{q},\underline{p})$ is also computable following above steps.

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[†] Tokyo Institute of Technology, Oh-okayama, Meguro-ku, Tokyo, 152, Japan

An application to "quantum" case: Putting

$$D(t,\underline{t};\bar{q},\underline{q}) = \det\left(\frac{\partial^2 S(t,\underline{t}\,;\bar{q},\underline{p})}{\partial\bar{q}_j\partial\underline{p}_k}\right),$$

called van Vleck determinant, we may define a Fourier integral operator through

$$F(t,\underline{t})u(\bar{q}) = c\int D(t,\underline{t};\bar{q},\underline{p})^{1/2}e^{iS(t,\underline{t}\,;\bar{q},\underline{p})}\hat{u}(t,\underline{p})d\underline{p}.$$

Here, in order to show the L^2-boundedness of this operator, we need to know the dependence on $(\bar{q},\underline{p})$ for $S(\bar{t},\underline{t}\,;\bar{q},\underline{p})$.

Dividing $[\underline{t},t]$ as

$$\Delta : \underline{t} = t_0 < t_1 < \cdots < t_{n-1} < t_n = t, \quad |\Delta| = \max|t_k - t_{k-1}|$$

we have, by the so-called Lie-Trotter-Kato procedure,

$$U(t,\underline{t}) = \lim_{|\Delta|\to 0} F(t_n,t_{n-1})\cdot F(t_{n-1},t_{n-2})\cdot\cdots\cdot F(t_1,t_0)$$

which satisfies

$$i\frac{\partial}{\partial t}U(t,\underline{t}) = H(t,q,D_q)U(t,\underline{t}), \quad U(\underline{t},\underline{t}) = I.$$

That is, $F(t,\underline{t})\underline{u}$ may be considered as a"parametrix" of the initial value problem for

$$i\frac{\partial}{\partial t}u(t) = H(t,q,D_q)u(t) \quad \text{with} \quad u(\underline{t}) = \underline{u}.$$

1.2 A Problem

Replace a scalar function $H(t,q,p)$ above with

$$\mathbb{H}(t,q,p) = \begin{pmatrix} a(t,q,p) & (c-id)(t,q,p) \\ (c+id)(t,q,p) & b(t,q,p) \end{pmatrix}$$

where $a,b,c,d \in \mathcal{B}_t^\infty(\mathrm{Symb}^1)$. Here, putting $\langle p\rangle = \sqrt{1+|p|^2}$,

$$\mathrm{Symb}^k = \big\{\phi(q,p) \in C^\infty(T^*\mathbb{R}^d : \mathbb{R}) \,\big|\, |\partial_q^\alpha\partial_p^\beta\phi(q,p)| \le C_{\alpha,\beta}\langle p\rangle^{k-|\beta|}\big\}.$$

Problem: Can we "introduce" the Hamilton-Jacobi equation corresponding to the above matrix $\mathbb{H}(t,q,p)$?

2 New look at matrix structure

In this section, we embody the following idea:
"A matrix is decomposed by Clifford algebra which is represented on Grassmann algebra".

2.1 Decomposition of matrix

Decompose a 2×2 matrix using the Clifford algebra as

$$\mathbb{A} = \frac{a+b}{2}\mathbb{I}_2 + \frac{a-b}{2}\boldsymbol{\sigma}_3 + \frac{c+d}{2}\boldsymbol{\sigma}_1 + i\frac{c-d}{2}\boldsymbol{\sigma}_2,$$

where the Pauli matrices $\{\boldsymbol{\sigma}_j\}_{j=1}^3$ are given by

$$\boldsymbol{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

with the following relations:

$$\boldsymbol{\sigma}_j\boldsymbol{\sigma}_k + \boldsymbol{\sigma}_k\boldsymbol{\sigma}_j = 2\delta_{jk}\mathbb{I}_2 \quad \text{for } j,k = 1,2,3, \quad \text{(Clifford relation)}$$

$$\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2 = i\boldsymbol{\sigma}_3, \quad \boldsymbol{\sigma}_2\boldsymbol{\sigma}_3 = i\boldsymbol{\sigma}_1, \quad \boldsymbol{\sigma}_3\boldsymbol{\sigma}_1 = i\boldsymbol{\sigma}_2.$$

Key point: In the following, we consider $\{\boldsymbol{\sigma}_j\}_{j=1}^3$ as something like "co-ordinates frame" representing "matrix structure". To make concrete this point of view, we need another notion "superanalysis" which is given in Appendices.

2.2 Pauli matrices are considered as differential operators

Preparing **odd variables** θ_1, θ_2 which satisfy $\theta_1^2 = \theta_2^2 = 0 = \theta_1\theta_2 + \theta_2\theta_1$, we introduce a Grassmann algebra and identification maps $\#$, $\flat$ as

$$\Gamma_0 = \{u_0 + u_1\theta_1\theta_2 \mid u_0, u_1 \in \mathbb{C}\} \overset{\flat}{\underset{\sharp}{\rightleftarrows}} \mathbb{C}^2, \quad \#\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = u_0 + u_1\theta_1\theta_2,\ \flat(u_0 + u_1\theta_1\theta_2) = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}.$$

Defining

$$\begin{cases} \sigma_1(\theta, \partial_\theta) = \theta_1\theta_2 - \dfrac{\partial^2}{\partial\theta_1\partial\theta_2} = \theta_1\theta_2 + D_{\theta_1}D_{\theta_2}, \\ \sigma_2(\theta, \partial_\theta) = i\Big[\theta_1\theta_2 + \dfrac{\partial^2}{\partial\theta_1\partial\theta_2}\Big] = i\Big[\theta_1\theta_2 - D_{\theta_1}D_{\theta_2}\Big], \\ \sigma_3(\theta, \partial_\theta) = 1 - \theta_1\dfrac{\partial}{\partial\theta_1} - \theta_2\dfrac{\partial}{\partial\theta_2} = 1 - i\theta_1 D_{\theta_1} - i\theta_2 D_{\theta_2}, \end{cases}$$

where $D_{\theta_k} = -i\partial_{\theta_k}$, we have

$$\sigma_j(\theta, \partial_\theta)\sigma_k(\theta, \partial_\theta) + \sigma_k(\theta, \partial_\theta)\sigma_j(\theta, \partial_\theta) = 2\delta_{jk}I \quad \text{on } \Gamma_0,$$

with

$$\begin{gathered} \sigma_1(\theta, \partial_\theta)\sigma_2(\theta, \partial_\theta) = i\sigma_3(\theta, \partial_\theta), \quad \sigma_2(\theta, \partial_\theta)\sigma_3(\theta, \partial_\theta) = i\sigma_1(\theta, \partial_\theta), \\ \sigma_3(\theta, \partial_\theta)\sigma_1(\theta, \partial_\theta) = i\sigma_2(\theta, \partial_\theta) \quad \text{on } \Gamma_0. \end{gathered}$$

Remark. We may regard that

$$\begin{gathered} \theta_j(u_0 + u_1\theta_1\theta_2) \sim dz_j \wedge (u_0 + u_1 dz_1 \wedge dz_2), \\ \frac{\partial}{\partial\theta_j}(u_0 + u_1\theta_1\theta_2) \sim \frac{d}{dz_j} \rfloor (u_0 + u_1 dz_1 \wedge dz_2). \end{gathered}$$

For example,

$$\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \sim u_0 + u_1\theta_1\theta_2 \Longrightarrow \Big[\theta_1\theta_2 - \frac{\partial^2}{\partial\theta_1\partial\theta_2}\Big][u_0 + u_1\theta_1\theta_2] = u_0\theta_1\theta_2 + u_1 \sim \begin{pmatrix} u_1 \\ u_0 \end{pmatrix}$$

That is,

$$\sigma_1(\theta, \partial_\theta) \sim \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \flat\sigma_1(\theta, \partial_\theta)\#\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}.$$

Analogously,

$$\sigma_3(\theta, \partial_\theta) = 1 - \theta_1\frac{\partial}{\partial\theta_1} - \theta_2\frac{\partial}{\partial\theta_2} \sim \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \flat\sigma_3(\theta, \partial_\theta)\#\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}.$$

Note. It takes some time for me to find out the origin of the term 1 in $\sigma_3(\theta, \partial_\theta)$, i.e., Weyl type quantization of the "symbol" $\theta_1\pi_1 + \theta_2\pi_2$ yields 1 in $\sigma_3(\theta, \partial_\theta)$!

Remark. To justify the above interpretation of matrix as differential operators acting on supersmooth functions, we need only the Grassmann properties of $\{\theta_j\}$, their finer properties like containing countably many Grassmann generators are not necessary.

3 Classical Mechanics on superspace

3.1 Reformulation

Using Fourier transformation, we put

$$\begin{cases} \sigma_1(\theta,\pi) = \theta_1\theta_2 + \pi_1\pi_2, \\ \sigma_2(\theta,\pi) = i(\theta_1\theta_2 - \pi_1\pi_2), \\ \sigma_3(\theta,\pi) = -i\langle\theta|\pi\rangle = -i(\theta_1\pi_1 + \theta_2\pi_2), \end{cases}$$

which yields $\mathcal{H}(t,x,\xi,\theta,\pi) \in \mathcal{C}_{\mathrm{SS}}(\mathbb{R}\times\mathcal{T}^*\mathfrak{R}^{d|2} : \mathfrak{R}_{\mathrm{ev}})$ as

$$\begin{aligned} \mathcal{H}(t,x,\xi,\theta,\pi) &= \mathfrak{a}(t,x,\xi) + \mathfrak{b}(t,x,\xi)\sigma_3(\theta,\pi) + c(t,x,\xi)\sigma_1(\theta,\pi) + d(t,x,\xi)\sigma_2(\theta,\pi) \\ &= \mathfrak{a}(t,x,\xi) - i\mathfrak{b}(t,x,\xi)\langle\theta|\pi\rangle + (c+id)(t,x,\xi)\theta_1\theta_2 + (c-id)(t,x,\xi)\pi_1\pi_2 \end{aligned}$$

where

$$\mathfrak{a}(t,x,\xi) = \frac{a(t,x,\xi)+b(t,x,\xi)}{2}, \quad \mathfrak{b}(t,x,\xi) = \frac{a(t,x,\xi)-b(t,x,\xi)}{2}.$$

Moreover, we have

$$\begin{cases} a(t,q,p) = \Big(\mathcal{H}(t,x,\xi,\theta,\pi) - \dfrac{i}{2}\displaystyle\sum_{j=1}^{2}\frac{\partial^2}{\partial\theta_j\partial\pi_j}\mathcal{H}(t,x,\xi,\theta,\pi)\Big)\Big|_{\substack{x_{\mathrm{B}}=q,\,\xi_{\mathrm{B}}=p \\ \theta=0,\,\pi=0}}, \\ b(t,q,p) = \Big(\mathcal{H}(t,x,\xi,\theta,\pi) + \dfrac{i}{2}\displaystyle\sum_{j=1}^{2}\frac{\partial^2}{\partial\theta_j\partial\pi_j}\mathcal{H}t,(x,\xi,\theta,\pi)\Big)\Big|_{\substack{x_{\mathrm{B}}=q,\,\xi_{\mathrm{B}}=p \\ \theta=0,\,\pi=0}}, \\ c(t,q,p) = \dfrac{\partial^2}{\partial\pi_2\partial\pi_1}\mathcal{H}(t,x,\xi,\theta,\pi)\Big|_{\substack{x_{\mathrm{B}}=q,\,\xi_{\mathrm{B}}=p \\ \theta=0,\,\pi=0}}, \\ d(t,q,p) = \dfrac{\partial^2}{\partial\theta_2\partial\theta_1}\mathcal{H}(t,x,\xi,\theta,\pi)\Big|_{\substack{x_{\mathrm{B}}=q,\,\xi_{\mathrm{B}}=p \\ \theta=0,\,\pi=0}}. \end{cases} \tag{2}$$

Then, the operator $\mathcal{H}(t,x,D_x,\theta,D_\theta)$ acting on $u(x,\theta) = u_0(x) + u_1(x)\theta_1\theta_2$, is given by

$$\mathcal{H}(t,x,D_x,\theta,D_\theta)u(x,\theta) = (2\pi)^{-d}\int_{\mathcal{T}^*\mathfrak{R}^{d|2}} d\xi dx' d\pi d\theta'\, e^{i\langle x-x'|\xi\rangle + i\langle\theta-\theta'|\pi\rangle}\mathcal{H}\big(t,\frac{x+x'}{2},\xi,\frac{\theta+\theta'}{2},\pi\big)u(x',\theta').$$

We should remark that

$$\begin{aligned} &\mathcal{H}(t,x,D_x,\theta,D_\theta)u(x,\theta)\big|_{\theta=0} = \hat{a}u_0(x) + (\hat{c} - i\hat{d})u_1(x), \\ &\frac{\partial^2}{\partial\theta_2\partial\theta_1}\mathcal{H}(t,x,\partial_x,\theta,\partial_\theta)u(x,\theta)\big|_{\theta=0} = (\hat{c}+i\hat{d})u_0(x) + \hat{b}u_1(x) \end{aligned}$$

with $\hat{a}u_0(x) = a(t,x,D_x)u_0(x)\big|_{x_{\mathrm{B}}=q} = (2\pi)^{-d}\iint dp\,dq' e^{i\langle q-q'|p\rangle}a(t,\frac{q+q'}{2},p)\psi_1(q')$, etc.

3.2 Classical mechanics on superspace

For $\mathcal{H}(t,x,\xi,\theta,\pi)$, we introduce the super-version of Hamilton canonical equation as

$$\begin{cases} \dfrac{d}{dt}x_i = \dfrac{\partial\mathcal{H}(t,x,\xi,\theta,\pi)}{\partial\xi_i}, & \dfrac{d}{dt}\xi_j = -\dfrac{\partial\mathcal{H}(t,x,\xi,\theta,\pi)}{\partial x_j}, \\ \dfrac{d}{dt}\theta_k = -\dfrac{\partial\mathcal{H}(t,x,\xi,\theta,\pi)}{\partial\pi_k}, & \dfrac{d}{dt}\pi_l = -\dfrac{\partial\mathcal{H}(t,x,\xi,\theta,\pi)}{\partial\theta_l}. \end{cases} \tag{3}$$

Theorem 1 *Let $a,b,c,d\in\mathcal{B}_t^{\infty}(\mathrm{Symb}^1)$. (0) There exists a unique local (in time) solution $(x(t),\xi(t),\theta(t),\pi(t))$ of (3) with any initial data $(x(\underline{t}),\xi(\underline{t}),\theta(\underline{t}),\pi(\underline{t}))=(\underline{x},\underline{\xi},\underline{\theta},\underline{\pi})\in\mathfrak{R}^{2d|4}$.*
(I) For any $T>0$, the solution $(x(t),\xi(t),\theta(t),\pi(t))$ of (3) on $I=\{t\mid |t-\underline{t}|\le T\}$ is "s-smooth" in $(t,\underline{x},\underline{\xi},\underline{\theta},\underline{\pi})$. That is, smooth in t for fixed $(\underline{x},\underline{\xi},\underline{\theta},\underline{\pi})$ and supersmooth in $(\underline{x},\underline{\xi},\underline{\theta},\underline{\pi})$ for fixed t.
(II) (II-0) $t,\underline{t}\in I$, $|a+b|=0$, $N=|\alpha+\beta|=0,1,2,\cdots$:

$$\begin{cases} |\pi_{\mathrm{B}}\partial_{\underline{x}}^{\alpha}\partial_{\underline{\xi}}^{\beta}(x(t;\underline{x},\underline{\xi},\underline{\theta},\underline{\pi})-\underline{x})|\le C_{\alpha,\beta}|t-\underline{t}|\langle\underline{p}\rangle^{-|\beta|}, \\ |\pi_{\mathrm{B}}\partial_{\underline{x}}^{\alpha}\partial_{\underline{\xi}}^{\beta}(\xi(t;\underline{x},\underline{\xi},\underline{\theta},\underline{\pi})-\underline{\xi})|\le C_{\alpha,\beta}|t-\underline{t}|\langle\underline{p}\rangle^{1-|\beta|}. \end{cases}$$

(II-1) $|t-\underline{t}|\le 1$, $|a+b|=1$, $l,k=1,2$:

$$\begin{cases} |\pi_{\mathrm{B}}\partial_{\underline{\theta}}^{a}\partial_{\underline{\pi}}^{b}(\theta_l(t;\underline{x},\underline{\xi},\underline{\theta},\underline{\pi})-\underline{\theta}_l))|\le 2, \\ |\pi_{\mathrm{B}}\partial_{\underline{\theta}}^{a}\partial_{\underline{\pi}}^{b}(\pi_k(t;\underline{x},\underline{\xi},\underline{\theta},\underline{\pi})-\underline{\pi}_k))|\le 2. \end{cases}$$

(II-2) $|t-\underline{t}|\le 1$, $|a+b|=2$:

$$\begin{cases} |\pi_{\mathrm{B}}\partial_{\underline{\theta}}^{a}\partial_{\underline{\pi}}^{b}(x(t;\underline{x},\underline{\xi},\underline{\theta},\underline{\pi})-\underline{x})|\le C|t-\underline{t}|^{1/2}, \\ |\pi_{\mathrm{B}}\partial_{\underline{\theta}}^{a}\partial_{\underline{\pi}}^{b}(\xi(t;\underline{x},\underline{\xi},\underline{\theta},\underline{\pi})-\underline{\xi})|\le C\langle\underline{p}\rangle|t-\underline{t}|^{1/2}. \end{cases}$$

(II-3) $|t-\underline{t}|\le 1$, $|a+b|=3$:

$$\begin{cases} |\pi_{\mathrm{B}}\partial_{\underline{\theta}}^{a}\partial_{\underline{\pi}}^{b}(\theta(t;\underline{x},\underline{\xi},\underline{\theta},\underline{\pi})-\underline{\theta}))|\le C\langle\underline{p}\rangle|t-\underline{t}|^{3/2}, \\ |\pi_{\mathrm{B}}\partial_{\underline{\theta}}^{a}\partial_{\underline{\pi}}^{b}(\pi(t;\underline{x},\underline{\xi},\underline{\theta},\underline{\pi})-\underline{\pi}))|\le C\langle\underline{p}\rangle|t-\underline{t}|^{3/2}. \end{cases}$$

(II-4) $|t-\underline{t}|\le 1$, $|a+b|=4$:

$$\begin{cases} |\pi_{\mathrm{B}}\partial_{\underline{\theta}}^{a}\partial_{\underline{\pi}}^{b}(x(t;\underline{x},\underline{\xi},\underline{\theta},\underline{\pi})-\underline{x})|\le C|t-\underline{t}|^{3/2}, \\ |\pi_{\mathrm{B}}\partial_{\underline{\theta}}^{a}\partial_{\underline{\pi}}^{b}(\xi(t;\underline{x},\underline{\xi},\underline{\theta},\underline{\pi})-\underline{\xi})|\le C\langle\underline{p}\rangle|t-\underline{t}|^{3/2}. \end{cases}$$

Remark. We don't mention here the estimates concerning $|\pi_{\mathrm{B}}\partial_{\underline{x}}^{\alpha}\partial_{\underline{\xi}}^{\beta}\partial_{\underline{\theta}}^{a}\partial_{\underline{\pi}}^{b}(\theta(t;\underline{x},\underline{\xi},\underline{\theta},\underline{\pi})-\underline{\theta}))|$, etc.

After solving above Hamilton equation, we may consider also its inverse functions, i.e. for given $(\bar{x},\bar{\theta})$, $\bar{x}=x(t;\underline{x},\underline{\xi},\underline{\theta},\underline{\pi})$ and $\bar{\theta}=\theta(t;\underline{x},\underline{\xi},\underline{\theta},\underline{\pi})$ have inverse functions $\underline{x}=y(t,\bar{x},\underline{\xi},\bar{\theta},\underline{\pi})$ and $\underline{\theta}=\omega(t,\bar{x},\underline{\xi},\bar{\theta},\underline{\pi})$ whose dependence on $(\bar{x},\bar{\theta},\underline{\xi},\underline{\pi})$ are not mentioned here.

Then, we put $\mathcal{S}_0(t\,;\underline{x},\underline{\xi},\underline{\theta},\underline{\pi})=\mathcal{S}_0(t,\underline{t}\,;\underline{x},\underline{\xi},\underline{\theta},\underline{\pi})$ such that

$$\mathcal{S}_0(t\,;\underline{x},\underline{\xi},\underline{\theta},\underline{\pi})=\int_{\underline{t}}^{t}\{\langle\dot{x}(s)|\xi(s)\rangle+\langle\dot{\theta}(s)|\pi(s)\rangle-\mathcal{H}(s,x(s),\xi(s),\theta(s),\pi(s))\}ds,$$

and $\mathcal{S}(t\,;\bar{x},\underline{\xi},\bar{\theta},\underline{\pi})=\mathcal{S}(t,\underline{t}\,;\bar{x},\underline{\xi},\bar{\theta},\underline{\pi})$, such that

$$\mathcal{S}(t\,;\bar{x},\underline{\xi},\bar{\theta},\underline{\pi})=\Big(\langle\underline{x}|\underline{\xi}\rangle+\langle\underline{\theta}|\underline{\pi}\rangle+\mathcal{S}_0(t\,;\underline{x},\underline{\xi},\underline{\theta},\underline{\pi})\Big)\Big|_{\substack{\underline{x}=y(t,\bar{x},\underline{\xi},\bar{\theta},\underline{\pi})\\ \underline{\theta}=\omega(t,\bar{x},\underline{\xi},\bar{\theta},\underline{\pi})}}$$

Theorem 2 *The above defined $\mathcal{S}(t\,;\bar{x},\underline{\xi},\bar{\theta},\underline{\pi})$ satisfies the following equation:*

$$\begin{cases} \dfrac{\partial}{\partial t}\mathcal{S}+\mathcal{H}\Big(\bar{x},\dfrac{\partial\mathcal{S}}{\partial\bar{x}},\bar{\theta},\dfrac{\partial\mathcal{S}}{\partial\bar{\theta}}\Big)=0 \\ \mathcal{S}(\underline{t}\,;\bar{x},\underline{\xi},\bar{\theta},\underline{\pi})=\langle\bar{x}|\underline{\xi}\rangle+\langle\bar{\theta}|\underline{\pi}\rangle. \end{cases}\tag{4}$$

By the way, whether may we have a matrix represetation corresponding to (4)?

Since we may decompose

$$\mathcal{S}(t,x,\xi,\theta,\pi)=\mathcal{S}_{\mathrm{B}}(t,x,\xi)+\mathcal{S}_{\mathrm{S}}(t,x,\xi,\theta,\pi) \tag{5}$$

with

$$\begin{aligned}\mathcal{S}_{\mathrm{B}}(t,x,\xi)&=\mathcal{S}(t,x,\xi,0,0)=\mathcal{S}_{0000}(t,x,\xi),\\ \mathcal{S}_{\mathrm{S}}(t,x,\theta,\xi,\pi)&=\mathcal{S}_{1100}(t,x,\xi)\theta_1\theta_2+\mathcal{S}_{1010}(t,x,\xi)\theta_1\pi_1+\mathcal{S}_{0101}(t,x,\xi)\theta_2\pi_2\\ &\qquad+\mathcal{S}_{0011}(t,x,\xi)\pi_1\pi_2+\mathcal{S}_{1111}(t,x,\xi)\theta_1\theta_2\pi_1\pi_2.\end{aligned} \tag{6}$$

Applying (2), we get

$$\mathcal{S}(t,x,\xi,\theta,\pi)\Longleftrightarrow\mathbb{S}(t,q,p)=\begin{pmatrix}S_{11}(t,q,p) & S_{12}(t,q,p)\\ S_{21}(t,q,p) & S_{22}(t,q,p)\end{pmatrix},$$

where

$$\begin{aligned}S_{11}(t,q,p)&=\mathcal{S}_{0000}(t,q,p)+\mathcal{S}_{1010}(t,q,p)+\mathcal{S}_{0101}(t,q,p),\\ S_{12}(t,q,p)&=\mathcal{S}_{0011}(t,q,p),\\ S_{21}(t,q,p)&=\mathcal{S}_{1100}(t,q,p),\\ S_{22}(t,q,p)&=\mathcal{S}_{0000}(t,q,p)-\mathcal{S}_{1010}(t,q,p)-\mathcal{S}_{0101}(t,q,p).\end{aligned}$$

We define operations $\circledcirc$ and $\bigstar$ as

$$\begin{aligned}(\mathcal{H}\circledcirc\mathcal{S})(t,x,\xi,\theta,\pi)\equiv&\mathcal{H}(x,\mathcal{S}_x(t,x,\xi,\theta,\pi),\theta,\mathcal{S}_\theta(t,x,\xi,\theta,\pi))\\ &\Longleftrightarrow\mathbb{H}\bigstar\mathbb{S}(t,q,p)\equiv\begin{pmatrix}(H\bigstar S)_{11}(t,q,p) & (H\bigstar S)_{12}(t,q,p)\\ (H\bigstar S)_{21}(t,q,p) & (H\bigstar S)_{22}(t,q,p)\end{pmatrix}.\end{aligned}$$

$$\begin{aligned}(H\bigstar S)_{11}(t,q,p)&\equiv\left((\mathcal{H}\circledcirc\mathcal{S})(t,x,\xi,\theta,\pi)-\frac{i}{2}\sum_{j=1}^{2}\frac{\partial^2}{\partial\theta_j\partial\pi_j}(\mathcal{H}\circledcirc\mathcal{S})(t,x,\xi,\theta,\pi)\right)\Bigg|_{\substack{x_{\mathrm{B}}=q,\,\xi_{\mathrm{B}}=p,\\ \theta=0,\,\pi=0}}\\ (H\bigstar S)_{12}(t,q,p)&\equiv\frac{\partial^2}{\partial\pi_2\partial\pi_1}(\mathcal{H}\circledcirc\mathcal{S})(t,x,\xi,\theta,\pi)\Big|_{\substack{x_{\mathrm{B}}=q,\,\xi_{\mathrm{B}}=p,\\ \theta=0,\,\pi=0}}\\ (H\bigstar S)_{21}(t,q,p)&\equiv\frac{\partial^2}{\partial\theta_2\partial\theta_1}(\mathcal{H}\circledcirc\mathcal{S})(t,x,\xi,\theta,\pi)\Big|_{\substack{x_{\mathrm{B}}=q,\,\xi_{\mathrm{B}}=p,\\ \theta=0,\,\pi=0}}\\ (H\bigstar S)_{22}(t,q,p)&\equiv\left((\mathcal{H}\circledcirc\mathcal{S})(t,x,\xi,\theta,\pi)+\frac{i}{2}\sum_{j=1}^{2}\frac{\partial^2}{\partial\theta_j\partial\pi_j}(\mathcal{H}\circledcirc\mathcal{S})(t,x,\xi,\theta,\pi)\right)\Bigg|_{\substack{x_{\mathrm{B}}=q,\,\xi_{\mathrm{B}}=p.\\ \theta=0,\,\pi=0}}\end{aligned}$$

Putting $S(t,q,p)=\mathcal{S}_{0000}(t,x,\xi)\big|_{x=q,\xi=p}$, we have

$$\begin{aligned}\frac{\partial^2\mathcal{H}\circledcirc\mathcal{S}}{\partial\theta_1\partial\pi_1}(t,q,p,0,0)=&-\mathcal{S}_{1010,x_k}(t,q,p)\mathfrak{a}_{\xi_k}(t,q,S_q(t,q,p))\\ &+\mathcal{S}_{1010}(t,q,p)[i\mathfrak{b}(t,q,S_q(t,q,p))+\mathcal{S}_{1100}(t,q,p)(c-id)(t,q,S_q(t,q,p))],,\quad\text{etc.}\end{aligned}$$

Using these, we have the precise expression of $(H\bigstar S)_{ij}(t,q,p)$ but we don't give them here.

Notation. Since on the superspace the meaning of $\mathcal{H}\circledcirc\mathcal{S}$ is transparent, we use $\circledcirc$. On the other hand, the meaning of matrix $\mathbb{H}\bigstar\mathbb{S}$ is not so clear or dark, therefore we use $\bigstar$.

Therefore, we have the matrix-version of Hamilton-Jacobi equation:

$$\begin{cases}\dfrac{\partial}{\partial t}\mathbb{S}(t,q,p)+\mathbb{H}\bigstar\mathbb{S}(t,q,p)=0,\\ \mathbb{S}(0,q,p)=\langle q|p\rangle\mathbb{I}_2-i\boldsymbol{\sigma}_3=\begin{pmatrix}\langle q|p\rangle-i & 0\\ 0 & \langle q|p\rangle+i\end{pmatrix}.\end{cases} \tag{7}$$

Remark. When we treat the matrix structures using diagonalization, we need to treat the dependence of each component of matrices in very condensed form. That is, the degeneracy or non-smoothness of eigenvalues depends very subtly to each components of matrices and to check every possibility is cumbersome. Moreover, diagonalizability of matrices is not always garanteed.

Claim Our method treats each component of matrices as equally as it is, though we need to introduce new notion to decompose matrix structure as admitting non-commutative but scalar-like treatise. Therefore may we say that the superspace formulation is finer than only treating matrices using diagonalization.

A Fundamentals of superanalysis

A.1 Supernumbers and Superspaces

Supernumbers: Preparing letters $\{\sigma_j\}_{j=1}^{\infty}$ satisfying the Grassmann relation $\sigma_j\sigma_k + \sigma_k\sigma_j = 0$ for $j,k = 1,2,\cdots$, we put

$$\begin{cases} \mathfrak{C} = \{X = \sum_{I\in\mathcal{I}} X_I\sigma^I \mid X_I \in \mathbb{C}\}, \quad \mathfrak{C} \cong \mathfrak{C}_{\text{ev}} \oplus \mathfrak{C}_{\text{od}} \cong \mathfrak{C}_{\text{ev}} \times \mathfrak{C}_{\text{od}} \\ \mathfrak{C}_{\text{ev}} = \{X = \sum_{I\in\mathcal{I},|I|=\text{even}} X_I\sigma^I \mid X_I \in \mathbb{C}\}, \\ \mathfrak{C}_{\text{od}} = \{X = \sum_{I\in\mathcal{I},|I|=\text{odd}} X_I\sigma^I \mid X_I \in \mathbb{C}\}, \end{cases}$$

where

$$\mathcal{I} = \{I = (i_k) \in \{0,1\}^{\mathbf{N}} \mid |I| = \sum_k i_k < \infty\},$$
$$\sigma^I = \sigma_1^{i_1}\sigma_2^{i_2}\cdots, \ I = (i_1, i_2, \cdots), \ \tilde{0} = (0,0,\cdots).$$

Proposition 1 (Inoue and Maeda [?]) *$\mathfrak{C}$ forms a ∞-dimensional Fréchet-Grassmann algebra over $\mathbb{C}$, that is, an associative, distributive and non-commutative ring with degree, which is endowed with the Fréchet topology.*

Remarks. (0-0) The space $\mathfrak{C}$ is identified topologically with the sequence space ω of Köthe [?]:

$$\begin{cases} \phi = \{\mathfrak{x} = (x_k) = (x_1, x_2, \cdots, x_k, \cdots) \mid x_k \in \mathbb{C} \text{ and } x_k = 0 \text{ effm } k\}, \\ \omega = \{\mathfrak{u} = (u_k) = (u_1, u_2, \cdots, u_k, \cdots) \mid u_k \in \mathbb{C}\} \quad \text{(effm=except for finitely many)}. \end{cases}$$

For any sequence space $\mathcal{X}$ containing ϕ, we define the space $\mathcal{X}^\times$ by

$$\mathcal{X}^\times = \left\{\mathfrak{u} = (u_k) \mid \sum_k |u_k||x_k| < \infty \quad \text{for any} \quad \mathfrak{x} = (x_k) \in \mathcal{X}\right\},$$

then, we get

$$\phi^\times = \omega \quad \text{and} \quad \omega^\times = \phi.$$

(0-1) Degree in $\mathfrak{C}$ is defined by introducing subspaces

$$\mathfrak{C}_{[j]} = \{X = \sum_{I\in\mathcal{I},|I|=j} X_I\sigma^I\} \quad \text{for} \quad j = 0,1,\cdots$$

which satisfy

$$\mathfrak{C} = \oplus_{j=0}^{\infty}\mathfrak{C}_{[j]}, \quad \mathfrak{C}_{[j]} \cdot \mathfrak{C}_{[k]} \subset \mathfrak{C}_{[j+k]}.$$

(1) Define $\text{proj}_I(X) = X_I$ for $X = \sum_{I\in\mathcal{I}} X_I\sigma^I \in \mathfrak{C}$. The topology in $\mathfrak{C}$ is given by; $X \to 0$ in $\mathfrak{C}$ iff for any $I \in \mathcal{I}$, $\text{proj}_I(X) \to 0$ in $\mathbb{C}$.

This topology is equivalent to the one introduced by the metric $\text{dist}\,(X,Y) = \text{dist}\,(X-Y)$ where $\text{dist}\,(X)$ is defined by

$$\text{dist}\,(X) = \sum_{I\in\mathcal{I}} \frac{1}{2^{r(I)}} \frac{|\,\text{proj}_I(X)|}{1 + |\,\text{proj}_I(X)|} \quad \text{with} \quad r(I) = 1 + \frac{1}{2}\sum_{k=1}^{\infty} 2^k i_k \quad \text{for} \quad I \in \mathcal{I}.$$

(2) We introduce parity in $\mathfrak{C}$ by setting

$$p(X)=0\text{if } X=\sum_{I\in\mathcal{I},|I|=\text{ev}}X_I\sigma^I, 1\text{if } X=\sum_{I\in\mathcal{I},|I|=\text{od}}X_I\sigma^I, \text{undefinedif otherwise.}$$

$X\in\mathfrak{C}$ is called homogeneous if it satisfies $p(X)=0$ or $=1$. We put

$$\begin{cases}\mathfrak{C}_{\text{ev}}=\{X\in\mathfrak{C}\,|\,p(X)=0\},\\ \mathfrak{C}_{\text{od}}=\{X\in\mathfrak{C}\,|\,p(X)=1\},\\ \mathfrak{C}\cong\mathfrak{C}_{\text{ev}}\oplus\mathfrak{C}_{\text{od}}\cong\mathfrak{C}_{\text{ev}}\times\mathfrak{C}_{\text{od}}.\end{cases}$$

Superspaces: Introducing the body (projection) map π_{B} by

$$\pi_{\text{B}}X=\text{proj}_{\tilde{0}}(X)=X_{\tilde{0}}=X_{\text{B}}\quad\text{for any}\quad X\in\mathfrak{C},$$

we put

$$\begin{cases}\mathfrak{R}=\{X\in\mathfrak{C}\,|\,\pi_{\text{B}}X\in\mathbb{R}\},\\ \mathfrak{R}_{\text{ev}}=\mathfrak{R}\cap\mathfrak{C}_{\text{ev}},\quad\mathfrak{R}_{\text{od}}=\mathfrak{R}\cap\mathfrak{C}_{\text{od}}=\mathfrak{C}_{\text{od}},\\ \mathfrak{R}\cong\mathfrak{R}_{\text{ev}}\oplus\mathfrak{R}_{\text{od}}\cong\mathfrak{R}_{\text{ev}}\times\mathfrak{R}_{\text{od}}.\end{cases}$$

We define the (real) superspace $\mathfrak{R}^{m|n}$ by

$$\mathfrak{R}^{m|n}=\mathfrak{R}^m_{\text{ev}}\times\mathfrak{R}^n_{\text{od}}\ni X=(x,\theta).$$

Here,

$$x=(x_1,\cdots,x_m),\quad x_j=\sum_{I,|I|=\text{even}}x_I\sigma^I\in\mathfrak{R}_{\text{ev}}\quad\text{and}$$
$$\theta=(\theta_1,\cdots,\theta_n),\quad\theta_k=\sum_{|I|=\text{odd}}\theta_{k,I}\sigma^I\in\mathfrak{R}_{\text{od}},$$
$$x=x_{\text{B}}+x_{\text{S}}=(x_{1,\text{B}}+x_{1,\text{S}},\cdots,x_{m,\text{B}}+x_{m,\text{S}})\in\mathfrak{R}^m_{\text{ev}}\quad\text{with}$$
$$x_{j,\text{B}}=\pi_{\text{B}}x_j=x_{j,\tilde{0}},\quad x_{j,\text{S}}=\sum_{|I|=\text{even}\geq 2}x_{j,I}\sigma^I.$$

We call $X_{\text{S}}=X-X_{\text{B}}$ as the soul part of X.

A.2 Grassmann extension and supersmooth functions

For $f(q)\in C^\infty(\mathbb{R}^m:\mathbb{C})$, we may put

$$f(x)=\sum_{|\alpha|=0}^{\infty}\frac{1}{\alpha!}\partial_q^\alpha f(x_{\text{B}})x_{\text{S}}^\alpha,$$

which is called the Grassmann continuation of $f(q)$.

We define a class of supersmooth functions $\mathcal{C}_{SS}(\mathfrak{R}^{m|n})$ with value in $\mathbb{C}$ by

$$u(X)=u(x,\theta)=\sum_{|a|\leq n}u_a(x)\theta^a,$$

with $u_a(x)$ is the Grassmann continuation of $u_a(q)\in C^\infty(\mathbb{R}^m:\mathbb{C})$.

Example. For $\xi=(\xi_1,\cdots,\xi_m)\in\mathfrak{R}^{m|0}=\mathfrak{R}^m_{\text{ev}}$, we define $|\xi|\in\mathfrak{R}_{\text{ev}}$ as follows: Putting

$$|\xi|=|\xi|_{\text{B}}+|\xi|_{\text{S}}\quad\text{with}\quad|\xi|_{\text{S}}=\sum_{|I|=\text{even}\geq 2}|\xi|_I\sigma^I,\quad|\xi|_{\text{B}}\geq 0,\ |\xi|_I\in\mathbb{R},$$

we should have

$$|\xi|^2=\sum_{j=1}^m(\xi_{j,\mathrm{B}}+\xi_{j,\mathrm{S}})(\overline{\xi_{j,\mathrm{B}}+\xi_{j,\mathrm{S}}})=\sum_{j=1}^m\xi_{j,\mathrm{B}}^2+\sum_{j=1}^m\xi_{j,\mathrm{B}}(\xi_{j,\mathrm{S}}+\overline{\xi_{j,\mathrm{S}}})+\sum_{j=1}^m\xi_{j,\mathrm{S}}\overline{\xi_{j,\mathrm{S}}},$$

$$\xi_{j,\mathrm{S}}=\sum_{|I|=\mathrm{even}\geq 2}\xi_{j,I}\sigma^I,\quad \overline{\xi_{j,\mathrm{S}}}=\sum_{|I|=\mathrm{even}\geq 2}\overline{\xi_{j,I}}\sigma^I$$

with $\overline{\xi_{j,I}}$ being the complex conjugate of $\xi_{j,I}$ in $\mathbb{C}$.

Therefore,

$$|\xi|_{\mathrm{B}}=\{\sum_{j=1}^m\xi_{j,\mathrm{B}}^2\}^{1/2},$$

$$2|\xi|_K|\xi|_{\mathrm{B}}+\sum_{I+J=K}|\xi|_I\overline{|\xi|_J}(-1)^{\tau(K;I,J)}=\sum_{j=1}^m 2\xi_{j,\mathrm{B}}\,\Re\xi_{j,K}+\sum_{I+J=K}\sum_{j=1}^m\xi_{j,I}\overline{\xi_{j,J}}(-1)^{\tau(K;I,J)}$$

which are solved by induction with respect to the length $|K|$.

For example, if $|K|=2$, we have

$$|\xi|_K=|\xi|_{\mathrm{B}}^{-1}\sum_{j=1}^m\xi_{j,\mathrm{B}}\,\Re\xi_{j,K}.$$

If $|K|=4$,

$$2|\xi|_K=|\xi|_{\mathrm{B}}^{-1}\big(2\sum_{j=1}^m\xi_{j,\mathrm{B}}\,\Re\xi_{j,K}+\sum_{I+J=K}\sum_{j=1}^m\xi_{j,I}\overline{\xi_{j,J}}(-1)^{\tau(K;I,J)}-\sum_{I+J=K}\sum_{j=1}^m|\xi|_I|\xi|_J(-1)^{\tau(K;I,J)}\big),\quad \text{etc.}$$

Proceeding analogously, we may define $\sin|\xi|$, $\cos|\xi|$ as

$$\sin|\xi|=\sum_{n=0}^\infty\frac{1}{n!}\sin\big(|\xi|_{\mathrm{B}}+\frac{n\pi}{2}\big)|\xi|_{\mathrm{S}}^n,\quad \cos|\xi|=\sum_{n=0}^\infty\frac{1}{n!}\cos\big(|\xi|_{\mathrm{B}}+\frac{n\pi}{2}\big)|\xi|_{\mathrm{S}}^n.$$

A.3 Derivatives

For a given supersmooth function $u(X)$ on $\Re^{m|n}$, we define its derivatives as follows: For $j=1,2,\cdots,m$ and $k=1,2,\cdots,n$, we put

$$\begin{cases} U_j(X)=\displaystyle\sum_{|a|\leq n}\partial_{x_j}u_a(x)\theta^a, \\ U_{k+m}(X)=\displaystyle\sum_{|a|\leq n}(-1)^{l_k(a)}u_a(x)\theta_1^{a_1}\cdots\theta_k^{a_k-1}\cdots\theta_n^{a_n} \end{cases}$$

where $l_k(a)=\sum_{j=1}^{k-1}a_j$ and $\theta_k^{-1}=0$. $U_\kappa(X)$ are called the partial derivatives of u with respect to X_κ at $X=(x,\theta)$ and are denoted by

$$U_j(X)=\frac{\partial}{\partial x_j}u(x,\theta)=\partial_{x_j}u(x,\theta)\quad \text{for}\quad j=1,2,\cdots,m,$$

$$U_{m+s}(X)=\frac{\partial}{\partial\theta_s}u(x,\theta)=\partial_{\theta_s}u(x,\theta)\quad \text{for}\quad s=1,2,\cdots,n$$

or simply by

$$U_\kappa(X)=\partial_{X_\kappa}u(X)\quad \text{for}\quad \kappa=1,\cdots,m+n.$$

For

$$\mathfrak{a} = (\alpha, a),\ \ \alpha = (\alpha_1, \cdots, \alpha_m) \in \mathbf{N}^m,\ \ a = (a_1, \cdots, a_n) \in \{0,1\}^n,$$
$$|\alpha| = \sum_{j=1}^m \alpha_j,\ \ |a| = \sum_{k=1}^n a_k,\ \ |\mathfrak{a}| = |\alpha| + |a|,$$

we put

$$\partial_X^{\mathfrak{a}} = \partial_x^\alpha \partial_\theta^a,\quad \partial_x^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_m}^{\alpha_m},\quad \partial_\theta^a = \partial_{\theta_1}^{a_1} \cdots \partial_{\theta_n}^{a_n}.$$

Though Taylor expansion and Implicit function theorem hold for supersmooth functions, we don't mention them here.

A.4 Integration

Using the integration by parts, we get the following fundamental result.

Proposition 2 *Let $u(t) \in C^\infty([\lambda_{\mathrm{B}}, \mu_{\mathrm{B}}] : \mathfrak{C})$ and let $u(x)$ be the Grassmann continuation of $u(t)$. Suppose that there exists a function $U(t) \in C^\infty([\lambda_{\mathrm{B}}, \mu_{\mathrm{B}}] : \mathfrak{C})$ satisfying $U'(t) = u(t)$ on $[\lambda_{\mathrm{B}}, \mu_{\mathrm{B}}]$. Then, for any continuous and piecewise C^1-curve $c : [\lambda_{\mathrm{B}}, \mu_{\mathrm{B}}] \to U_{ev} \subset \mathfrak{R}^{1|0}$ such that $c(\lambda_{\mathrm{B}}) = \lambda$, $c(\mu_{\mathrm{B}}) = \mu$, we have*

$$\int_c dx\, u(x) = U(\lambda) - U(\mu). \tag{8}$$

Remark. This proposition comes from the very definition of the Grassmann continuation which looks-like "analytic in soul direction".

The integration w.r.t. odd variables: There exists a well-known formula (Berezin integral),

$$\int_{\mathfrak{R}^{0|n}} d\theta_n \cdots d\theta_1\, \theta_1 \cdots \theta_n = 1,\quad \int_{\mathfrak{R}^{0|n}} d\theta_n \cdots d\theta_1\, 1 = 0.$$

The integration w.r.t. even and odd variables:

$$\begin{aligned}\int_{\mathfrak{R}^{m|n}} dx d\theta\, u(x,\theta) &= \int_{\mathfrak{R}^{m|0}} dx \left\{ \int_{\mathfrak{R}^{0|n}} d\theta\, u(x,\theta) \right\} \\ &= \int_{\mathfrak{R}^{0|n}} d\theta \left\{ \int_{\mathfrak{R}^{m|0}} dx\, u(x,\theta) \right\} = \int_{\mathfrak{R}^{m|n}} d\theta dx\, u(x,\theta).\end{aligned}$$

A.5 Fourier transformation

$$(F_e v)(\xi) = (2\pi\hbar)^{-m/2} \int_{\mathfrak{R}^{m|0}} dx\, e^{-i\hbar^{-1}\langle x|\xi\rangle} v(x),\quad (\bar{F}_e w)(x) = (2\pi\hbar)^{-m/2} \int_{\mathfrak{R}^{m|0}} d\xi\, e^{i\hbar^{-1}\langle x|\xi\rangle} w(\xi),$$
$$(F_o v)(\pi) = \hbar^{n/2}\iota_n \int_{\mathfrak{R}^{0|n}} d\theta\, e^{-i\hbar^{-1}\langle\theta|\pi\rangle} v(\theta),\quad (\bar{F}_o w)(\theta) = \hbar^{n/2}\iota_n \int_{\mathfrak{R}^{0|n}} d\pi\, e^{i\hbar^{-1}\langle\theta|\pi\rangle} w(\pi).$$
$$\langle x|\xi\rangle = \sum_{j=1}^m x_j \xi_j,\ \langle\theta|\pi\rangle = \sum_{k=1}^n \theta_k \pi_k,\ \iota_n = e^{-\frac{\pi i}{4} n(n-2)}.$$

Putting also,

$$\langle X|\Xi\rangle = \langle x|\xi\rangle + \langle\theta|\pi\rangle \in \mathfrak{R}_{\mathrm{ev}},\ c_{m,n} = (2\pi\hbar)^{-m/2}\hbar^{n/2}\iota_n,$$

we rewrite

$$\begin{aligned}(\mathcal{F}u)(\xi,\pi) =& c_{m,n} \int_{\mathfrak{R}^{m|n}} dX\, e^{-i\hbar^{-1}\langle X|\Xi\rangle} u(X) = \sum_a [(F_e u_a)(\xi)][(F_o \theta^a)(\pi)], \\ (\bar{\mathcal{F}}v)(x,\theta) =& c_{m,n} \int_{\mathfrak{R}^{m|n}} d\Xi\, e^{i\hbar^{-1}\langle X|\Xi\rangle} v(\Xi) = \sum_a [(\bar{F}_e v_a)(x)][(\bar{F}_o \pi^a)(\theta)].\end{aligned}$$

B The free Weyl equation—an example

B.1 Hamilton-Jacobi equation corresponding to $\left(\begin{smallmatrix} p_3 & p_1-ip_2 \\ p_1+ip_2 & -p_3 \end{smallmatrix}\right)$

From the free Weyl equation

$$i\hbar\frac{\partial}{\partial t}\psi = -ic\sum_{j=1}^{3}\boldsymbol{\sigma}_j\frac{\partial}{\partial q_j}\psi \quad \text{with} \quad \psi(0,q)=\underline{\psi}(q), \tag{9}$$

putting $c=1=\hbar$, we may derive $\mathbb{H}(q,p)$ as

$$\mathbb{H}(q,p) = \begin{pmatrix} p_3 & p_1-ip_2 \\ p_1+ip_2 & -p_3 \end{pmatrix}.$$

Then the superspace version corresponding to $\mathbb{H}(q,p)$ is denoted by

$$\begin{aligned}\mathcal{H}(\xi,\theta,\pi) &= (\theta_1\theta_2+\pi_1\pi_2)\xi_1 + i(\theta_1\theta_2-\pi_1\pi_2)\xi_2 - i(\theta_1\pi_1+\theta_2\pi_2)\xi_3 \\ &= (\xi_1+i\xi_2)\theta_1\theta_2 + (\xi_1-i\xi_2)\pi_1\pi_2 - i\xi_3(\theta_1\pi_1+\theta_2\pi_2).\end{aligned}$$

In this case, $\mathcal{S}(t,\bar{x},\underline{\xi},\bar{\theta},\underline{\pi})$ is obtained as

$$\mathcal{S}(t,\bar{x},\underline{\xi},\bar{\theta},\underline{\pi}) = \langle\bar{x}|\underline{\xi}\rangle + \delta^{-1}\big[|\underline{\xi}|\langle\bar{\theta}|\underline{\pi}\rangle - (\underline{\xi}_1+i\underline{\xi}_2)\sin(t|\underline{\xi}|)\bar{\theta}_1\bar{\theta}_2 - (\underline{\xi}_1-i\underline{\xi}_2)\sin(t|\underline{\xi}|)\underline{\pi}_1\underline{\pi}_2\big],$$

where

$$\delta = |\underline{\xi}|\cos(t|\underline{\xi}|) - i\underline{\xi}_3\sin(t|\underline{\xi}|).$$

Therefore, we have

$$\begin{aligned}\mathcal{H}(\mathcal{S}_{\bar{x}},\bar{\theta},\mathcal{S}_{\bar{\theta}}) &= (\underline{\xi}_1+i\underline{\xi}_2)\bar{\theta}_1\bar{\theta}_2 + (\underline{\xi}_1-i\underline{\xi}_2)\mathcal{S}_{\bar{\theta}_1}\mathcal{S}_{\bar{\theta}_2} - i\underline{\xi}_3(\bar{\theta}_1\mathcal{S}_{\bar{\theta}_1}+\bar{\theta}_2\mathcal{S}_{\bar{\theta}_2}) \\ &= |\underline{\xi}|^2\delta^{-2}\big[(\underline{\xi}_1+i\underline{\xi}_2)\bar{\theta}_1\bar{\theta}_2 - (|\underline{\xi}|\sin(t|\underline{\xi}|)+i\underline{\xi}_3\cos(t|\underline{\xi}|))\langle\bar{\theta}|\underline{\pi}\rangle + (\underline{\xi}_1-i\underline{\xi}_2)\underline{\pi}_1\underline{\pi}_2\big],\end{aligned}$$

and

$$\mathcal{S}_t = |\underline{\xi}|^2\delta^{-2}(|\underline{\xi}|\sin(t|\underline{\xi}|)+i\underline{\xi}_3\cos(t|\underline{\xi}|))\langle\bar{\theta}|\underline{\pi}\rangle - |\underline{\xi}|^2\delta^{-2}\big[(\underline{\xi}_1+i\underline{\xi}_2)\bar{\theta}_1\bar{\theta}_2 + (\underline{\xi}_1-i\underline{\xi}_2)\underline{\pi}_1\underline{\pi}_2\big].$$

Rewriting above with $\delta = |\underline{p}|\cos(t|\underline{p}|) - i\underline{p}_3\sin(t|\underline{p}|)$, we have

$$\mathbb{S}(t,\bar{q},\underline{p}) = \begin{pmatrix} \bar{q}{\cdot}\underline{p} & 0 \\ 0 & \bar{q}{\cdot}\underline{p} \end{pmatrix} + \delta^{-1}\begin{pmatrix} i|\underline{p}| & -(\underline{p}_1-i\underline{p}_2)\sin(t|\underline{p}|) \\ (\underline{p}_1+i\underline{p}_2)\sin(t|\underline{p}|) & -i|\underline{p}| \end{pmatrix},$$

$$\mathbb{S}_t(t,\bar{q},\underline{p}) = |\underline{p}|^2\delta^{-2}\begin{pmatrix} i(|\underline{p}|\sin(t|\underline{p}|)+i\underline{p}_3\cos(t|\underline{p}|)) & -(\underline{p}_1-i\underline{p}_2) \\ -(\underline{p}_1+i\underline{p}_2) & -i(|\underline{p}|\sin(t|\underline{\xi}|)+i\underline{p}_3\cos(t|\underline{p}|)) \end{pmatrix},$$

and

$$\mathbb{H}\bigstar\mathbb{S}(t,\bar{q},\underline{p}) = -|\underline{p}|^2\delta^{-2}\begin{pmatrix} i(|\underline{p}|\sin(t|\underline{p}|)+i\underline{p}_3\cos(t|\underline{p}|)) & -(\underline{p}_1-i\underline{p}_2) \\ -(\underline{p}_1+i\underline{p}_2) & -i(|\underline{p}|\sin(t|\underline{p}|)+i\underline{p}_3\cos(t|\underline{p}|)) \end{pmatrix}.$$

That is,

$$\mathcal{S}_t(t,\bar{x},\bar{\theta}) + \mathcal{H}(\mathcal{S}_{\bar{x}},\bar{\theta},\mathcal{S}_{\bar{\theta}}) = 0 \Longleftrightarrow \mathbb{S}_t(t,\bar{q},\underline{p}) + \mathbb{H}\bigstar\mathbb{S}(t,\bar{q},\underline{p}) = 0. \tag{10}$$

For reader's sake, we give a "quantum" version on $\mathfrak{R}^{6|4} = \mathcal{T}^*\mathfrak{R}^{3/2}$: Now, we put

$$\mathcal{D}(t,\bar{x},\underline{\xi},\bar{\theta},\underline{\pi}) = \operatorname{sdet}\begin{pmatrix} \frac{\partial^2\mathcal{S}}{\partial\bar{x}\,\partial\underline{\xi}} & \frac{\partial^2\mathcal{S}}{\partial\bar{x}\,\partial\underline{\pi}} \\ \frac{\partial^2\mathcal{S}}{\partial\bar{\theta}\,\partial\underline{\xi}} & \frac{\partial^2\mathcal{S}}{\partial\bar{\theta}\,\partial\underline{\pi}} \end{pmatrix} = |\underline{\xi}|^{-2}\big[|\underline{\xi}|\cos(t|\underline{\xi}|) - i\underline{\xi}_3\sin(t|\underline{\xi}|)\big]^2. \tag{11}$$

where "sdet" stands for the super-determinant. Then, it satisfies the following continuity equation:

$$\begin{cases} \dfrac{\partial}{\partial t}\mathcal{D} + \dfrac{\partial}{\partial\bar{x}}\Big(\mathcal{D}\dfrac{\partial\mathcal{H}}{\partial\xi}\Big) + \dfrac{\partial}{\partial\bar{\theta}}\Big(\mathcal{D}\dfrac{\partial\mathcal{H}}{\partial\pi}\Big) = 0, \\ \mathcal{D}(0,\bar{x},\underline{\xi},\bar{\theta},\underline{\pi}) = 1. \end{cases} \tag{12}$$

In the above, the argument of $\mathcal{D}$ is $(t, \bar{x}, \underline{\xi}, \bar{\theta}, \underline{\pi})$, those of $\partial\mathcal{H}/\partial\xi$ and $\partial\mathcal{H}/\partial\pi$ are $(\mathcal{S}_{\bar{x}}, \bar{\theta}, \mathcal{S}_{\bar{\theta}})$, respectively.

We define a Fourier integral operator

$$(\mathcal{U}(t)u)(\bar{x}, \bar{\theta}) = (2\pi\hbar)^{-3/2}\hbar \iint d\underline{\xi} d\underline{\pi}\, \mathcal{D}^{1/2}(t, \bar{x}, \bar{\theta}, \underline{\xi}, \underline{\pi}) e^{i\mathcal{S}(t,\bar{x},\bar{\theta},\underline{\xi},\underline{\pi})} \mathcal{F}u(\underline{\xi}, \underline{\pi}), \tag{13}$$

where $\mathcal{F}$ stands for the Fourier transformation defined for functions on the superspace. The function $u(t, \bar{x}, \bar{\theta}) = (\mathcal{U}(t)\underline{u})(\bar{x}, \bar{\theta})$ will be shown as a desired solution for (**??**).

Theorem 3 (Path-integral representation of a solution for the free Weyl equation)

$$\psi(t, q) = \flat\Big((2\pi)^{-3/2} \iint_{\mathfrak{R}^{3|2}} d\underline{\xi} d\underline{\pi}\, \mathcal{D}^{1/2}(t, \bar{x}, \bar{\theta}, \underline{\xi}, \underline{\pi}) e^{i\mathcal{S}(t,\bar{x},\bar{\theta},\underline{\xi},\underline{\pi})} \mathcal{F}(\sharp\underline{\psi})(\underline{\xi}, \underline{\pi})\Big)\Big|_{\bar{x}_{\mathrm{B}}=q}. \tag{14}$$

Here, $\mathcal{S}(t, \bar{x}, \bar{\theta}, \underline{\xi}, \underline{\pi})$ and $\mathcal{D}(t, \bar{x}, \bar{\theta}, \underline{\xi}, \underline{\pi})$ are solutions of the Hamilton-Jacobi and continuity equations, (**??**) *and* (**??**) *respectively.*

Remark. We may extend the expression above for the Weyl equation with external electro-magnetic potentials, at least in the parametrix level, see [**?**].

B.2 Structure change of matrices

Putting

$$\begin{cases} \sigma_1(t) = \theta_1(t)\theta_2(t) + \pi_1(t)\pi_2(t), \\ \sigma_2(t) = i(\theta_1(t)\theta_2(t) - \pi_1(t)\pi_2(t)), \\ \sigma_3(t) = -i(\theta_1(t)\pi_1(t) + \theta_2(t)\pi_2(t)), \end{cases} \quad \text{with} \quad \begin{cases} \sigma_1(0) = \underline{\theta}_1\underline{\theta}_2 + \underline{\pi}_1\underline{\pi}_2, \\ \sigma_2(0) = i(\underline{\theta}_1\underline{\theta}_2 - \underline{\pi}_1\underline{\pi}_2), \\ \sigma_3(0) = -i(\underline{\theta}_1\underline{\pi}_1 + \underline{\theta}_2\underline{\pi}_2), \end{cases}$$

and differentiating with respect to t, we get easily

$$\frac{d}{dt}\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix} = 2\mathbb{Y}\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix} \quad \text{where} \quad \mathbb{Y} = \begin{pmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{pmatrix}.$$

As

$$\mathbb{Y}^2 = \begin{pmatrix} -\xi_3^2 - \xi_2^2 & \xi_2\xi_1 & \xi_3\xi_1 \\ \xi_1\xi_2 & -\xi_3^2 - \xi_1^2 & \xi_3\xi_2 \\ \xi_1\xi_3 & \xi_2\xi_3 & -\xi_2^2 - \xi_1^2 \end{pmatrix} \quad \text{and} \quad \mathbb{Y}^3 = -|\xi|^2\mathbb{Y},$$

we have

$$e^{2t\mathbb{Y}} = \mathbb{I}_3 + |\xi|^{-1}\sin(2t|\xi|)\mathbb{Y} + |\xi|^{-2}(1 - \cos(2t|\xi|))\mathbb{Y}^2.$$

This implies

$$\begin{aligned} \sigma_1(s) &= \underline{\sigma}_1 + \sin(2s|\underline{\xi}|)|\underline{\xi}|^{-1}(-\underline{\xi}_3\underline{\sigma}_2 + \underline{\xi}_2\underline{\sigma}_3) \\ &\qquad + (1 - \cos(2s|\underline{\xi}|))|\underline{\xi}|^{-2}\big[-(\underline{\xi}_2^2 + \underline{\xi}_3^2)\underline{\sigma}_1 + \underline{\xi}_1\underline{\xi}_2\underline{\sigma}_2 + \underline{\xi}_1\underline{\xi}_3\underline{\sigma}_3\big], \\ \sigma_2(s) &= \underline{\sigma}_2 + \sin(2s|\underline{\xi}|)|\underline{\xi}|^{-1}(\underline{\xi}_3\underline{\sigma}_1 - \underline{\xi}_1\underline{\sigma}_3) \\ &\qquad + (1 - \cos(2s|\underline{\xi}|))|\underline{\xi}|^{-2}\big[\underline{\xi}_1\underline{\xi}_2\underline{\sigma}_1 - (\underline{\xi}_1^2 + \underline{\xi}_3^2)\underline{\sigma}_2 + \underline{\xi}_2\underline{\xi}_3\underline{\sigma}_3\big], \\ \sigma_3(s) &= \underline{\sigma}_3 + \sin(2s|\underline{\xi}|)|\underline{\xi}|^{-1}(-\underline{\xi}_2\underline{\sigma}_1 + \underline{\xi}_1\underline{\sigma}_2) \\ &\qquad + (1 - \cos(2s|\underline{\xi}|))|\underline{\xi}|^{-2}\big[\underline{\xi}_1\underline{\xi}_3\underline{\sigma}_1 + \underline{\xi}_2\underline{\xi}_3\underline{\sigma}_2 - (\underline{\xi}_1^2 + \underline{\xi}_2^2)\underline{\sigma}_3\big]. \end{aligned}$$

Remark. In this sense, we give the change in time of the each matrix $\sigma_j(t)$ compared with the one at time $t = \underline{t}$, i.e. $\{\sigma_j(t)\}_{j=1}^3$ may be considered as "moving frame" if we regard $\{\boldsymbol{\sigma}_j\}_{j=1}^3$ as frame of matrix structure as explained before.

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