Introduction to

Superanalysis and its Applications

—

Systems of Partial Differential Equations
and
Random Matrix Theory

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Preface

The motivation of writing this book is two folded:

(I) **Giving Foundations on Superanalysis**:

Though there are so many mathematico-physico papers with the prefix "super", but their foundation is not so familiar to mathematicians. Therefore, we need to develop from scratch the superanalysis, that is, the elementary analysis (elementary calculus, linear algebra, etc) and real analysis (Fourier analysis, pseudo-differential operators, Fourier integral operators, etc) on the superspace.

Here, the superspace is the space based on the ∞-dimensional Fréchet-Grassmann algebra $\mathfrak{R}$ or $\mathfrak{C}$ which play the role of $\mathbb{R}$ or $\mathbb{C}$, respectively. On the other hand, because the so-called differential calculus is not extended freely from Euclidian space to the Fréchet space, we need a care to develop such theory.

The Part I is devoted to this foundation. Though this part is partially presented in Inoue-Maeda [117], I rewrite it here because not only the journal containing that paper is unfamiliar but also the contents of it shows the unmatureness of our thought at that time, especially the estimates of error terms should be modified completely.

(II) **Giving Applications of Supernalysis**:

If there were no application of the superanalysis, it had been possibly a useless artifact to develop such analysis.

We give some applications, hoping that these provide the evidence of the superanalysis being an indespensable tool for the future mathematical study;

(a) a new treatise of systems of PDEs(=Partial Differential Equations) which includes the resolution of the Feynman's problem, a system version of Egorov's theorem, and also an explanation of Atiyah-Singer's index theorem as the super-version of Weyl thoerem, which is based on the idea of Witten in 1980's, etc.

(b) new treatise of RMT(=Random Matrix Theory) using the "slowness variables" represented by matrices with elements in $\mathfrak{C}$.

(III) In explaining these applications, I list some problems, hoping those not so trivial, which I have interest but probably not enough time to solve.

Contents

Chapter 1

Introduction

1.1 A new treatise of systems of PDEs

In the theory of linear PDE(=Partial Differential Equation) of scalar type, the main problem is to reduce the non-commutativity inherited from the so-called Heisenberg's uncertainty principle

$$\left[\frac{\hbar}{i}\frac{\partial}{\partial q_j}, q_k\right] = \frac{\hbar}{i}\delta_{jk},$$

to the one where commutative algebraic calculation is available. This is done using Fourier transformations, that is, the non-commutativity caused by the Heisenberg's uncertainty, is reduced to the commutative one with error terms on the phase space by Fourier transformation, and then this transformed one is analyzed there, and by the inverse Fourier transformation, it is transformed back to the original setting. This procedure is done modulo error terms with suitable estimates.

This procedure is extended also to a system of PDEs if one may diagonalize that system and apply the standard method to its component. If it is hard to diagonalize straightforwordly, then one imposes certain conditions on the characteristic roots associated to that system in order to assure that one may essentially reduce that system to the scalar pseudo differential operators. But if this procedure fails, is there any detour? For example, if we need a "Hamilton flow" for the given system, how do we associate that object without diagonalization?

On the other hand, if the phenomenon is truely describable using a system of PDEs, it seems natural to adandone the idea of reducing it to the scalar one. Of course, treating that case, we need new idea to attack the non-commutativity of matrices.

1.1.1 Feynman's problem

Though the path integrals are now freely used by physicists, as there exists mathematically no non-trivial Feynman measure[1] on the path space, it is even now important to justify mathematically the results obtained by manipulating path integrals. (See, Proposition 3.1 of Smolyanov and Fomin [205] for the non-existence of the non-trivial, strongly quasi-invariant measure on an infinite dimensional barrelled locally convex space, and p.435 of Abraham and Marsden [1] for the non-existence of the full-quantization.)

A certain part of this situation (relation between the Schrödinger equations and the Feynman's kernel representations by path-integrals) is now ameliorated slightly by Fujiwara [78, 79]. More precisely

[1]Feynman measure $\sim$ Lebesgue-like Borel measure

speaking, a fundamental solution or a parametrix of the Schrödinger equation with a suitable potential (smooth and quadratic at ∞) is constructed by using Feynman's idea of path integral combined with the L^2-boundedness of certain Fourier integral operators. We may consider this construction as a mathematical procedure of quantization of certain Lagrangian or Hamiltonian functions (see, Feynman [74] and also Inoue & Maeda [114]). One of the most characteristic thing of this construction is the very explicit dependence on the Planck constant, in other word, the so-called Bohr correspondence principle is nicely inherited assuming that the stationary phase method works well in that representation. Though Fujiwara used the Lagrangian formulation, Kitada [132] and Intissar [120] tried to formulate it in the Hamiltonian one. See also, Kumano-go [?], Inoue [112].

On the other hand, as is written in p.355 of Feynman & Hibbs [75], 'spin' has been the object outside Feynman's procedures at that time:

> $\cdots$ path integrals suffer grievously from a serious defect. They do not permit a discussion of spin operators or other such operators in a simple and lucid way. They find their greatest use in systems for which coordinates and their conjugate momenta are adequate. Nevertheless, spin is a simple and vital part of real quantum-mechanical systems. It is a serious limitation that the half-integral spin of the electron does not find a simple and ready representation. It can be handled if the amplitudes and quantities are considered as quarternions instead of ordinary complex numbers, but the lack of commutativity of such numbers is a serious complication.

A part of our aim (II-a), explained in the preface, is to extend the idea of above mentioned works to cover Schrödinger equations with spin (for example, the equations named after Dirac, Pauli and Weyl) using analysis on the superspace, which answers partly the problem above raised by Feynman. Main ingredient is, how to associate a Hamiltonian function and classical mechanics to a system of PDEs.

Roughly speaking, we proceed as follow:

By the general accordance with the formulation of quantum mechanics, in order to describe the dynamics of a quantum spinning particle, we should construct a 1-parameter family of bounded invertible linear operators on a certain Hilbert space associated with some inherent classical mechanics. To derive such a mechanics for a system of PDEs, we use the special representation of the Clifford algebra to the even part of the Grassmann algebra. Using that representation for a given Schrödinger equation with spin, we may associate a function, called super Hamiltonian (or simply, Hamiltonian), on the cotangent superspace $\mathcal{T}^*\mathfrak{R}^{m|n} \equiv \mathfrak{R}^{2m|2n}$ (m is the dimension of the configuration space and n is related to the size of matrices appeared in that system of PDEs). From that super Hamiltonian, we may construct the classical mechanics on the cotangent superspace (called the pseudo mechanics by Casalbouni [45]), which also defines an extended version of the Hamilton–Jacobi equation and the continuity equation. Constructing phase and amplitude functions from these equations, we define an integral transformation in $\mathcal{C}_{\rm SS,ev}(\mathfrak{R}^{m|n})$ (=the algebra of even super smooth functions on $\mathfrak{R}^{m|n}$) and investigate the convergence of its product integral with respect to the time slicing in the L^2-norm. Furthermore, by making a correspondence between $\mathcal{C}_{\rm SS,ev}(\mathfrak{R}^{m|n})$ and $\Gamma^\infty(\mathcal{S})$ (=the set of smooth spin fields on $\mathbb{R}^m$), we get a desired 1-parameter family of bounded invertible linear operators on $L^2(\mathcal{S})$ (=the Hilbert space of square integrable spin fields on $\mathbb{R}^m$). In order to make a general scheme similar to the one used in ordinary Euclidean space, we must develop a some part of calculus of oscillatory integrals defined on the superspace (called super oscillatory integrals).

1.1.1.1 Free Weyl equation

We construct a fundamental solution of the initial value probelm of the free Weyl equation by using "classical quantities" and we express that solution as explicitly as possible. This outlines not only a solution of the problem posed by Feynman but also exhibitting the new treatise of the non-commutativity caused by matrices. As a by-product, we develop a new road to extend the method of characteristic of the first order PDE, to the one with matrix coefficients. (For more general system of PDEs, we present a part of some examples.)

$$\begin{cases} i\hbar\dfrac{\partial}{\partial t}\psi(t,q)=\mathbb{H}\psi(t,q), \quad \mathbb{H}=-ic\hbar\displaystyle\sum_{j=1}^{3}\boldsymbol{\sigma}_j\frac{\partial}{\partial q_j}, \\ \psi(0,q)=\underline{\psi}(q). \end{cases} \tag{1.1}$$

Here, $\psi(t,q)={}^t(\psi_1(t,q),\psi_2(t,q)):\mathbb{R}\times\mathbb{R}^3\to\mathbb{C}^2$, Pauli matrices $\{\boldsymbol{\sigma}_j\}_{j=1}^3$ are commonly represented by

$$\boldsymbol{\sigma}_1=\begin{bmatrix}0&1\\1&0\end{bmatrix}, \quad \boldsymbol{\sigma}_2=\begin{bmatrix}0&-i\\i&0\end{bmatrix}, \quad \boldsymbol{\sigma}_3=\begin{bmatrix}1&0\\0&-1\end{bmatrix}.$$

They satisfy the following Clifford relations:

$$\boldsymbol{\sigma}_j\boldsymbol{\sigma}_k+\boldsymbol{\sigma}_k\boldsymbol{\sigma}_j=2\delta_{jk}\mathbb{I}_2, \quad (\mathbb{I}_m=m\times m \text{ unit matrix}) \tag{1.2}$$

which satisfy also

$$\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2=i\boldsymbol{\sigma}_3, \quad \boldsymbol{\sigma}_2\boldsymbol{\sigma}_3=i\boldsymbol{\sigma}_1, \quad \boldsymbol{\sigma}_3\boldsymbol{\sigma}_1=i\boldsymbol{\sigma}_2.$$

Applying the Fourier transformation to (1.1) w.r.t. $q\in\mathbb{R}^3$, we get

$$i\hbar\frac{\partial}{\partial t}\hat{\psi}(t,p)=\hat{\mathbb{H}}\hat{\psi}(t,p). \tag{1.3}$$

Here,

$$\hat{\mathbb{H}}=c\boldsymbol{\sigma}_jp_j=c\begin{bmatrix}p_3&p_1-ip_2\\p_1+ip_2&-p_3\end{bmatrix} \quad \text{and} \quad \hat{\mathbb{H}}^2=c^2|p|^2\mathbb{I}_2.$$

Therefore, we get readily

Proposition 1.1.1 *For any $t\in\mathbb{R}$ and $\underline{\psi}\in L^2(\mathbb{R}^3:\mathbb{C}^2)$, we have*

$$e^{-i\hbar^{-1}t\mathbb{H}}\underline{\psi}(q)=(2\pi\hbar)^{-3/2}\int_{\mathbb{R}^3}dp\,e^{i\hbar^{-1}qp}e^{-i\hbar^{-1}t\hat{\mathbb{H}}}\hat{\underline{\psi}}(p). \tag{1.4}$$

Moreover, for $\underline{\psi}\in\mathcal{S}(\mathbb{R}^3:\mathbb{C}^2)$, we get

$$e^{-i\hbar^{-1}t\mathbb{H}}\underline{\psi}(q)=\mathbb{E}*\underline{\psi}(t,q)=\int_{\mathbb{R}^3}dq'\,\mathbb{E}(t,q-q')\underline{\psi}(q'), \tag{1.5}$$

with

$$\mathbb{E}(t,q)=(2\pi\hbar)^{-3}\int_{\mathbb{R}^3}dp\,e^{i\hbar^{-1}qp}\big[\cos(c\hbar^{-1}t|p|)\mathbb{I}_2-i\frac{\sin(c\hbar^{-1}t|p|)}{c|p|}\hat{\mathbb{H}}\big]\in\mathcal{S}'(\mathbb{R}^3:\mathbb{C}^2). \tag{1.6}$$

Pauli claimed one day that there exists no classical counter-part corresponding to quantum spinning particle, and surely it is difficult to imagine from (1.6) the classical mechanics corresponding to (1.1). In spite of these, we construct a classical mechanics corresponding to the right-hand side of (1.1). That is, we may formulate Fourier integral operators whose phase function is defined by an action integral of that classical mechanics, which gives a representation of the solution of (1.1) as follows:

Theorem 1.1.2 (Path-integral representation of a solution for the Weyl equation)

$$\psi(t,q)=\flat\Big((2\pi)^{-3/2}\hbar^{-1/2}\iint_{\mathfrak{R}^{3|2}}d\underline{\xi}d\underline{\pi}\,\mathcal{D}^{1/2}(t,\bar{x},\bar{\theta},\underline{\xi},\underline{\pi})e^{i\hbar^{-1}\mathcal{S}(t,\bar{x},\bar{\theta},\underline{\xi},\underline{\pi})}\mathcal{F}(\#\underline{\psi})(\underline{\xi},\underline{\pi})\Big)\Big|_{\bar{x}_{\mathrm{B}}=q}. \tag{1.7}$$

Here, $\mathcal{S}(t,\bar{x},\bar{\theta},\underline{\xi},\underline{\pi})$ and $\mathcal{D}(t,\bar{x},\bar{\theta},\underline{\xi},\underline{\pi})$ are given by

$$\begin{aligned}\mathcal{S}(t,\bar{x},\underline{\xi},\bar{\theta},\underline{\pi})&=\langle\bar{x}|\underline{\xi}\rangle+\big[|\underline{\xi}|\cos(c\hbar^{-1}t|\underline{\xi}|)-i\underline{\xi}_3\sin(c\hbar^{-1}t|\underline{\xi}|)\big]^{-1}\big[|\underline{\xi}|\langle\bar{\theta}|\underline{\pi}\rangle\\&\quad-\hbar\sin(c\hbar^{-1}t|\underline{\xi}|)(\underline{\xi}_1+i\underline{\xi}_2)\bar{\theta}_1\bar{\theta}_2+\hbar^{-1}\sin(c\hbar^{-1}t|\underline{\xi}|)(\underline{\xi}_1-i\underline{\xi}_2)\underline{\pi}_1\underline{\pi}_2\big].\end{aligned} \tag{1.8}$$

$$\mathcal{D}(t,\bar{x},\underline{\xi},\bar{\theta},\underline{\pi})=|\underline{\xi}|^{-2}\big[|\underline{\xi}|\cos(c\hbar^{-1}t|\underline{\xi}|)-i\underline{\xi}_3\sin(c\hbar^{-1}t|\underline{\xi}|)\big]^2.$$

Moreover, $\mathcal{S}(t,\bar{x},\bar{\theta},\underline{\xi},\underline{\pi})$ and $\mathcal{D}(t,\bar{x},\bar{\theta},\underline{\xi},\underline{\pi})$ are solutions of the Hamilton-Jacobi equation

$$\begin{cases}\dfrac{\partial}{\partial t}\mathcal{S}(t,\bar{x},\underline{\xi},\bar{\theta},\underline{\pi})+\mathcal{H}\Big(\dfrac{\partial\mathcal{S}}{\partial\bar{x}},\bar{\theta},\dfrac{\partial\mathcal{S}}{\partial\bar{\theta}}\Big)=0,\\ \mathcal{S}(0,\bar{x},\underline{\xi},\bar{\theta},\underline{\pi})=\langle\bar{x}|\underline{\xi}\rangle+\langle\bar{\theta}|\underline{\pi}\rangle,\end{cases} \tag{1.9}$$

and the continuity equation,

$$\begin{cases}\dfrac{\partial}{\partial t}\mathcal{D}+\dfrac{\partial}{\partial\bar{x}}\Big(\mathcal{D}\dfrac{\partial\mathcal{H}}{\partial\xi}\Big)+\dfrac{\partial}{\partial\bar{\theta}}\Big(\mathcal{D}\dfrac{\partial\mathcal{H}}{\partial\pi}\Big)=0,\\ \mathcal{D}(0,\bar{x},\underline{\xi},\bar{\theta},\underline{\pi})=1,\end{cases} \tag{1.10}$$

respectively. In the above, the argument of $\mathcal{D}$ is $(t,\bar{x},\underline{\xi},\bar{\theta},\underline{\pi})$, while those of $\mathcal{H}_\xi$ and $\mathcal{H}_\pi$ are $(\mathcal{S}_{\bar{x}},\bar{\theta},\mathcal{S}_{\bar{\theta}})$, respectively. $\mathcal{F}$ is the Fourier transformation for functions on the superspace $\mathfrak{R}^{3|2}$.

Remark 1. Notations such as $\mathfrak{R}^{3|2}$, $\underline{x}_{\mathrm{B}}$, $\bar{\theta}$, $\underline{\pi}$, $\#$, $\flat$, $\mathcal{F}$, are explained in Part I. To treat matrices as non-commutative scalar differential operators, we need these new notions, which are basic to analysis on the superspace (alias superanalysis).

Remark 2. The difference between $(2\pi\hbar)^{-3/2}$ in (1.4) and $(2\pi\hbar)^{-3/2}\hbar$ in (1.7), is remarkable in the sense that the singularity when $\hbar=0$ is apparently diminishes by the existence of odd variables. After integrating w.r.t. odd variables $\underline{\pi}$ in (1.7), we get the representation which is equivalent to (1.4). But from (1.5) we can't get (1.7).

1.1.2 Problems for systems of PDEs

1.1.2.1 How should be the system version of Egorov's theorem?

It is well-known that Egorov's theorem concerning the conjugation of ΨDO with FIO is a very powerfull tool analysing properties of solutions of PDEs. Using superanalysis, we try to extend that theorem to systems of PDOs.

Problem: *Calculate the symbol of*

$$e^{i\hbar^{-1}t\mathbb{H}(q,D_q)}\mathbb{P}(q,D_q)e^{-i\hbar^{-1}t\mathbb{H}(q,D_q)},\quad D_q=-i\hbar\partial_q,$$

and find a relation to that of $\sigma(\mathbb{P}(q,D_q))$ composed of the flow induced by $\sigma(\mathbb{H}(q,D_q))$.

Here $\mathbb{H}(q,D_q)$ is a 2×2 matrix of PDO, and $\mathbb{P}(q,D_q)$ is a 2×2-matrix valued PDO given by

$$\mathbb{P}(q,D_q)=\begin{pmatrix}a(q,D_q)&c(q,D_q)\\d(q,D_q)&b(q,D_q)\end{pmatrix},$$

and $\mathbb{U}(t) = e^{-i\hbar^{-1}t\mathbb{H}(q,D_q)}$ *represents formally the solution operator of the initial value problem*

$$i\hbar\frac{\partial}{\partial t}\psi = \mathbb{H}(q,D_q)\psi \quad \textit{with} \quad \psi(0,q) = \underline{\psi}(q)$$

To make this problem more concretely, we give an example here: Remarking that

$$(-\alpha\partial_x^2+\beta x^2)e^{-i\gamma x^2/2} = [i\alpha\gamma+(\beta+\alpha\gamma^2)x^2]e^{-i\gamma x^2/2},$$
$$(c(x)\partial_x+\partial_x c(x))e^{-i\gamma x^2/2} = (c'(x)-2ic(x)\gamma x)e^{-i\gamma x^2/2},$$

we have

$$\begin{gathered}\begin{pmatrix} e^{i\gamma x^2/2} & 0 \\ 0 & e^{i\tilde\gamma x^2/2}\end{pmatrix}\begin{pmatrix} -\alpha\partial_x^2+\beta x^2 & 2^{-1}(c(x)\partial_x+\partial_x c(x)) \\ 2^{-1}(d(x)\partial_x+\partial_x d(x)) & -\tilde\alpha\partial_x^2+\tilde\beta x^2\end{pmatrix}\begin{pmatrix} e^{-i\gamma x^2/2} & 0 \\ 0 & e^{-i\tilde\gamma x^2/2}\end{pmatrix} \\ = \begin{pmatrix} i\alpha\gamma+(\beta+\alpha\gamma^2)x^2 & 2^{-1}(c'(x)-2i\tilde\gamma x c(x))e^{i(\gamma-\tilde\gamma)x^2/2} \\ 2^{-1}(d'(x)-2i\gamma x d(x))e^{-i(\gamma-\tilde\gamma)x^2/2} & i\tilde\alpha\tilde\gamma+(\tilde\beta+\tilde\alpha\tilde\gamma^2)x^2\end{pmatrix}.\end{gathered} \tag{1.11}$$

In February 2001, Bernardi[2] asked me whether it is possible to explain (1.11), especially the origin of the off-diagonal part, using superanalysis. We re-interpret (1.11) as follows: For $u, v \in C^\infty(\mathbb{R})$,

$$\begin{gathered}\begin{pmatrix} e^{i\gamma x^2/2} & 0 \\ 0 & e^{i\tilde\gamma x^2/2}\end{pmatrix}\begin{pmatrix} -\alpha\partial_x^2+\beta x^2 & \frac{c(x)\partial_x+\partial_x c(x)}{2} \\ \frac{d(x)\partial_x+\partial_x d(x)}{2} & -\tilde\alpha\partial_x^2+\tilde\beta x^2\end{pmatrix}\begin{pmatrix} e^{-i\gamma x^2/2} & 0 \\ 0 & e^{-i\tilde\gamma x^2/2}\end{pmatrix}\begin{pmatrix} u \\ v\end{pmatrix} \\ = \begin{pmatrix} (\beta+\alpha\gamma^2)x^2+i\alpha\gamma+2i\gamma x\alpha\partial_x-\alpha\partial_x^2 & e^{i(\gamma-\tilde\gamma)x^2/2}(\frac{c'(x)}{2}+c(x)\partial_x-i\tilde\gamma x c(x)) \\ e^{-i(\gamma-\tilde\gamma)x^2/2}(\frac{d'(x)}{2}+d(x)\partial_x-i\gamma x d(x)) & (\tilde\beta+\tilde\alpha\tilde\gamma^2)x^2+i\tilde\alpha\tilde\gamma+2i\tilde\gamma x\tilde\alpha\partial_x-\tilde\alpha\partial_x^2\end{pmatrix}\begin{pmatrix} u \\ v\end{pmatrix}.\end{gathered} \tag{1.12}$$

Since (1.12) with $u = v = 1$ gives (1.11), we should explain the meaning of (1.12) instead of (1.11).

Since we have

$$\big((\beta+\alpha\gamma^2)x^2+i\alpha\gamma+2i\gamma x\alpha\partial_x-\alpha\partial_x^2\big)u(x) = (2\pi)^{-1}\int_{\mathbb{R}^2} d\xi dy\, e^{i(x-y)\xi}a\left(\frac{x+y}{2},\xi\right)u(y),$$
$$e^{2^{-1}i(\gamma-\tilde\gamma)x^2}(c'/2+c\partial_x-i\tilde\gamma x c)u(x) = (2\pi)^{-1}\int_{\mathbb{R}^2} d\xi dy\, e^{i(x-y)\xi}c\left(\frac{x+y}{2},\xi\right)u(y)$$

with Weyl symbols

$$a(x,\xi) = \alpha(\xi-\gamma x)^2+\beta x^2 = (\beta+\alpha\gamma^2)x^2-2\alpha\gamma x\xi+\alpha\xi^2,$$
$$c(x,\xi) = ie^{2^{-1}i(\gamma-\tilde\gamma)x^2}c(x)\left(\xi-\frac{\gamma+\tilde\gamma}{2}x\right),$$

we get the Weyl symbol of the right-hand side of (1.12), given by

$$\begin{aligned}&\begin{pmatrix} \alpha(\xi-\gamma x)^2+\beta x^2 & ie^{i(\gamma-\tilde\gamma)x^2/2}c(x)(\xi-\frac{\gamma+\tilde\gamma}{2}x) \\ ie^{-i(\gamma-\tilde\gamma)x^2/2}d(x)(\xi-\frac{\gamma+\tilde\gamma}{2}x) & \tilde\alpha(\xi-\tilde\gamma x)^2+\tilde\beta x^2\end{pmatrix} \\ &\sim \frac{\alpha+\tilde\alpha}{2}\xi^2-(\alpha\gamma+\tilde\alpha\tilde\gamma)x\xi+\frac{(\beta+\alpha\gamma^2)+(\tilde\beta+\tilde\alpha\tilde\gamma^2)}{2}x^2 \\ &\quad -i\left[\frac{\alpha-\tilde\alpha}{2}\xi^2-(\alpha\gamma-\tilde\alpha\tilde\gamma)x\xi+\frac{(\beta+\alpha\gamma^2)-(\tilde\beta+\tilde\alpha\tilde\gamma^2)}{2}x^2\right]\langle\theta|\pi\rangle \\ &\quad + ie^{-i(\gamma-\tilde\gamma)x^2/2}d(x)\left(\xi-\frac{\gamma+\tilde\gamma}{2}x\right)\theta_1\theta_2+ie^{i(\gamma-\tilde\gamma)x^2/2}c(x)\left(\xi-\frac{\gamma+\tilde\gamma}{2}x\right)\pi_1\pi_2.\end{aligned} \tag{1.13}$$

[2] as a chairman of a session where I gave a talk

Superspace interpretation: On the other hand, putting

$$\sigma(\mathbb{P})(x,\xi) = \begin{pmatrix} \alpha\xi^2+\beta x^2 & ic(x)\xi \\ id(x)\xi & \tilde\alpha\xi^2+\tilde\beta x^2 \end{pmatrix}, \quad \sigma(\mathbb{H})(x) = \begin{pmatrix} 2^{-1}\gamma x^2 & 0 \\ 0 & 2^{-1}\tilde\gamma x^2 \end{pmatrix},$$

we have

$$\begin{aligned} \sigma(\sharp\mathbb{P}\flat)(x,\xi,\theta,\pi) &= \sigma(\hat{\mathcal{P}})(x,\xi,\theta,\pi) = \mathcal{P}(x,\xi,\theta,\pi) \\ &= \frac{\alpha+\tilde\alpha}{2}\xi^2 + \frac{\beta+\tilde\beta}{2}x^2 - i\left[\frac{\alpha-\tilde\alpha}{2}\xi^2+\frac{\beta-\tilde\beta}{2}x^2\right]\langle\theta|\pi\rangle + id(x)\xi\theta_1\theta_2 + ic(x)\xi\pi_1\pi_2, \\ \sigma(\sharp\mathbb{H}\flat)(x,\theta,\pi) &= \sigma(\hat{\mathcal{H}})(x,\theta,\pi) = \mathcal{H}(x,\theta,\pi) = \frac{(\gamma+\tilde\gamma)x^2}{4} - i\frac{(\gamma-\tilde\gamma)x^2}{4}\langle\theta|\pi\rangle. \end{aligned} \tag{1.14}$$

Therefore, we have

$$\begin{cases} \dot x = \mathcal{H}_\xi = 0, \\ \dot\xi = -\mathcal{H}_x = -\dfrac{\gamma+\tilde\gamma}{2}x + i\dfrac{\gamma-\tilde\gamma}{2}x\langle\theta|\pi\rangle, \\ \dot\theta_j = -\mathcal{H}_{\pi_j} = -i\dfrac{\gamma-\tilde\gamma}{4}x^2\theta_j \quad (j=1,2), \\ \dot\pi_j = -\mathcal{H}_{\theta_1} = i\dfrac{\gamma-\tilde\gamma}{4}x^2\pi_j \quad (j=1,2), \end{cases} \quad \text{with} \quad (x(0),\xi(0),\theta(0),\pi(0)) = (\underline{x},\underline{\xi},\underline{\theta},\underline{\pi})$$

which yields the Hamilton flow corresponding to $\mathcal{H}$ as

$$\begin{gathered} \mathcal{C}(t)(\underline{x},\underline{\xi},\underline{\theta},\underline{\pi}) = (x(t),\xi(t),\theta(t),\pi(t)) \quad \text{with} \\ x(t) = x(t,\underline{x},\underline{\xi},\underline{\theta},\underline{\pi}) = \underline{x}, \quad \xi(t) = \xi(t,\underline{x},\underline{\xi},\underline{\theta},\underline{\pi}) = \underline{\xi} - \frac{\gamma+\tilde\gamma}{2}\underline{x}t + i\frac{\gamma-\tilde\gamma}{2}\underline{x}t\langle\underline{\theta}|\underline{\pi}\rangle, \\ \theta_j(t) = \theta_j(t,\underline{x},\underline{\xi},\underline{\theta},\underline{\pi}) = e^{-i(\gamma-\tilde\gamma)t\underline{x}^2/4}\underline{\theta}_j, \quad \pi_j(t) = \pi(t,\underline{x},\underline{\xi},\underline{\theta},\underline{\pi}) = e^{i(\gamma-\tilde\gamma)t\underline{x}^2/4}\underline{\pi}_j \quad (j=1,2). \end{gathered}$$

Putting operators

$$\hat A = -i\partial_{\underline{x}} - \frac{\gamma+\tilde\gamma}{2}\underline{x}t, \quad \hat B = \frac{\gamma-\tilde\gamma}{2}\underline{x}t, \quad \widehat{\sigma}_3 = 1 - \underline{\theta}_1\frac{\partial}{\partial\underline{\theta}_1} - \underline{\theta}_2\frac{\partial}{\partial\underline{\theta}_2}, \quad \widehat{\xi(t)} = \hat A - \hat B\widehat{\sigma}_3,$$

with Weyl symbols

$$\begin{gathered} \sigma(\hat A)(\underline{x},\underline{\xi}) = \xi - \frac{\gamma+\tilde\gamma}{2}\underline{x}t, \quad \sigma(\hat B)(\underline{x},\underline{\xi}) = \frac{\gamma-\tilde\gamma}{2}\underline{x}t, \quad \sigma(\widehat{\sigma}_3)(\underline{\theta},\underline{\pi}) = -i\langle\underline{\theta}|\underline{\pi}\rangle, \\ \sigma(\widehat{\xi(t)})(\underline{x},\underline{\xi},\underline{\theta},\underline{\pi}) = \underline{\xi} - \frac{\gamma+\tilde\gamma}{2}\underline{x}t + i\frac{\gamma-\tilde\gamma}{2}\underline{x}t\langle\underline{\theta}|\underline{\pi}\rangle, \quad \langle\theta(t)|\pi(t)\rangle = \langle\underline{\theta}|\underline{\pi}\rangle, \end{gathered}$$

we have the following:

$$\sigma(\hat A\hat B + \hat B\hat A)(\underline{x},\underline{\xi}) = (\gamma-\tilde\gamma)\underline{x}t\left(\underline{\xi} - \frac{\gamma+\tilde\gamma}{2}\underline{x}t\right), \quad \sigma(\widehat{\sigma}_3^2)(\underline{\theta},\underline{\pi}) = 1.$$

Remark 1. Let $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$ be non-commutative or commutative "operators". For monomials $p_2(x,y) = xy$ and $p_3(x,y,z) = xyz$, we define

$$p_{2,\mathrm{s}}(\mathfrak{a},\mathfrak{b}) = \begin{cases} \frac{1}{2!}(\mathfrak{ab}+\mathfrak{ba}) & \text{if } [\mathfrak{a},\mathfrak{b}] \neq 0, \\ \mathfrak{ab} & \text{if } [\mathfrak{a},\mathfrak{b}] = 0, \end{cases}$$

$$p_{3,\mathrm{s}}(\mathfrak{a},\mathfrak{b},\mathfrak{c}) = \begin{cases} \frac{1}{3!}(\mathfrak{abc}+\mathfrak{acb}+\mathfrak{bca}+\mathfrak{bac}+\mathfrak{cba}+\mathfrak{cab}) & \text{if } \mathfrak{a},\mathfrak{b},\mathfrak{c} \text{ are non-commutative each other}, \\ \frac{1}{2!}(\mathfrak{abc}+\mathfrak{bca}) & \text{if } [\mathfrak{a},\mathfrak{b}] \neq 0, \text{ but } [\mathfrak{a},\mathfrak{c}] = 0 \text{ and } [\mathfrak{b},\mathfrak{c}] = 0, \\ \mathfrak{abc} & \text{if } \mathfrak{a},\mathfrak{b},\mathfrak{c} \text{ are commutative each other}. \end{cases}$$

From these, we get

$$\sigma\big(\widehat{\xi(t)}^2\big)(\underline{x},\underline{\xi},\underline{\theta},\underline{\pi}) = \underline{\xi}^2 - (\gamma+\tilde\gamma)\underline{x}t\underline{\xi} + \frac{\gamma^2+\tilde\gamma^2}{2}\underline{x}^2t^2 + i\bigg(\underline{\xi} - \frac{\gamma+\tilde\gamma}{2}\underline{x}t\bigg)(\gamma-\tilde\gamma)\underline{x}t\langle\underline{\theta}|\underline{\pi}\rangle.$$

In fact,

$$\widehat{\xi(t)}^2 = (\hat A - \hat B\hat\sigma_3)(\hat A - \hat B\hat\sigma_3) = \hat A^2 - (\hat A\hat B + \hat B\hat A)\hat\sigma_3 + \hat B^2\hat\sigma_3^2,$$

with

$$\hat A^2 + \hat B^2 = -\partial_{\underline{x}}^2 + i\partial_{\underline{x}}\bigg(\frac{\gamma+\tilde\gamma}{2}\underline{x}t\bigg) + \bigg(\frac{\gamma+\tilde\gamma}{2}\underline{x}t\bigg)i\partial_{\underline{x}} + \frac{\gamma^2+\tilde\gamma^2}{2}\underline{x}^2t^2,$$

and

$$\sigma(\hat A^2 + \hat B^2) = \underline{\xi}^2 - i\underline{\xi}(\gamma+\tilde\gamma)\underline{x}t + \frac{\gamma^2+\tilde\gamma^2}{2}\underline{x}^2t^2.$$

Since $[\widehat{\xi(t)}^2, \langle\widehat{\theta(t)|\pi(t)}\rangle] = 0$, we have

$$\begin{aligned}\big(\widehat{\xi(t)}^2\langle\widehat{\theta(t)|\pi(t)}\rangle\big)(\underline{x},\underline{\xi},\underline{\theta},\underline{\pi}) &= \sigma\big(\widehat{\xi(t)}^2\langle\widehat{\theta(t)|\pi(t)}\rangle\big)(\underline{x},\underline{\xi},\underline{\theta},\underline{\pi})\\ &= \bigg(\underline{\xi}^2 - (\gamma+\tilde\gamma)\underline{x}t\underline{\xi} + \frac{\gamma^2+\tilde\gamma^2}{2}\underline{x}^2t^2\bigg)\langle\underline{\theta}|\underline{\pi}\rangle - i\bigg(\underline{\xi} - \frac{\gamma+\tilde\gamma}{2}\underline{x}t\bigg)(\gamma-\tilde\gamma)\underline{x}t\bigg(\frac{1}{2} + 2\underline{\theta}_1\underline{\theta}_2\underline{\pi}_1\underline{\pi}_2\bigg).\end{aligned}$$

Though $[\hat\sigma_3, \widehat{\theta_1\theta_2}] \neq 0$ and $[\hat\sigma_3, \widehat{\pi_1\pi_2}] \neq 0$, we have $[\widehat{\xi(t)}, \theta_1\widehat{(t)\theta_2}(t)] = 0$ and $[\widehat{\xi(t)}, \pi_1\widehat{(t)\pi_2}(t)] = 0$. Moreover, we get

$$\begin{aligned}\widehat{\xi(t)}\theta_1\widehat{(t)\theta_2}(t)(u_0 + u_1\underline{\theta}_1\underline{\theta}_2) &= e^{-i(\gamma-\tilde\gamma)\underline{x}^2t/2}(-i\partial_{\underline{x}} - \gamma\underline{x}t)u_0\underline{\theta}_1\underline{\theta}_2,\\ \widehat{\xi(t)}\pi_1\widehat{(t)\pi_2}(t)(u_0 + u_1\underline{\theta}_1\underline{\theta}_2) &= e^{i(\gamma-\tilde\gamma)\underline{x}^2t/2}(-i\partial_{\underline{x}} - \tilde\gamma\underline{x}t)u_1.\end{aligned}$$

Therefore, we get

$$\begin{aligned}\big(\widehat{\xi(t)}\theta_1\widehat{(t)\theta_2}(t)\big)(\underline{x},\underline{\xi},\underline{\theta},\underline{\pi}) &= \sigma\big(\widehat{\xi(t)}\theta_1\widehat{(t)\theta_2}(t)\big)(\underline{x},\underline{\xi},\underline{\theta},\underline{\pi})\\ &= \bigg(\underline{\xi} - \frac{\gamma+\tilde\gamma}{2}\underline{x}t\bigg)e^{-i(\gamma-\tilde\gamma)\underline{x}^2t/2}\underline{\theta}_1\underline{\theta}_2,\\ \big(\widehat{\xi(t)}\pi_1\widehat{(t)\pi_2}(t)\big)(\underline{x},\underline{\xi},\underline{\theta},\underline{\pi}) &= \sigma\big(\widehat{\xi(t)}\pi_1\widehat{(t)\pi_2}(t)\big)(\underline{x},\underline{\xi},\underline{\theta},\underline{\pi})\\ &= \bigg(\underline{\xi} - \frac{\gamma+\tilde\gamma}{2}\underline{x}t\bigg)e^{i(\gamma-\tilde\gamma)\underline{x}^2t/2}\underline{\pi}_1\underline{\pi}_2.\end{aligned}$$

On the other hand, since $[\widehat{d(x(t))}, \widehat{\xi(t)}] \neq 0$ but $[\widehat{d(x(t))}, \theta_1\widehat{(t)\theta_2}(t)] = 0$ and $[\widehat{\xi(t)}, \theta_1\widehat{(t)\theta_2}(t)] = 0$, we have

$$p_{3,\mathrm{s}}(\widehat{d(x(t))}, \widehat{\xi(t)}, \theta_1\widehat{(t)\theta_2}(t)) = \frac{1}{2}(\widehat{d(x(t))}\widehat{\xi(t)}\theta_1\widehat{(t)\theta_2}(t) + \widehat{\xi(t)}\theta_1\widehat{(t)\theta_2}(t)\widehat{d(x(t))}),$$

that is,

$$p_{3,\mathrm{s}}(\widehat{d(x(t))}, \widehat{\xi(t)}, \theta_1\widehat{(t)\theta_2}(t))(u_0 + u_1\underline{\theta}_1\underline{\theta}_2) = e^{-i(\gamma-\tilde\gamma)\underline{x}^2t/2}/2\{d(\underline{x})(-i\partial_{\underline{x}} - \gamma\underline{x}t) + (-i\partial_{\underline{x}} - \gamma\underline{x}t)d(\underline{x})\}u_0\underline{\theta}_1\underline{\theta}_2.$$

Analogous holds for $p_{3,\mathrm{s}}(\widehat{c(x(t))}\widehat{\xi(t)}\pi_1\widehat{(t)\pi_2}(t))$.

From these, we have

$$\begin{aligned}
\mathcal{P}\big[\mathcal{C}(t)(\underline{x},\underline{\xi},\underline{\theta},\underline{\pi})\big] &= \frac{\alpha+\tilde{\alpha}}{2}(\widehat{\xi(t)}^2) + \frac{\beta+\tilde{\beta}}{2}(\widehat{x(t)}^2) - i\frac{\alpha-\tilde{\alpha}}{2}(\widehat{\xi(t)}^2\langle\widehat{\theta(t)|\pi(t)}\rangle) - i\frac{\alpha-\tilde{\alpha}}{2}(\widehat{x(t)}^2\langle\widehat{\theta(t)|\pi(t)}\rangle) \\
&\quad + i(\widehat{d(x(t))}\widehat{\xi(t)}\widehat{\theta_1(t)\theta_2(t)}) + i(\widehat{c(x(t))}\widehat{\xi(t)}\widehat{\pi_1(t)\pi_2(t)})(\underline{x},\underline{\xi},\underline{\theta},\underline{\pi}) \\
&= \frac{\alpha+\tilde{\alpha}}{2}\underline{\xi}^2 + \frac{\beta+\tilde{\beta}}{2}\underline{x}^2 + \frac{\alpha\gamma^2t^2+\tilde{\alpha}\tilde{\gamma}^2t^2}{2}\underline{x}^2 - (\alpha\gamma t+\tilde{\alpha}\tilde{\gamma}t)\underline{x}\underline{\xi} \\
&\quad - i\bigg[\frac{\alpha-\tilde{\alpha}}{2}\underline{\xi}^2 + \frac{\beta-\tilde{\beta}}{2}\underline{x}^2 - (\alpha\gamma t-\tilde{\alpha}\tilde{\gamma}t)\underline{x}\underline{\xi} + \frac{\alpha\gamma^2t^2-\tilde{\alpha}\tilde{\gamma}^2t^2}{2}\underline{x}^2\bigg]\langle\underline{\theta}|\underline{\pi}\rangle \\
&\quad + id(\underline{x})\bigg(\underline{\xi} - \frac{\gamma+\tilde{\gamma}}{2}\underline{x}t\bigg)e^{-i(\gamma t-\tilde{\gamma}t)\underline{x}^2/2}\underline{\theta}_1\underline{\theta}_2 + ic(\underline{x})\bigg(\underline{\xi} - \frac{\gamma t+\tilde{\gamma}t}{2}\underline{x}\bigg)e^{i(\gamma t-\tilde{\gamma}t)\underline{x}^2/2}\underline{\pi}_1\underline{\pi}_2.
\end{aligned} \tag{1.15}$$

This equals to (1.13) after replacing $\gamma \to \gamma t$ and $\tilde{\gamma} \to \tilde{\gamma}t$.

Therefore, denoting $\underline{x}$ simply by x, etc, we have proved

$$\sigma(\sharp e^{it\mathbb{H}}\mathbb{P}e^{-it\mathbb{H}}\flat)(x,\xi,\theta,\pi) = \sigma(e^{it\hat{\mathcal{H}}}\hat{\mathcal{P}}e^{-it\hat{\mathcal{H}}})(x,\xi,\theta,\pi) = \mathcal{P}\big[\mathcal{C}(t)(x,\xi,\theta,\pi)\big]. \qquad \square \tag{1.16}$$

This is the interpretation of (1.13) using superanalysis and which is the typical example of a system version of Egorov's theorem.

1.1.2.2 WKB approach to Dirac equation by Pauli, de Broglie, Rubinow & Keller

The modified Dirac equation with an anomalous magnetic moment, may be written in the form

$$i\hbar\frac{\partial}{\partial t}\psi = [c\boldsymbol{\alpha}_j(\frac{\hbar}{i}\frac{\partial}{\partial q_j} - \frac{e}{c}A_j) + e\Phi + \boldsymbol{\beta}mc^2]\psi + g\frac{ie\hbar}{2mc}F_{kl}(\boldsymbol{\alpha}^k\boldsymbol{\alpha}^l - \boldsymbol{\alpha}^l\boldsymbol{\alpha}^k)\psi \tag{1.17}$$

where

$$F_{kl} = \frac{\partial A_k}{\partial x_l} - \frac{\partial A_l}{\partial x_k}.$$

Pauli tried to have a solution of (1.17) in the following form:

$$\psi \sim e^{i\hbar^{-1}S}\sum_{n=0}^{\infty}(-i\hbar)^n a_n,$$

where S is a scalar function, a_n are matrix-valued functions. Though Pauli didn't calculate generic terms completely, Rubinow & Keller [195] claimed that his procedure yields the correct result in inhomogeneous field regions and fixed finite distances from them, but not at all distances of the order $\hbar^{-1}$ from them.

Our problem is to apply our method to the superversion of (1.17) and to get the corresponding result mathematically.

For the case of the free Dirac equation, that is, when $A_j = \Phi = 0$, we have the result [110]: Given $\underline{\psi}(q)$, find a good representation of $\psi(t,q) : \mathbb{R}\times\mathbb{R}^3 \to \mathbb{C}^4$, satisfying

$$\begin{cases} i\hbar\dfrac{\partial}{\partial t}\psi(t,q) = \mathbb{H}\psi(t,q) \\ \psi(0,q) = \underline{\psi}(q) \end{cases} \tag{1.18}$$

with

$$\mathbb{H} = -i\hbar c\boldsymbol{\alpha}_k\frac{\partial}{\partial q_k} + mc^2\boldsymbol{\beta}.$$

Here, $\hbar$ is the Planck's constant, c, m are constants, $\psi(t,q) = {}^t(\psi_1(t,q), \psi_2(t,q), \psi_3(t,q), \psi_4(t,q))$, the summation with respect to $k = 1,2,3$ is abbrebiated, and the matrices $\{\boldsymbol{\alpha}_k, \boldsymbol{\beta}\}$ satisfy the Clifford relation:

$$\boldsymbol{\alpha}_j\boldsymbol{\alpha}_k + \boldsymbol{\alpha}_k\boldsymbol{\alpha}_j = 2\delta_{jk}\mathbb{I}_4, \quad \boldsymbol{\alpha}_k\boldsymbol{\beta} + \boldsymbol{\beta}\boldsymbol{\alpha}_k = 0, \quad \boldsymbol{\beta}^2 = \mathbb{I}_4 \quad j,k = 1,2,3. \tag{1.19}$$

In the following, we use the Dirac representation of matrices

$$\boldsymbol{\beta} = \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{pmatrix}, \quad \boldsymbol{\alpha}_k = \begin{pmatrix} 0 & \boldsymbol{\sigma}_k \\ \boldsymbol{\sigma}_k & 0 \end{pmatrix} \quad \text{for} \quad k = 1,2,3.$$

Applying formally the Fourier transformation with respect to $q \in \mathbb{R}^3$ to (1.18), we get

$$i\hbar\frac{\partial}{\partial t}\hat{\psi}(t,p) = \mathbb{H}(p)\hat{\psi}(t,p) \tag{1.20}$$

where

$$\mathbb{H}(p) = c\boldsymbol{\alpha}_j p_j + mc^2\boldsymbol{\beta} = c\begin{pmatrix} mc & 0 & p_3 & p_1 - ip_2 \\ 0 & mc & p_1 + ip_2 & -p_3 \\ p_3 & p_1 - ip_2 & -mc & 0 \\ p_1 + ip_2 & -p_3 & 0 & -mc \end{pmatrix}. \tag{1.21}$$

Remarking $\mathbb{H}^2(p) = c^2\|p\|^2\mathbb{I}_4$ with $\|p\| = \sqrt{m^2c^2 + |p|^2}$, we have,

$$e^{-i\hbar^{-1}t\mathbb{H}(p)} = \cos(c\hbar^{-1}t\|p\|)\mathbb{I}_4 - \frac{i}{c\|p\|}\sin(c\hbar^{-1}t\|p\|)\mathbb{H}(p). \tag{1.22}$$

Therefore, we have readily

Proposition 1.1.3 *For any $t \in \mathbb{R}$ and $\underline{\psi} \in L^2(\mathbb{R}^3 : \mathbb{C})^4 = L^2(\mathbb{R}^3 : \mathbb{C}^4)$,*

$$\psi(t,q) = e^{-i\hbar^{-1}t\mathbb{H}}\underline{\psi}(q) = (2\pi\hbar)^{-3/2}\int_{\mathbb{R}^3} dp\, e^{i\hbar^{-1}qp}e^{-i\hbar^{-1}t\mathbb{H}(p)}\hat{\underline{\psi}}(p). \tag{1.23}$$

For $\underline{\psi} \in \mathcal{S}(\mathbb{R}^3 : \mathbb{C})^4$, we have formally

$$e^{-i\hbar^{-1}t\mathbb{H}}\underline{\psi}(q) = \int_{\mathbb{R}^3} dq'\, \mathbb{E}(t, q - q')\underline{\psi}(q') \tag{1.24}$$

with

$$\mathbb{E}(t,q) = (2\pi\hbar)^{-3}\int_{\mathbb{R}^3} dp\, e^{i\hbar^{-1}qp}\big[\cos(c\hbar^{-1}t\|p\|)\mathbb{I}_4 - \frac{i}{c\|p\|}\sin(c\hbar^{-1}t\|p\|)\mathbb{H}(p)\big] \in \mathcal{S}'(\mathbb{R}^3 : \mathbb{C})^4. \tag{1.25}$$

Applying our analysis on superspace, we have the following.

Theorem 1.1.4 (Path-integral representation of a solution for the free Dirac equation)

$$\psi(t,q) = \flat\Big((2\pi)^{-3/2}e^{\pi i/4}\iint_{\mathfrak{R}^{3|3}} d\underline{\xi}d\underline{\pi}\, \mathcal{D}^{1/2}(t,\bar{x},\bar{\theta},\underline{\xi},\underline{\pi})e^{i\hbar^{-1}\mathcal{S}(t,\bar{x},\bar{\theta},\underline{\xi},\underline{\pi})}\mathcal{F}(\sharp\underline{\psi})(\underline{\xi},\underline{\pi})\Big)\Big|_{\bar{x}_{\mathrm{B}}=q}. \tag{1.26}$$

Here, $\mathcal{S}(t,\bar{x},\bar{\theta},\underline{\xi},\underline{\pi})$ and $\mathcal{D}(t,\bar{x},\bar{\theta},\underline{\xi},\underline{\pi})$ are given by,

$$\begin{aligned} &\mathcal{S}(t,\bar{x},\underline{\xi},\bar{\theta},\underline{\pi}) = \langle\bar{x}|\underline{\xi}\rangle + \langle\bar{\theta}|\underline{\pi}\rangle + \bar{\mathcal{B}}(t)[2imc\bar{\theta}_3\underline{\pi}_3 + (\hbar\bar{\Theta} - i\underline{\Pi})(\bar{\theta}_3 + i\hbar^{-1}\underline{\pi}_3)], \\ &\mathcal{D}(t,\bar{x},\underline{\xi},\bar{\theta},\underline{\pi}) = \bar{\delta}(t), \end{aligned} \tag{1.27}$$

where

$$\bar{\mathcal{B}}(t) = \bar{\mathcal{A}}(t)\bar{\delta}^{-1}(t), \quad \bar{\mathcal{A}}(t) = \mathrm{a}(t) - 2imc\mathrm{b}(t), \quad \bar{\delta}(t) = 1 - 2\mathrm{b}(t)|\underline{\xi}|^2 - 2imc\bar{\mathcal{A}}(t),$$

$$\mathrm{a}(t) = \frac{\sin 2\nu t}{2\|\underline{\xi}\|}, \quad \mathrm{b}(t) = \frac{1 - \cos 2\nu t}{4\|\underline{\xi}\|^2} \quad \nu = c\hbar^{-1}\|\underline{\xi}\|, \quad \|\underline{\xi}\|^2 = |\underline{\xi}|^2 + m^2c^2 \quad \textit{and}$$

$$\bar{\Theta} = (\underline{\xi}_1 + i\underline{\xi}_2)\bar{\theta}_1 - \underline{\xi}_3\bar{\theta}_2, \quad \underline{\Pi} = (\underline{\xi}_1 - i\underline{\xi}_2)\underline{\pi}_1 - \underline{\xi}_3\underline{\pi}_2.$$

Moreover, they are solutions of Hamilton-Jacobi equation and continuity equation, respectively.

Problem. *Extend the procedure mentioned above for the free Dirac equation to* (1.17) *(hint: see [*?, ?*] which treats the analogous case for the Weyl equation).*

1.1.2.3 Sung's example for a system version of Melin's inequality

Let $H(q,p) = \sum_{|\alpha+\beta|\le 2} a_{\alpha\beta} q^\alpha p^\beta$ where $a_{\alpha\beta} \in \mathbb{R}$ and $(q,p) \in \mathbb{R}^{2m}$. Let $H_2(q,p) = \sum_{|\alpha+\beta|=2} a_{\alpha\beta} q^\alpha p^\beta$ and $P((q,p),(q',p'))$ be the polarized form of $H_2(q,p)$. Let $\sigma(\cdot,\cdot)$ be the standard symplectic form on $\mathbb{R}^{2m}$. F is the Hamiltonian map of H_2 defined by $\sigma((q,p),F(q',p')) = P((q,p),(q',p'))$ and $\mathrm{tr}^+ p_2$ is defined as the sum of the positive eigenvalues of $-iF$.

Let

$$H^W(q,D_q)u(q) = (2\pi)^{-2m} \iint dq'dp\, H\Big(\frac{q+q'}{2},p\Big) e^{i(q-q')p} u(q') \quad \text{for} \quad u \in \mathcal{S}(\mathbb{R}^m).$$

Theorem 1.1.5 (Melin []) *$\langle H^W(q,D_q)u,u\rangle \ge 0$ for any $u \in \mathcal{S}(\mathbb{R}^m)$ if and only if $\inf H(q,p) + \mathrm{tr}^+ H_2 \ge 0$. In particular, if $H(q,\xi) \ge 0$, then $H^W(q,D) \ge 0$.*

This claim is not generalized straight fowardly to the system of PDE:

Example.(Hörmander [97]). Let

$$\mathbb{P}(q,p) = \begin{pmatrix} q^2 & qp \\ qp & p^2 \end{pmatrix} \quad \text{for} \quad (q,p) \in \mathbb{R}^2,$$

then $\mathbb{P}(q,p) \ge 0$ but for $u_1 = v''$, $u_2 = i(v - qv')$ and $0 \not\equiv v \in \mathcal{S}(\mathbb{R})$,

$$\langle \mathbb{P}^W(q,D_q)\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\rangle = -\frac{1}{2}\int dq\,(v')^2 < 0.$$

Problem. *Is it posssible to characterize vectors v such that $\langle \mathbb{P}^W(q,D_q)v,v\rangle \le 0$?*

Let

$$\mathbb{H}(q,p) = \begin{pmatrix} aq^2+bp^2 & \alpha qp \\ \alpha qp & cq^2+dp^2 \end{pmatrix} \quad \text{for} \quad (q,p) \in \mathbb{R}^2,\ a,\ b,\ c,\ d \ge 0 \quad \text{and} \quad ad+bc \ne 0.$$

Theorem 1.1.6 (Sung [207]) *Let a, b, c, $d \ge 0$ and $ad+bc \ne 0$. For $\mathbb{H}^W(q,D_q) \ge 0$, it is neccessary and sufficient that $(\lambda_1,\lambda_2) \in \Omega$ or $(\lambda_2,\lambda_1) \in \Omega$ where*

$$\lambda_1 = \frac{\sqrt{ad}-\sqrt{bc}+\alpha}{\sqrt{ad}+\sqrt{bc}}, \quad \lambda_2 = \frac{\sqrt{ad}-\sqrt{bc}-\alpha}{\sqrt{ad}+\sqrt{bc}}, \quad \Omega = \{(x,y)\,|\,N(x,y) \ge 0\},$$

and

$$N(x,y) = \begin{pmatrix} 1 & \zeta_0 x & 0 & 0 & 0 & 0 & \cdots \\ \zeta_0 x & 1 & \zeta_1 y & 0 & 0 & 0 & \cdots \\ 0 & \zeta_1 y & 1 & \zeta_2 x & 0 & 0 & \cdots \\ 0 & 0 & \zeta_2 x & 1 & \zeta_3 y & 0 & \cdots \\ 0 & 0 & 0 & \zeta_3 y & 1 & \zeta_4 x & \cdots \\ 0 & 0 & 0 & 0 & \zeta_4 x & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \quad \textit{with} \quad \zeta_n = \Big(\frac{(2n+1)(2n+2)}{(4n+1)(4n+5)}\Big)^{1/2}.$$

Problems. *(1) Construct a good parametrix for the following operators:*

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \mathbb{H}^W(q,-i\hbar\partial_q)\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},$$

$$\frac{\partial^2}{\partial t^2}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \mathbb{H}^W(q,-i\partial_q)\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},$$

$$\frac{\partial}{\partial t}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \mathbb{H}^W(q,\partial_q)\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}.$$

(2) Extend the result of Sung to more general positive definite matrices? Find the condition like Melin's characterization.

1.1.2.4 Gelfand's question for the meaning of ellipticity

Let a matrix be given by

$$\mathbb{B}(p) = \begin{pmatrix} p_1^2 - p_2^2 & -2p_1p_2 \\ 2p_1p_2 & p_1^2 - p_2^2 \end{pmatrix}$$

which is weakly but not strongly elliptic system. How about the characteristic behavior of the solution caused by "weakly but not strongly elliptic system" of the following equations?

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \mathbb{B}^W(-i\hbar\partial_q)\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix},$$

$$\frac{\partial^2}{\partial t^2}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \mathbb{B}^W(-i\partial_q)\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix},$$

$$\frac{\partial}{\partial t}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \mathbb{B}^W(\partial_q)\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}.$$

Problem. *Can we characterize the ellipticity of the systems of PDE by checking the behavior of solutions of the heat type for $t \to \infty$?*

1.1.2.5 Is the Euler equation attackable by superanalysis?

The Euler equation on $\mathbb{R}^3$ is given by

$$\begin{cases} u_t + (u\cdot\nabla)u + \nabla p = 0, \\ \operatorname{div} u = 0, \\ u(0,x) = \underline{u}(x), \quad \text{where} \quad u = {}^t(u_1(t,x), u_2(t,x), u_3(t,x)). \end{cases} \tag{1.28}$$

This equation is the one of the most charming one which is not solved for the long time.

Taking the rotation $du = v$, we get

$$\begin{cases} v_t + (u\cdot\nabla)v = (v\cdot\nabla)u, \\ v(0,x) = \underline{v}(x). \end{cases} \tag{1.29}$$

Putting $\begin{pmatrix}v_1\\ v_2\\ v_3\end{pmatrix} = du = \begin{pmatrix}u_{2,3}-u_{3,2}\\ u_{3,1}-u_{1,3}\\ u_{1,2}-u_{2,1}\end{pmatrix}$, $u_{i,j} = \dfrac{\partial u_i}{\partial x_j}$, we have, for each $i = 1,2,3$,

$$\sum_{j=1}^3 v_j u_{i,j} = \sum_{j=1}^3 d_{ij} v_j \quad \text{where} \quad d_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}). \tag{1.30}$$

$D = (d_{ij})$ is called the deformation matrix of the fluid flow with $\sum_{i=1}^3 d_{ii} = \operatorname{div} u = 0$.

Therefore

$$\frac{\partial}{\partial t}\begin{pmatrix}v_1\\ v_2\\ v_3\end{pmatrix} + \sum_{j=1}^3 u_j \mathbb{I}_3 \frac{\partial}{\partial x_j}\begin{pmatrix}v_1\\ v_2\\ v_3\end{pmatrix} = \begin{pmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33}\end{pmatrix}\begin{pmatrix}v_1\\ v_2\\ v_3\end{pmatrix}. \tag{1.31}$$

Problem. *The above equation* (1.29) *in* $\mathbb{R}^2$ *has no right-hand side and solved nicely which garantees the classical solution for* (1.28) *in dimension* 2. *In spite of this fact, whether one can make use of the solution of this vorticity equation nicely to the Euler equation in* $\mathbb{R}^3$ *?*

On the other hand, it is well-known that we may apply the method of characteristics to

$$\sum_{j=1}^n a_j(q,u)\mathbb{I}_l \frac{\partial u_k}{\partial q_j} = b_k(q,u) \quad \text{for} \quad k=1,2,\cdots,l, \tag{1.32}$$

assuming $(a_1(q,u),\cdots,a_n(q,u)) \neq 0$.

Especially, we have the following:

Theorem 1.1.7 *Let* $a_j(t,q)$ *be* C^1 *near* $(\underline{t},\underline{q})$, *and let* $b_k(t,q,u)$ *be* C^1 *near* $(\underline{t},\underline{q},\underline{u})$, $\underline{u}=\phi(\underline{q})$, *and* ϕ *is* C^1 *near* $\underline{q}$. *If* $q=x(t,\underline{t}\,;\underline{q})$ *is a solution of*

$$\dot{q}_j = a_j(t,q), \quad q_j(\underline{t},\underline{t}\,;\underline{q}) = \underline{q}_j,$$

and $U(t,\underline{q})=(U_1(t,\underline{q}),\cdots,U_l(t,\underline{q}))$ *is a solution of*

$$\dot{U}_k = b_k(t,x(t,\underline{t}\,;\underline{q}),U), \quad U_k(\underline{t},\underline{q}) = \phi_k(\underline{q}).$$

Putting $u(t,\bar{q})=U(t,y(t,\underline{t}\,;\bar{q}))$ *where* $y=y(t,\underline{t}\,;\bar{q})$ *is the inverse function of* $\bar{q}=x(t,\underline{t}\,;\underline{q})$, *then it satisfies*

$$\frac{\partial u_k}{\partial t} + \sum_{j=1}^n a_j(t,q)\mathbb{I}_l \frac{\partial u_k}{\partial q_j} = b_k(t,q,u) \quad \textit{with} \quad u(\underline{t},\underline{q}) = \phi(\underline{q}) \tag{1.33}$$

Probelm. *Extends the above theorem to the case* $a_j(t,q)$ *are* $l\times l$*-matrices.*

1.1.2.6 The generalized Hopf-Cole transformation of Maslov

Let $V(t,q)\in C^\infty(\mathbb{R}_+\times\mathbb{R}^3:\mathbb{R})$ be given. For a solution $\psi\in C^2(\mathbb{R}\times\mathbb{R}^3:\mathbb{R})$ satisfying

$$\begin{cases} \nu\psi_t = \dfrac{\nu^2}{2}\Delta\psi + V\psi, \\ \psi(0) = \underline{\psi} = e^{-\nu^{-1}\phi}, \end{cases} \tag{1.34}$$

we put $u(t,q)=-\nu\nabla\log\psi(t,q)$, that is, $u={}^t(u_1,u_2,u_3)={}^t(-\nu\frac{\psi_{q_1}}{\psi},-\nu\frac{\psi_{q_2}}{\psi},-\nu\frac{\psi_{q_3}}{\psi})$. Then, u satisfies

$$\begin{cases} u_t + (u\cdot\nabla)u + \nabla V = \dfrac{\nu}{2}\Delta u, \\ u(0) = \nabla\phi. \end{cases} \tag{1.35}$$

Example. Let $V(t,q)=\sum_{j=1}^3 \frac{1}{2}\omega_j^2 q_j^2$. We have a solution of (1.34) as

$$\begin{aligned} \psi(t,\bar{q}) = (E_t\underline{\psi})(\bar{q}) &= (2\pi\nu)^{-3/2}\int_{\mathbb{R}^3} d\underline{q}\, D(t,\bar{q},\underline{q})^{1/2} e^{-\nu^{-1}S(t,\bar{q},\underline{q})}\underline{\psi}(\underline{q}) \\ &= \prod_{j=1}^3 \left(\frac{\omega_j}{2\pi\nu\sin(\omega_j t)}\right)^{1/2} \int_{\mathbb{R}^3} d\underline{q}\, e^{-\nu^{-1}S(t,\bar{q},\underline{q})}\underline{\psi}(\underline{q}). \end{aligned}$$

Here, we put

$$S(t,\bar{q},\underline{q}) = \sum_{j=1}^3 \left[\frac{\omega_j}{2}(\cot(\omega_j t))(\underline{q}_j^2+\bar{q}_j^2) - \frac{\omega_j}{\sin(\omega_j t)}\underline{q}_j\bar{q}_j\right] \quad \text{and} \quad D(t,\bar{q},\underline{q}) = \prod_{j=1}^3 \frac{\omega_j}{\nu\sin(\omega_j t)}.$$

Therefore, we get

$$u_j(t,\bar{q}) = \Big(\int_{\mathbb{R}^3} d\underline{q}\,(\omega_j \cot(\omega_j t)\bar{q}_j - \frac{\omega_j}{\sin(\omega_j t)}\underline{q}_j)e^{-\nu^{-1}S(t,\bar{q},\underline{q})}\underline{\psi}(\underline{q})\Big)\Big(\int_{\mathbb{R}^3} d\underline{q}\,e^{-\nu^{-1}S(t,\bar{q},\underline{q})}\underline{\psi}(\underline{q})\Big)^{-1}$$
$$= \omega_j \cot(\omega_j t)\bar{q}_j - \Big(\int_{\mathbb{R}^3} d\underline{q}\,\frac{\omega_j}{\sin(\omega_j t)}\underline{q}_j e^{-\nu^{-1}S(t,\bar{q},\underline{q})}\underline{\psi}(\underline{q})\Big)\Big(\int_{\mathbb{R}^3} d\underline{q}\,e^{-\nu^{-1}S(t,\bar{q},\underline{q})}\underline{\psi}(\underline{q})\Big)^{-1}.$$

Taking especially

$$\phi(q) = \frac{1}{2}\phi_{jk}q_j q_k \quad \text{with} \quad (\phi_{jk})\text{: a symmetric, positive matrix,}$$

we calculate $u_j(t,\bar{q})$ explicitly which is linear w.r.t. $\bar{q}$ and ν-independent, moreover, it satisfies (1.35) with $\nu = 0$.

Problem. *Does there exists Ehrenfest type theorems for the above* (1.34) *and what does it imply in* (1.35)*? (see, Hepp [**?**]).*

1.1.2.7 Gelfand's problem for dynamical systems

Outline of the problem: The study of dynamical systems geverned by

$$\frac{d}{dt}q_j(t) = F_j(q_1(t),\cdots,q_n(t)) \quad (j = 1,2,\cdots,n) \tag{1.36}$$

is related to that of a partial differential equation(PDE) of the first order

$$\frac{\partial}{\partial t}u(t,q) = \sum_{j=1}^{n} F_j(q_1,\cdots,q_n)\frac{\partial}{\partial q_j}u(t,q). \tag{1.37}$$

By the so-called spectral method of the theory of dynamical systems due to Koopman, the theory of dynamical systems may to a significant degree be interpreted as a theory relative to a linear partial differential equation of first order.

For example, if Ω is an invariant set of the flow defined by (1.36) (i.e. if T_t is defined by $q(t) = T_t q(0)$, Ω should satisfy $T_t\Omega = \Omega$), there exists an invariant measure μ of the flow T_t (i.e. for any Borel set $\omega \subset \Omega$, $\mu(T_{-t}\omega) = \omega$) such that $i\sum_{j=1}^n F_j(q)\partial/\partial q_j$ is self adjoint on $L^2(\Omega, d\mu)$.

Gelfand [**?**] asked whether in the above story, we may replace (1.37) by

$$\frac{\partial}{\partial t}u_j(t,q) = \sum_{k=1}^{d} A^{(k)}_{j,\ell}(q)\frac{\partial}{\partial q_k}u_\ell(t,q) \quad \text{for} \quad j,\ell = 1,2,\cdots,n, \tag{1.38}$$

where $A^{(k)}$ are $n\times n$-matrices whose elemants are denoted by $A^{(k)}_{j,\ell}$. Gelfand's first question in this direction is, whether there exists an invariant measure $\tilde{\mu}$ on an invariant set $\tilde{\Omega}$ such that $i\sum_{k=1}^d A^{(k)}_{j,\ell}(q)\frac{\partial}{\partial q_k}$ becomes self adjoint on $L^2(\tilde{\Omega}; d\tilde{\mu})$?

Our formulation by an example: Here, we may take 2×2-systems of PDE and explain our formulation for Gelfand's problem.

We consider the initial value problem

$$\frac{\partial}{\partial t}\begin{pmatrix}\psi_1(t,q)\\ \psi_2(t,q)\end{pmatrix} = \sum_{j=1}^{3}\begin{pmatrix}a^j(q) & c^j(q)\\ d^j(q) & b^j(q)\end{pmatrix}\frac{\partial}{\partial q_j}\begin{pmatrix}\psi_1(t,q)\\ \psi_2(t,q)\end{pmatrix} \quad \text{with} \quad \begin{pmatrix}\psi_1(0,q)\\ \psi_2(0,q)\end{pmatrix} = \begin{pmatrix}\underline{\psi}_1(q)\\ \underline{\psi}_2(q)\end{pmatrix}. \tag{1.39}$$

For the hyperbolicity, we assume

$$(a(q)p - b(q)p)^2 + 4(c(q)p)(d(q)p) \geq 0 \quad \text{for} \quad |p| = 1. \tag{1.40}$$

Here, we abbribiate $\sum_{j=1}^{3} a^j(q)p_j = a(q)p$, etc.

For the matrix

$$\mathbb{H}(q,p) = -\begin{pmatrix} a(q)p & c(q)p \\ d(q)p & b(q)p \end{pmatrix}$$
$$= -\frac{a(q)p + b(q)p}{2} - \frac{a(q)p - b(q)p}{2}\boldsymbol{\sigma}_3 - \frac{c(q)p + d(q)p}{2}\boldsymbol{\sigma}_1 - \frac{c(q)p - d(q)p}{2}\boldsymbol{\sigma}_2,$$

we may associate a Hamiltonian $\mathcal{H}(x,\xi,\theta,\pi)$ on $\mathcal{T}^*(\mathfrak{R}^{3|2}) = \mathfrak{R}^{6|4}$ given by

$$\mathcal{H}(x,\xi,\theta,\pi) = -\mathfrak{a}(x)\xi + i\mathfrak{b}(x)\xi\langle\theta|\pi\rangle - c(x)\xi\theta_1\theta_2 - d(x)\xi\pi_1\pi_2, \tag{1.41}$$

with

$$\mathfrak{a}^j(x) = \frac{a^j(x) + b^j(x)}{2},\ \mathfrak{b}^j(x) = \frac{a^j(x) - b^j(x)}{2},\ \mathfrak{a}(x)\xi = \sum_{j=1}^{3}\mathfrak{a}^j(x)\xi_j,\ \mathfrak{b}(x)\xi = \sum_{j=1}^{3}\mathfrak{b}^j(x)\xi_j.$$

It yields the superspace version of the equation (1.39) represented by

$$i\frac{\partial}{\partial t}u(t,x,\theta) = \mathcal{H}\Big(x, -i\frac{\partial}{\partial x}, \theta, -i\frac{\partial}{\partial\theta}\Big)u(t,x,\theta) \quad \text{with} \quad u(0,x,\theta) = \underline{u}(x,\theta). \tag{1.42}$$

As $\mathcal{H}$ is even, we may consider the classical mechanics corresponding to $\mathcal{H}(x,\xi,\theta,\pi)$:

$$\begin{cases} \dfrac{d}{dt}x_j = \dfrac{\partial\mathcal{H}(x,\xi,\theta,\pi)}{\partial\xi_j} = -\mathfrak{a}^j(x) + i\mathfrak{b}^j(x)\langle\theta|\pi\rangle - c^j(x)\theta_1\theta_2 - d^j(x)\pi_1\pi_2, \\ \dfrac{d}{dt}\xi_j = -\dfrac{\partial\mathcal{H}(x,\xi,\theta,\pi)}{\partial x_j} = \mathfrak{a}_{x_j}(x)\xi - i\mathfrak{b}_{x_j}(x)\xi\langle\theta|\pi\rangle + c_{x_j}(x)\xi\theta_1\theta_2 + d_{x_j}(x)\xi\pi_1\pi_2 \end{cases} \tag*{(5.8)$_{ev}$}$$

$$\begin{cases} \dfrac{d}{dt}\theta_1 = -\dfrac{\partial\mathcal{H}(x,\xi,\theta,\pi)}{\partial\pi_1} = i\mathfrak{b}(x)\xi\theta_1 + d(x)\xi\pi_2, \\ \dfrac{d}{dt}\theta_2 = -\dfrac{\partial\mathcal{H}(x,\xi,\theta,\pi)}{\partial\pi_2} = i\mathfrak{b}(x)\xi\theta_2 - d(x)\xi\pi_1, \\ \dfrac{d}{dt}\pi_1 = -\dfrac{\partial\mathcal{H}(x,\xi,\theta,\pi)}{\partial\theta_1} = -i\mathfrak{b}(x)\xi\pi_1 + c(x)\xi\theta_2, \\ \dfrac{d}{dt}\pi_2 = -\dfrac{\partial\mathcal{H}(x,\xi,\theta,\pi)}{\partial\theta_2} = -i\mathfrak{b}(x)\xi\pi_2 - c(x)\xi\theta_1 \end{cases} \tag*{(5.8)$_{od}$}$$

and at time $t = 0$, the initial data are given by

$$(x(0),\xi(0),\theta(0),\pi(0)) = (\underline{x},\underline{\xi},\underline{\theta},\underline{\pi}). \tag{1.44}$$

If there exists a unique solution of (5.8) with (5.9), we denote it by

$$\mathcal{T}_t(\underline{x},\underline{\xi},\underline{\theta},\underline{\pi}) = (x(t),\xi(t),\theta(t),\pi(t)) = (x(t\,;\underline{x},\underline{\xi},\underline{\theta},\underline{\pi}),\xi(t\,;\underline{x},\underline{\xi},\underline{\theta},\underline{\pi}),\theta(t\,;\underline{x},\underline{\xi},\underline{\theta},\underline{\pi}),\pi(t\,;\underline{x},\underline{\xi},\underline{\theta},\underline{\pi})).$$

Therefore, it is natural to ask whether there exists a set $\tilde{\Omega} \subset \mathfrak{R}^{6|4} = \mathcal{T}^*\mathfrak{R}^{3|2}$ such that

$$\mathcal{T}_t\tilde{\Omega} = \tilde{\Omega}\,?$$

whether $-i\mathcal{H}(x,-i\partial_x,\theta,-i\partial_\theta)$ is self-adjoint on $L^2(\tilde{\Omega},\tilde{\mu})$? Here, $\tilde{\mu}$ is an invariant measure on $\tilde{\Omega}$ related to the symplectic measure $dx \wedge d\xi + d\theta \vee d\pi$ on $\mathcal{T}^*\mathfrak{R}^{3|2}$.

1.1.2.8 Another example

If we consider

$$P_1 = -a_1\Delta + V_1(x), \quad P_2 = -a_2\Delta + V_2(x)$$

with $V_j(x)$ are real-valued, we ask whether

$$\mathbb{P} = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}$$

is self adjoint on $L^2(\mathbb{R}^m)^2$ even if P_j are not necessarily self-adjoint on $L^2(\mathbb{R}^m)$?

1.1.3 Supersymmetry

After E. Witten re-explained Morse theory from his point of view, it is rather popular in mathematical physics, to handle supersymmetry. Though it is easy to introduce the notion of supersymetric pairs in quantum fields, but its classical correspondence is not so clear mathematically.

Let (M, g) be a Riemannian manifold with d, d^* being usual exterior derivative and its adjoint. Let ϕ be a smooth real valued functions on M and λ a real number. Define

$$d_\lambda = e^{-\lambda\phi} d e^{\lambda\phi}, \quad d^*_\lambda = e^{\lambda\phi} d^* e^{-\lambda\phi}$$

and put

$$Q_{1\lambda} = d_\lambda + d^*_\lambda, \quad Q_{2\lambda} = i(d_\lambda - d^*_\lambda), \quad \mathbb{H}_\lambda = d_\lambda d^*_\lambda + d^*_\lambda d_\lambda.$$

Then, a triplet $(Q_{1\lambda}, Q_{2\lambda}, \mathbb{H}_\lambda)$ satisfies the algebra

$$Q^2_{1\lambda} = Q^2_{2\lambda} = \mathbb{H}_\lambda, \quad Q_{1\lambda}Q_{2\lambda} + Q_{2\lambda}Q_{1\lambda} = 0.$$

Though these are rather easily recognized, but in [220], Witten claimed innocently that the triplet $(Q_{1\lambda}, Q_{2\lambda}, \mathbb{H}_\lambda)$ is canonically obtained by quantizing the Lagrangian density

$$\begin{aligned}\mathcal{L}_\lambda = \frac{1}{2}\Big[\sum_{jk} g_{jk}\Big(\frac{dq^j}{dt}\frac{dq^k}{dt} + i\bar\psi^j\frac{D\psi^k}{Dt}\Big) + \frac{1}{4}R_{jklm}\bar\psi^j\psi^l\bar\psi^k\psi^m \\ - \lambda^2 g^{jk}\frac{d\phi}{dq^j}\frac{d\phi}{dq^k} - \lambda\frac{D^2\phi}{Dq^jDq^k}\bar\psi^j\psi^k\Big].\end{aligned}$$

Here, q^j are local coordinates of M, g_{jk}, R_{jklm} are the metric and curvature tensors of M, $\bar\psi^k$, ψ^m are anticommuting fields tangent to M, which are the creation and annihilation operators after quantization.

Problem for us is **to clarify why** "the triplet $(Q_{1\lambda}, Q_{2\lambda}, \mathbb{H}_\lambda)$ is canonically obtained by quantizing the Lagrangian density $\mathcal{L}_\lambda$".

In this book, we give a partial answer of this question, which is described as follows:

Let a Riemannian metric $g_{jk}(q)$ be given on $\mathbb{R}^m$. We may extend the euclidian space $\mathbb{R}^m$ to the super euclidian space $\mathfrak{R}^{m|m}$ and the supersymmetric extension $\tilde g_{jk}$ as follows:

For a given Lagrangian

$$L = \frac{1}{2}g_{ij}(q)\dot q^i\dot q^j + A_j(q)\dot q^j - V(q) \in C^\infty(T\mathbb{R}^m;\mathbb{R}),$$

using Legendre transformation, we associate a Hamiltonian

$$H(q,p) = \frac{1}{2}g^{ij}(q)(p_i - A_i(q))(p_j - A_j(q)) + V(q) \in C^\infty(T^*\mathbb{R}^m;\mathbb{R}).$$

To such a Hamiltonian, via extending formally that Lagrangian, we may associate a supersymmetric extension

$$\begin{aligned}
&\mathcal{H}(x,\xi,\theta,\pi)\\
&\quad=\frac{1}{2}g^{ij}(x)\big(\xi_i-\frac{i}{2}(g_{ik,l}(x)-g_{il,k}(x))\theta^k\pi^l-A_i(x)\big)\big(\xi_j-\frac{i}{2}(g_{jm,n}(x)-g_{jn,m}(x))\theta^m\pi^n-A_j(x)\big)\\
&\qquad+\frac{1}{2}R_{ikjl}\theta^j\theta^l\pi^i\pi^k+\frac{1}{2}g^{jk}(x)W_{,j}(x)W_{,k}(x)-W_{;ij}(x)\theta^i\pi^j
\end{aligned}$$

which belongs to $\mathcal{C}_{\rm SS}(\mathfrak{R}^{2m|2m};\mathfrak{R}_{\rm ev})$.

We study the properties of solutions of

$$\begin{cases}
\dfrac{d}{dt}x^j(t)=\dfrac{\partial}{\partial\xi_j}\mathcal{H}(x,\xi,\theta,\pi),\\
\dfrac{d}{dt}\xi_j(t)=-\dfrac{\partial}{\partial x_j}\mathcal{H}(x,\xi,\theta,\pi),\\
\dfrac{d}{dt}\theta_k(t)=-\dfrac{\partial}{\partial\pi_k}\mathcal{H}(x,\xi,\theta,\pi),\\
\dfrac{d}{dt}\pi_k(t)=-\dfrac{\partial}{\partial\theta_k}\mathcal{H}(x,\xi,\theta,\pi)
\end{cases}$$

with

$$(x^j(0),\xi_j(0),\theta_k(0),\pi_k(0))=(\underline{x}^j,\underline{\xi}^j,\underline{\theta}_k,\underline{\pi}_k)\in\mathfrak{R}^{2m|2m}.$$

After solving these equations, we apply the Feynman's procedure explained before, and we produce a triplet as above in this case. Supercharges are transformed into Q's and $\mathcal{H}$ yields the Hamilton operator $\mathbb{H}_\lambda$.

We stress here that we apply the Feynman's procedure only **in imaginary time case** because L^2-boundedness theorem of a Fourier integral operator with highly oscillating phase is not available in general.

1.1.4 Atiyah-Singer Theorem

I want to give a model how to extend the given Lagrangian to a supersymmetric one, and calculate its index.

I take a damped harmonic oscillator on $\mathbb{R}^2$ as the simplest example:

$$L(q,\dot q)=\frac{1}{2}(\dot q_1^2+\dot q_2^2)+a_1q_2\dot q_1-a_2q_2\dot q_1-[\frac{1}{2}{\omega_1}^2q_1^2+\lambda q_1q_2+\frac{1}{2}{\omega_2}^2q_2^2]\in C^\infty(T\mathbb{R}^2). \tag{1.45}$$

Using Legendre transformation, we have a Hamiltonian $H(q,p)$, $q=(q_1,q_2)$, $p=(p_1,p_2)$ given by

$$H(q,p)=\frac{1}{2}[(p_1-a_1q_2)^2+(p_2-a_2q_1)^2]+\frac{1}{2}{\omega_1}^2q_1^2+\lambda q_1q_2+\frac{1}{2}{\omega_2}^2q_2^2\in C^\infty(T^*\mathbb{R}^2). \tag{1.46}$$

Then using Lagrangian path-integral method, we get a fundamental solution E_t^L of the following initial value problem:

$$\begin{cases}
i\hbar\dfrac{\partial}{\partial t}u(t,q)=H^\hbar u(t,q),\\
u(0,q)=v(q).
\end{cases} \tag{1.47}$$

That is,

$$u(t,q)=E_t^Lv(q)=\frac{1}{\sqrt{2\pi i\hbar}}\int_{\mathbb{R}}dq'\,D_L(t,q,q')^{1/2}e^{i\hbar^{-1}S_L(t,q,q')}v(q'). \tag{1.48}$$

Or by Hamiltonian path-integral method, we have another expression as

$$u(t,q)=E_t^Hv(q)=\frac{1}{\sqrt{2\pi\hbar}}\int_{\mathbb{R}}dp\,D_H(t,q,p)^{1/2}e^{i\hbar^{-1}S_H(t,q,p)}\hat v(p). \tag{1.49}$$

1.1.4.1 The index of the super extended damped harmonic oscillator

Introducing operators

$$D_\alpha = \frac{\partial}{\partial \rho_\alpha} - i\rho_\alpha \frac{\partial}{\partial t} \quad \text{with} \quad \alpha = 1, 2,$$

and $\epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha}$, $\epsilon_{12} = 1$, we extend $L(q, \dot{q})$ as

$$\tilde{L}_0 = -\frac{1}{4}(D_\alpha \Phi)\epsilon_{\alpha\beta}(D_\beta \Phi) + \frac{i}{2} A\rho_\alpha \epsilon_{\alpha\beta} D_\beta \Phi - iW(\Phi). \tag{1.50}$$

In the above, $A(q)$ is extended from $q \in \mathbb{R}$ to $\Phi = x + i(\rho_1\psi_2 - \rho_2\psi_1) + i\rho_1\rho_2 F \in \mathfrak{R}_{\text{ev}}$ as

$$A(\Phi) = A(x) + iA'(x)(\rho_1\psi_2 - \rho_2\psi_1 + \rho_1\rho_2 F) + A''(x)\rho_1\rho_2\psi_1\psi_2. \tag{1.51}$$

with the Grassmann extension given by, for $k = 0, 1, \cdots$,

$$\partial_x^k A(x) = \sum_{\ell=0}^{\infty} \frac{1}{\ell!} A^{(k+\ell)}(x) x^\ell \quad \text{where} \quad x = x + x \in \mathfrak{R}_{\text{ev}}, \quad x = q \in \mathbb{R}.$$

$W(\Phi)$ is analogously extended from $W(q)$ whose relation to $V(q)$ will be given later.

Remark. The following relation will be worth noticing:

$$\left(\frac{\partial}{\partial \rho_\alpha} - i\rho_\alpha \frac{\partial}{\partial t}\right)^2 = -i\frac{\partial}{\partial t} \quad \text{for each } \alpha = 1, 2.$$

Now, we have

$$\begin{aligned} L_0' &:= \int d\rho_2 d\rho_1\, \tilde{L}_0 \\ &= \frac{1}{2}\dot{x}^2 + A(x)\dot{x} + \frac{1}{2}F^2 + \frac{i}{2}\left(\psi_2\dot{\psi}_2 - \dot{\psi}_1\psi_1\right) + W'(x)F - iW''(x)\psi_1\psi_2. \end{aligned} \tag{1.52}$$

Assuming that the "auxilliary field F" should satisfy

$$0 = \frac{\delta L_0'}{\delta F} = F + W', \tag{1.53}$$

we arrived at

$$L_0 = \frac{1}{2}\dot{x}^2 + A(x)\dot{x} + \frac{i}{2}\left(\psi_2\dot{\psi}_2 - \dot{\psi}_1\psi_1\right) - \frac{1}{2}W'(x)^2 - iW''(x)\psi_1\psi_2. \tag{1.54}$$

This is the desired Lagrangian with variables $x, \dot{x}, \psi_\alpha, \dot{\psi}_\alpha$, but variables $\psi_\alpha, \dot{\psi}_\alpha$ are not independent each other. In fact, they satisfy

$$\{\psi_\alpha, \psi_\beta\} = \psi_\alpha\psi_\beta + \psi_\beta\psi_\alpha = 0, \quad \{\psi_\alpha, \dot{\psi}_\beta\} = 0 \quad \text{and} \quad \{\dot{\psi}_\alpha, \dot{\psi}_\beta\} = 0.$$

To find out "independent Grassmann variables" in (3.10), we introduce new variables by the following two methods:

(I) Defining new variables as

$$\begin{cases} \xi = \dfrac{\delta L_0}{\delta \dot{x}} = \dot{x} + ax, \\ \phi_\alpha = \dfrac{\delta L_0}{\delta \dot{\psi}_\alpha} = -\dfrac{i}{2}\psi_\alpha \quad \text{for} \quad \alpha = 1, 2, \end{cases} \tag{1.55}$$

we put

$$\begin{aligned} H(x, \xi, \psi_1, \psi_2) &= \dot{x}\xi + \dot{\psi}_\alpha\phi_\alpha - L_0 \\ &= \frac{1}{2}(\xi - ax)^2 + \frac{1}{2}W'(x)^2 + \frac{i}{2}W''(x)\psi_\alpha\epsilon_{\alpha\beta}\psi_\beta. \end{aligned}$$

Rewriting the variables ψ_1, ψ_2 as θ, π, respectively, we get

$$H(x,\xi,\theta,\pi) = \frac{1}{2}(\xi - ax)^2 + \frac{1}{2}W'(x)^2 + iW''(x)\theta\pi. \tag{1.56}$$

(II) In the above, we use the "real" odd variables ψ_α. We "complexify" these variables by putting

$$\psi = \frac{1}{\sqrt{2}}(\psi_1 + i\psi_2), \quad \bar\psi = \frac{1}{\sqrt{2}}(\psi_1 - i\psi_2), \quad i.e. \quad \psi_1 = \frac{1}{\sqrt{2}}(\psi + \bar\psi), \quad \psi_2 = \frac{1}{\sqrt{2}i}(\psi - \bar\psi), \tag{1.57}$$

and then we rewrite L_0 as

$$\bar L_0 = \frac{1}{2}\dot x^2 + A(x)\dot x + \frac{i}{2}(\psi\dot{\bar\psi} + \bar\psi\dot\psi) - \frac{1}{2}W'(x)^2 - W''(x)\bar\psi\psi. \tag{1.58}$$

Introducing new variables as

$$\xi = \frac{\delta \bar L_0}{\delta \dot x} = \dot x + A(x), \quad \phi = \frac{\delta \bar L_0}{\delta \dot\psi} = -\frac{i}{2}\bar\psi, \quad \bar\phi = \frac{\delta \bar L_0}{\delta \dot{\bar\psi}} = -\frac{i}{2}\psi, \tag{1.59}$$

we put

$$\begin{aligned} H(x,\xi,\psi,\bar\psi) &:= \dot x\xi + \dot\psi\phi + \dot{\bar\psi}\bar\phi - \bar L_0 \\ &= \frac{1}{2}(\xi - A(x))^2 + \frac{1}{2}W'(x)^2 + W''(x)\bar\psi\psi. \end{aligned}$$

Rewriting ψ and $\bar\psi$ by θ and π, respectively, we get finally a function

$$H(x,\xi,\theta,\pi) = \frac{1}{2}(\xi - A(x))^2 + \frac{1}{2}W'(x)^2 - W''(x)\theta\pi \in C_{SS}(\mathfrak{R}^{2|2} : \mathfrak{R}_{\mathrm{ev}}). \tag{1.60}$$

Here, $(x,\theta) \in \mathfrak{R}^{1|1}$, $(\xi,\pi) \in \mathfrak{R}^{1|1}$.

Remarks. (0) The difference between (3.12) and (3.16) is the existence of i in front of the term $W''(x)\theta\pi$. This difference is rather significant when we consider Witten index for supersymmetric quantum mechanics using the kernel representation of the corresponding evolution operator.
(1) As there is no preference at this stage to take π and θ instead of θ and π, there is no significance of the sign $\pm$ in front of the terms $iW''(x)\theta\pi$ in (3.12) or $W''(x)\theta\pi$ in (3.16) in these cases.
(2) We may regard $H(x,\xi,\theta,\pi)$ as a Hamiltonian in $C_{SS}(T^*\mathfrak{R}^{1|1} : \mathfrak{R}_{\mathrm{ev}})$.
(3) These Hamiltonians (3.12) and (3.16) are called supersymmetric extensions of (3.1) because they give supersymmetric quantum mechanics after quantization (see §4). The procedure above is author's unmatured understanding of amalgam of physics papers such as Cooper and Freedman [CF], Davis, Macfarlane, Popat and van Holten [DMPvH] etc. But supersymmetry in superspace $\mathfrak{R}^{m|n}$ will be studied separately.
(4) On the other hand, using the identification (1.13), we have

$$H^\hbar_\pm(x,\partial_x,\theta,\partial_\theta) = \#\begin{pmatrix} H^\hbar_\pm - \frac{\hbar}{2}b & 0 \\ 0 & H^\hbar_\pm + \frac{\hbar}{2}b \end{pmatrix}\flat = \#\mathbb{H}^{\hbar,b}_\pm\flat.$$

Moreover, in this case, the "complete Weyl symbol of the above $H^\hbar(x,\partial_x,\theta,\partial_\theta)$" is calculated by

$$\mathcal{H}_\pm(x,\xi,\theta,\pi) = (e^{-i\hbar^{-1}(x\xi+\theta\pi)}H^\hbar_\pm(x,\partial_x,\theta,\partial_\theta)e^{i\hbar^{-1}(x\xi+\theta\pi)}\Big|_{\hbar=0} = \frac{1}{2}(\xi - ax)^2 \pm \frac{1}{2}b^2x^2 + Ib\theta\pi. \tag{1.61}$$

$\mathcal{H}_+$ equals to (3.12) when $A(q) = aq$ and $W(x) = \frac{1}{2}bx^2$, and $\mathcal{H}_-$ is obtained from (3.16) with $A(q) = aq$ and $W(x) = -\frac{i}{2}bx^2$. These give the relation between $W(q)$ and $V(q)$. (See SUSY Q.M. defined in §4.)
(5) Witten [W1] considered as a quantum mechanical operator

$$\mathbb{H}(q,\partial_q) = \left(-\frac{1}{2}\partial_q^2 + v(q)\right)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - \frac{1}{2}w(q)\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}. \tag{1.62}$$

This operator is supersymmetric when there exists a function $\psi(q)$ such that

$$v(q) = \frac{1}{2}\psi'(q)^2, \quad w(q) = \psi''(q).$$

1.2 Random Matrix Theory

(II-b) has rather recent origin, that is in 1983, Efetov [69] published a paper entitled "Supersymmetry and theory of disordered metals". Moreover, there are other ramifications related to quantum colour dynamics. By the limitation of author's ability, we take an example which gives a precise remainder estimate (which is also obtaind by the ordinary method) concerning the Wigner's semi-circle law.

1.2.1 Wigner's semi-circle law

Let $\mathfrak{H}_N$ be a set of Hermitian $N \times N$ matrices, which is identified with $\mathbb{R}^{N^2}$ as a topological space. In this set, we introduce a probability measure $d\mu_N(H)$ on $\mathfrak{H}_N$ by

$$\begin{gathered} d\mu_N(H) = \prod_{k=1}^N d(\Re H_{kk}) \prod_{j<k}^N d(\Re H_{jk}) d(\Im H_{jk}) P_{N,J}(H), \\ P_{N,J}(H) = Z_{N,J}^{-1} \exp\big[-\frac{N}{2J^2} \operatorname{tr} H^* H\big] \end{gathered} \tag{1.63}$$

where $H = (H_{jk})$, $H^* = (H^*_{jk}) = (\overline{H}_{kj}) = {}^t\overline{H}$, $\prod_{k=1}^N d(\Re H_{kk}) \prod_{j<k}^N d(\Re H_{jk}) d(\Im H_{jk})$ being the Lebesgue measure on $\mathbb{R}^{N^2}$, and $Z_{N,J}^{-1}$ is the normalizing constant given by $Z_{N,J} = 2^{N/2}(J^2\pi/N)^{3N/2}$.

Let $E_\alpha = E_\alpha(H)$ $(\alpha = 1, \cdots, N)$ be real eigenvalues of $H \in \mathfrak{H}_N$.

We put

$$\rho_N(\lambda) = \rho_N(\lambda; H) = N^{-1} \sum_{\alpha=1}^N \delta(\lambda - E_\alpha(H)), \tag{1.64}$$

where δ is the Dirac's delta. Denoting

$$\big\langle f \big\rangle_N = \big\langle f(\cdot) \big\rangle_N = \int_{\mathfrak{H}_N} d\mu_N(H)\, f(H),$$

for a function f on $\mathfrak{H}_N$, we get

Theorem 1.2.1 (Wigner's semi-circle law)

$$\lim_{N\to\infty} \big\langle \rho_N(\lambda) \big\rangle_N = w_{sc}(\lambda) = \begin{cases} (2\pi J^2)^{-1}\sqrt{4J^2 - \lambda^2} & \text{for } |\lambda| < 2J, \\ 0 & \text{for } |\lambda| > 2J. \end{cases} \tag{1.65}$$

Seemingly, there exist several methods to prove this fact. Here, we want to explain a new derivation of this fact using odd variables obtained by Efetov [**?**].

(A) One of the key expression obtained by introducing new auxiliary variables, is

$$\big\langle \rho_N(\lambda) \big\rangle_N = \pi^{-1}\Im \int_{\mathfrak{Q}} dQ \left(\{(\lambda - i0)I_2 - Q\}^{-1}\right)_{bb} \exp\left[-N\mathcal{L}(Q)\right] \tag{1.66}$$

where I_n stands for $n \times n$-identity matrix and

$$\begin{gathered} \mathcal{L}(Q) = \operatorname{str}\left[(2J^2)^{-1}Q^2 + \log((\lambda - i0)I_2 - Q)\right], \\ \mathfrak{Q} = \big\{Q = \begin{pmatrix} x_1 & \rho_1 \\ \rho_2 & ix_2 \end{pmatrix} \bigm| x_1, x_2 \in \mathfrak{R}_{\rm ev},\ \rho_1, \rho_2 \in \mathfrak{R}_{\rm od}\big\} \cong \mathfrak{R}^{2|2}, \quad dQ = \frac{dx_1 dx_2}{2\pi} d\rho_1 d\rho_2, \\ \left(((\lambda - i0)I_2 - Q)^{-1}\right)_{bb} = \frac{(\lambda - i0 - x_1)(\lambda - i0 - ix_2) + \rho_1\rho_2}{(\lambda - i0 - x_1)^2(\lambda - i0 - ix_2)}. \end{gathered} \tag{1.67}$$

Here in (1.66), **the parameter N appears only in one place**. This formula is formidably charming but **not yet directly justified**, like Feynman's expression of certain quantum objects using his measure.

(B) In physics literatures, for example in [**?**], [**?**], they claim without proof that they may apply the method of steepest descent to (1.66) when $N \to \infty$. More precisely, as

$$\delta\mathcal{L}(Q)\tilde{Q} = \frac{d}{d\epsilon}\mathcal{L}(Q+\epsilon\tilde{Q})\Big|_{\epsilon=0},$$

they seek solutions of

$$\delta\mathcal{L}(Q) = \mathrm{str}\,\big(\frac{Q}{J^2} - \frac{1}{\lambda - Q}\big) = 0.$$

As a candidate of effective saddle points, they take

$$Q_c = (\frac{1}{2}\lambda + \frac{1}{2}\sqrt{\lambda^2 - 4J^2})I_2,$$

and they have

$$\lim_{N\to\infty}\big\langle \rho_N(\lambda)\big\rangle_N = \pi^{-1}\Im(\lambda - Q_c)^{-1}_{bb} = w_{sc}(\lambda). \qquad \square$$

Remark. Not only the expression (1.66) nor the applicability of the saddle point method to it are not so clear. To get the mathematical rigour, we **dare to loose such a beautiful expression** like (1.66). To prove that such expression holds for real λ, I conjecture to develop new integration theorem such as Henstock-Kurzweil's integral.

1.2.2 Relation between RMT and Painlevé transcendents

It is known rather recently that there is a mysterious connection between RMT and Painlevé functions.

Let $\mathfrak{U}_n$ be a set of unitary $n \times n$ matrices with Haar measure. Then, the characteristic function of the randomvariable tr U for $U \in \mathfrak{U}_n$ is

$$E_n(e^{t(\mathrm{tr}\ U + \mathrm{tr}\ \bar{U})}$$

For any g with Fourier coefficients $\{g_l\}$, putting eigenvalues of $U \in \mathfrak{U}_n$ as $\{e^{i\lambda_j}\}_{j=1}^n$, we have

$$E_n(\prod_{j=1}^N g(e^{i\lambda_j})) = \det T_n(g)$$

where

$$T_n(g) = (g_{j-k})_{j,k=0,\cdots,n-1} \quad \text{with} \quad g_l = \frac{1}{2\pi}\int_{S^1} e^{-il\phi}g(\phi)d\phi.$$

In this case, for $g(z,t) = e^{t(z+z^{-1})}$

$$D_n(t) = \det T_n(g(\cdot,t)) = E_n(e^{t(\mathrm{tr}\ U + \mathrm{tr}\ \bar{U})}.$$

Now put

$$U_n(t) = (T_n(g(\cdot,t))^{-1}f^+, \delta^-) = (T_n(g(\cdot,t))^{-1}f^-, \delta^+), \quad \Phi_n(t) = 1 - U_n(t)^2$$

where

$$\delta^+ = \begin{pmatrix}1\\0\\\vdots\\0\\0\end{pmatrix}, \quad \delta^+ = \begin{pmatrix}0\\0\\\vdots\\0\\1\end{pmatrix}, \quad f^+ = \begin{pmatrix}f_1\\f_2\\\vdots\\f_{n-1}\\f_n\end{pmatrix}, \quad f^+ = \begin{pmatrix}f_n\\f_{n-1}\\\vdots\\f_2\\f_1\end{pmatrix}.$$

Then, Φ_n satisfies a variant of the Painlevé V equation:

$$\Phi_n'' =$$

1.2.3 Matytsin's procedure

(1) Putting

$$I(X_N, Y_N; t) = \frac{1}{t^{N^2/2}} \int_{U(N)} dU \ \exp\Big(-\frac{N}{2t} \operatorname{tr} (X_N - UY_N U^\dagger)^2 \Big), \tag{1.68}$$

we have readily

$$I(X_N, Y_N; t) = \exp\Big(-\frac{N}{2t} \operatorname{tr} (X_N^2 + Y_N^2) \Big) I\Big(\frac{X_N}{\sqrt{t}}, \frac{Y_N}{\sqrt{t}} \Big). \tag{1.69}$$

Since we have

$$2N \frac{\partial I(X_N, Y_N; t)}{\partial t} = \frac{1}{\Delta(X_N)} \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} [\Delta(X_N) I(X_N, Y_N; t)], \tag{1.70}$$

putting

$$\tilde{I}(X_N, Y_N; t) = \Delta(X_N) I(X_N, Y_N; t), \tag{1.71}$$

we get

$$2N \frac{\partial \tilde{I}(X_N, Y_N; t)}{\partial t} = \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} [\tilde{I}(X_N, Y_N; t)]. \tag{1.72}$$

This yields

$$\tilde{I}(X_N, Y_N; t) = \frac{1}{((t-s)/N)^{N/2}} \int_{\mathbb{R}^N} dx \exp \Big[-\frac{(x_1 - z_1)^2 + (x_2 - z_2)^2 + \cdots + (x_N - z_n)^2}{2t/N} \Big] \tilde{I}(Z_N, Y_N; s), \tag{1.73}$$

Introducing the Ansatz

$$I(X, Y; t) = \exp\ (N^2 W(\rho, \sigma, t))$$

we get

$$2\frac{\partial W}{\partial t} = \frac{1}{N} \sum_{j=1}^N \frac{\partial^2 W}{\partial x_j^2} + N \sum_{j=1}^N \Big(\frac{\partial W}{\partial x_j} \Big) + 2 \sum_{j=1}^N V(x_j) \frac{\partial W}{\partial x_j},$$

where

$$V(x_j) = \frac{1}{N} \sum_{k \neq j} \frac{1}{x_j - x_k}.$$

Assuming that

$$\lim_{N \to \infty} W(\rho, \sigma; t) = \tilde{W}(\rho, \sigma; t),$$

we have

$$\frac{\partial W}{\partial x_j} = \frac{1}{N} \Big(\frac{\partial}{\partial x} \frac{\delta \tilde{W}}{\delta \rho(x)} bigg) \Big|_{x = x_j}$$

$$2\frac{\partial W}{\partial t} = N \sum_{j=1}^N \Big(\frac{\partial W}{\partial x_j} \Big) + 2 \sum_{j=1}^N V(x_j) \frac{\partial W}{\partial x_j},$$

$$W = S - \frac{1}{2N^2} \sum_{j \neq k} \log |x_j - x_k| - \frac{1}{2N^2} \sum_{j \neq k} \log |y_j - y_k|$$

$$2\frac{\partial S}{\partial t} = N \sum_{j=1}^N \Big(\frac{\partial S}{\partial x_j} \Big) - \frac{1}{N} \sum_{j=1}^N V^2(x_j),$$

$$\frac{1}{N} \sum_{j=1}^N V^2(x_j) = \frac{1}{N^3} \sum_{j \neq k} \frac{1}{(x_j - x_k)^2}$$

Part I

Elementary Analysis on superspaces

Chapter 2

Superspaces based on the Fréchet-Grassmann algebras

2.1 The ∞-dimensional Fréchet-Grassmann algebras

2.1.1 The Grassmann generators

We prepare[1] a set of countably infinite distinct letters $\{\sigma_j\}_{j\in\mathbb{N}}$ satisfying the Grassmann relation,

$$\sigma_i\sigma_j + \sigma_j\sigma_i = 0 \quad \text{for any} \quad i,j = 1,2,\cdots. \tag{2.1}$$

We consider following sets rather formally (but soon later "proved as meaningfull"):

$$\mathfrak{C} = \left\{ X = \sum_{I\in\mathcal{I}} X_I\sigma^I \mid X_I \in \mathbb{C} \right\}, \tag{2.2}$$

$$\begin{cases} \mathfrak{C}_{(0)} = \mathfrak{C}_{[0]} = \mathbb{C}, \\ \mathfrak{C}_{(j)} = \left\{ X = \sum_{|I|\le j} X_I\sigma^I \right\} \quad \text{and} \\ \mathfrak{C}_{[j]} = \left\{ X = \sum_{|I|=j} X_I\sigma^I \right\} = \mathfrak{C}_{(j)}/\mathfrak{C}_{(j-1)}, \end{cases} \tag{2.3}$$

where

$$\begin{gathered} \mathcal{I} = \{I = (i_1, i_2, \cdots, i_k, \cdots) \in \{0,1\}^{\mathbb{N}} \mid |I| = \sum_k i_k < \infty\}, \\ \sigma^I = \sigma_1^{i_1}\sigma_2^{i_2}\cdots \quad \text{with} \quad \sigma^{\tilde{0}} = 1, \quad \tilde{0} = (0,0,\cdots). \end{gathered} \tag{2.4}$$

Remark 2.1.1 *If you are unfamiliar with the formal introduction of 'letter' $\{\sigma_j\}_{j\in\mathbb{N}}$ with multiplication satisfying* (2.1)*, you may regard, for a while, σ_j as dz_j (the differential of z_j) with the exterior product $\wedge$ as the multiplication where $z = (z_1, z_2, \cdots) \in \prod_{j=1}^{\infty}\mathbb{R} = \mathbb{R}^\infty$. An explicit construction of $\{\sigma_j\}_{j\in\mathbb{N}}$ is given by Rogers in §2.1.4. Berezin [**?**] gave another realization of it as operators in the Fock space. See, for more algebraic treatment, Kostant & Sternberg [135].*

[1] that is, we may find or may construct such letters

Sequence spaces and their topologies: To give the concrete meaning of the above summation expressions in (2.2) and (2.3), we recall the sequence spaces ω and ϕ in the terminology of Köthe [136]. That is, we define

$$\begin{cases} \phi = \{\mathfrak{x} = (x_k) = (x_1, x_2, \cdots, x_k, \cdots) \mid x_k \in \mathbb{C} \text{ and } x_k = 0 \text{ except for finitely many } k\}, \\ \omega = \{\mathfrak{u} = (u_k) = (u_1, u_2, \cdots, u_k, \cdots) \mid u_k \in \mathbb{C}\}. \end{cases} \tag{2.5}$$

For any sequence space $\mathcal{X}$ containing ϕ, we define the space $\mathcal{X}^\times$ by

$$\mathcal{X}^\times = \left\{\mathfrak{u} = (u_k) \mid \sum_k |u_k||x_k| < \infty \quad \text{for any} \quad \mathfrak{x} = (x_k) \in \mathcal{X}\right\},$$

then, we get

$$\phi^\times = \omega \quad \text{and} \quad \omega^\times = \phi.$$

We introduce the (normal) topology in $\mathcal{X}$ and $\mathcal{X}^\times$ by defining the seminorms

$$p_{\mathfrak{u}}(\mathfrak{x}) = \sum_k |u_k||x_k| = p_{\mathfrak{x}}(\mathfrak{u}) \quad \text{for } \mathfrak{x} \in \mathcal{X} \text{ and } \mathfrak{u} \in \mathcal{X}^\times. \tag{2.6}$$

Especially, $\mathfrak{x}^{(n)}$ converges to $\mathfrak{x}$ in ϕ, that is, $p_{\mathfrak{u}}(\mathfrak{x}^{(n)} - \mathfrak{x}) \to 0$ as $n \to \infty$ for each $\mathfrak{u} \in \omega$ if and only if for any $\epsilon > 0$, there exist L and n_0 such that

$$\begin{cases} \text{(i)} \quad x_k^{(n)} = x_k = 0 \quad \text{for} \quad k > L \quad \text{when} \quad n \geq n_0, \quad \text{and} \\ \text{(ii)} \quad |x_k^{(n)} - x_k| < \epsilon \quad \text{for} \quad k \leq L \quad \text{when} \quad n \geq n_0. \end{cases} \tag{2.7}$$

Analogously, $\mathfrak{u}^{(n)}$ converges to $\mathfrak{u}$ in ω, that is, $p_{\mathfrak{x}}(\mathfrak{u}^{(n)} - \mathfrak{u}) \to 0$ as $n \to \infty$ for each $\mathfrak{x} \in \phi$ if and only if for any $\epsilon > 0$ and each k, there exists $n_0 = n_0(\epsilon, k)$ such that

$$|u_k^{(n)} - u_k| < \epsilon \quad \text{when} \quad n \geq n_0. \tag{2.8}$$

Clearly, ω forms a Fréchet space because the above topology in ω is equivalent to the one defined by countable seminorms: $\{p_k(\mathfrak{u})\}_{k\in\mathbb{N}}$ where $p_k(\mathfrak{u}) = |u_k|$ for $\mathfrak{u} = (u_1, u_2, \cdots) = \sum_{j=1}^\infty u_j \mathfrak{e}_j \in \omega$ with $\mathfrak{e}_j = (\overbrace{0, \cdots, 0, 1}^{j}, 0, \cdots) \in \omega$.

Now, we define the isomorphism from $\mathcal{I}$ onto $\mathbb{N}$ (diadic-decomposition) defined by

$$r : \mathcal{I} \ni I = (i_k) \to r(I) = 1 + \frac{1}{2}\sum_{k=1}^\infty 2^k i_k \in \mathbb{N} \quad \text{where} \quad i_k = 0 \text{ or } 1. \tag{2.9}$$

Using $r(I)$ in (2.9), we define a map

$$T: \ \sigma^I \to \mathfrak{e}_{r(I)} \quad \text{for} \quad I = (i_k) \in \mathcal{I}.$$

Extending this linearly, we put

$$T(X) = \sum x_{r(I)}\mathfrak{e}_{r(I)} \in \omega \quad \text{for} \quad X = \sum_{|I|\leq j} X_I \sigma^I \in \mathfrak{C}_{(j)}. \tag{2.10}$$

More explicitely, we have the following first few terms:

$$\sum x_{r(I)}\mathfrak{e}_{r(I)} = (X_{(0,0,0,\cdots)}, X_{(1,0,0,\cdots)}, X_{(0,1,0,\cdots)}, X_{(1,1,0,\cdots)}, X_{(0,0,1,\cdots)}, X_{(1,0,1,\cdots)}, X_{(0,1,1,\cdots)}, \cdots).$$

Then, we have

$$\bigcup_{j=0}^{\infty} T(\mathfrak{C}_{(j)}) = \sum_{j=0}^{\infty} T(\mathfrak{C}_{[j]}) = \omega \tag{2.11}$$

because $T(\mathfrak{C}_{[j]})$ and $T(\mathfrak{C}_{[k]})$ are disjoint sets in ω if $j \neq k$. Therefore, it is reasonable to write as in (2.2) and more precisely,

$$\mathfrak{C} = \oplus_{j=0}^{\infty} \mathfrak{C}_{[j]}, \quad \text{that is,} \quad X = \sum_{j=0}^{\infty} X_{[j]} \quad \text{with} \quad X_{[j]} = \sum_{|I|=j} X_I \sigma^I. \tag{2.12}$$

Here, $X_{[j]}$ is called the j-th degree component of $X \in \mathfrak{C}$. We have just given the meaning of the summations in (2.2) and (2.3) by using the summation in ω. By definition, we get

$$\begin{cases} \mathfrak{C}_{(j)} \subset \mathfrak{C}_{(k)} \quad \text{for} \quad j \leq k, \\ \mathfrak{C} = \cup_{j=0}^{\infty} \mathfrak{C}_{(j)} \quad \text{with} \quad \cap_{j=0}^{\infty} \mathfrak{C}_{(j)} = \mathbb{C}, \end{cases} \tag{2.13}$$

$$\mathfrak{C}_{[j]} \cdot \mathfrak{C}_{[k]} \subset \mathfrak{C}_{[j+k]} \quad \text{and} \quad \mathfrak{C}_{(j)} \cdot \mathfrak{C}_{(k)} \subset \mathfrak{C}_{(j+k)}. \tag{2.14}$$

Remark 2.1.2 *The second relation with $\mathfrak{C}_{(*)}$ in* (2.14) *also holds for the Clifford algebras but the first one with $\mathfrak{C}_{[\cdot]}$ is specific to the Grassmann algebras satisfying* (2.1)*. Here, the Clifford relation for $\{e_j\}$ is defined by*

$$e_i e_j + e_j e_i = 2\delta_{ij}\mathbb{I} \quad \textit{for any} \quad i, j = 1, 2, \cdots. \tag{2.15}$$

Typical examples are the 2×2-Pauli matrices $e_j = \{\boldsymbol{\sigma}_j\}_{j=1,2,3}$ and the 4×4-Dirac matrices $\{e_j\}_{j=0,1,2,3} = \{\boldsymbol{\beta}, \boldsymbol{\alpha}_j\}$.

2.1.2 Topology

We introduce the weakest topology in $\mathfrak{C}$ which makes the map T continuous from $\mathfrak{C}$ to ω, that is, $X = \sum_{I\in\mathcal{I}} X_I \sigma^I \to 0$ in $\mathfrak{C}$ if and only if $\mathrm{proj}_I(X) \to 0$ for each $I \in \mathcal{I}$ with $\mathrm{proj}_I(X) = X_I$; it is equivalent to the metric $\mathrm{dist}(X, Y) = \mathrm{dist}(X - Y)$ defined by

$$\mathrm{dist}(X) = \sum_{I\in\mathcal{I}} \frac{1}{2^{r(I)}} \frac{|\,\mathrm{proj}_I(X)|}{1 + |\,\mathrm{proj}_I(X)|} \quad \text{for any} \quad X \in \mathfrak{C}. \tag{2.16}$$

For example, $X^{(\ell)} = f(\ell)\sigma_1 \cdots \sigma_\ell \to 0$ in $\mathfrak{C}$ even if $f(\ell) \to \infty$ because $\mathrm{dist}(X^{(\ell)}) \leq 2^{-2^\ell+1}$.

2.1.3 Algebraic operations

For any $X, Y \in \mathfrak{C}$, we define

$$X + Y = \sum_{j=0}^{\infty} (X + Y)_{[j]} \quad \text{with} \quad (X + Y)_{[j]} = X_{[j]} + Y_{[j]} \quad \text{for} \quad j \geq 0 \tag{2.17}$$

and

$$XY = \sum_{j=0}^{\infty} (XY)_{[j]} \quad \text{where} \quad (XY)_{[j]} = \sum_{k=0}^{j} X_{[j-k]} Y_{[k]} = \sum_{|I|=j} (XY)_I \sigma^I. \tag{2.18}$$

Here, $(XY)_I = \sum_{I=J+K} (-1)^{\tau(I;J,K)} X_J Y_K \in \mathbb{C}$ is well-defined because for any set $I \in \mathcal{I}$, there exist only finitely many decompositions by sets J, K satisfying $I = J \dot{+} K$ (*i.e.* $I = J \cup K,\ J \cap K = \emptyset$). Here, the indeces $\tau(I; J, K)$, or more generally $\tau(I; J_1, \cdots, J_k)$ are defined by

$$(-1)^{\tau(I;J_1,\cdots,J_k)} \sigma^{J_1} \cdots \sigma^{J_k} = \sigma^I \quad \text{with} \quad I = J_1 \dot{+} J_2 \dot{+} \cdots \dot{+} J_k. \tag{2.19}$$

But for notational simplicity, we will use $(-1)^{\tau(*)}$ without specifying the decomposition if there occurs no confusion.

Exercise 2.1.3 *For sets J, K satisfying $I = J + K$,*

$$(-1)^{|J||K|}(-1)^{\tau(I;J,K)} = (-1)^{\tau(I;K,J)}.$$

Moreover, we get

Lemma 2.1.4 *The product defined by* (2.18) *is continuous from $\mathfrak{C} \times \mathfrak{C} \to \mathfrak{C}$.*

Proof. It is simple by noting that there exist $2^{|I|}$ elements $J \in \mathcal{I}$ satisfying $J \subset I$ and that

$$|(XY)_I| \le \sum_{I=J+K} |X_J||Y_K| \le 2^{r(I)}(\max_{J\subset I}|X_J|)(\max_{K\subset I}|Y_K|) \quad \text{for any} \quad X, Y \in \mathfrak{C}. \qquad \square$$

To summarize, we get

Theorem 2.1.5 *$\mathfrak{C}$ forms a Fréchet-Grassmann algebra over $\mathbb{C}$, that is, an associative, distributive and non-commutative ring with degree, which is endowed with the Fréchet topology.*

Proof. Clearly, we get

$$\begin{cases} X(YZ) = (XY)Z & (\textit{associativity}), \\ X(Y+Z) = XY + XZ & (\textit{distributivity}). \end{cases}$$

Other properties have been proved. $\square$

Comment. We may introduce another set denoted by $\Lambda^{\mathbb{C}}$ as the projective limit of sets $\Lambda^{\mathbb{C}}_L$. Here, $\Lambda^{\mathbb{C}}_L$ is defined by

$$\begin{aligned} \Lambda^{\mathbb{C}}_L &= \Big\{X = \sum_{I\in\mathcal{I}_L} X_I\sigma^I \,|\, X_I \in \mathbb{C}\Big\} \quad \text{with} \quad \mathcal{I}_L = \{I = (i_1, i_2, \cdots, i_L, 0, \cdots)\} \subset \mathcal{I} \\ &\cong \Lambda(\mathbb{R}^L : \mathbb{C}) \\ &= \text{the exterior algebra of forms on } \mathbb{R}^L \text{ with coefficients in } \mathbb{C} \cong \mathbb{C}^{2^L}. \end{aligned} \tag{2.20}$$

In fact, for $M > L$, defining maps $\psi_{LM} : \mathcal{I}_M \to \mathcal{I}_L$ by $\psi_{LM}(\sum_{I\in\mathcal{I}_M} X_I\sigma^I) = \sum_{I\in\mathcal{I}_L} X_I\sigma^I$, we have the set $(\Lambda^{\mathbb{C}}_L, \psi_{LM})$ which forms a projective system and yields a projective limit $\Lambda^{\mathbb{C}}$. More precisely, the topology of $\Lambda^{\mathbb{C}}$ is defined as follows: Elements $X^{(n)}$ converges to X in $\Lambda^{\mathbb{C}}$ if and only if for any $\epsilon > 0$ and I, there exists an integer $n_0 = n_0(\epsilon, I)$ such that $|X^{(n)}_I - X_I| < \epsilon$ when $n > n_0$.

Remark 2.1.6 *We may consider that an element of $X \in \mathfrak{C}$ stands for the 'state' such that the position labeled by σ^I is occupied by $X_I \in \mathbb{C}$. In other word, considering $\{\sigma_i\}$ as the countable indeterminate letters, it seems reasonable to regard $\mathfrak{C}$ as the set of certain formal power series* [2] *with simple topology. Therefore, it is permitted to reorder the terms freely under 'summation sign'. That is, the summation $\sum_{I\in\mathcal{I}} X_I\mathfrak{e}_{r(I)}$ is 'unconditionally (though not absolutely) convergent'* [3] *and so is $\sum_{I\in\mathcal{I}} X_I\sigma^I$. We use such a big space $\mathfrak{C}$ with rather weak topology because this algebra is considered as the ambient space for reordering the places. We feel such a big ambient space will be prefarable and tractable for our future use.*

[2] with the special property that same letter appears only once in each monomials
[3] diverting the terminology of the basis problem in the Banach spaces

Remark 2.1.7 *(1) As* $\{\mathfrak{C}_{(j)}\}$ *forms a filter by* (2.13) *and* (2.14), *it gives a 0-neighbourhood base of the linear topology of* $\mathfrak{C}$ *which is equivalent to the above one defined by* (2.8). *(See [136] for the linear topology of vector spaces.)*
(2) We may introduce a stronger topology in $\mathfrak{C}$ *called the topology by degree, that is,* $X^{(n)} \xrightarrow{s} X$ *in* $\mathfrak{C}$ *means that*

(i) there exists $d \geq 0$ *such that* $X_I^{(n)} = X_I = 0$ *for any* n *and* I *when* $|I| > d$ *and*
(ii) $|X_I^{(n)} - X_I| \to 0$ *as* $n \to \infty$ *when* $|I| \leq d$.

2.1.4 Banach-Grassmann algebra

Let ℓ^1 be a Banach space of sequences $w = (w_1, w_2, \cdots)$ satisfying $w_j \in \mathbb{C}$ and $\|w\| = \sum_{j=1}^\infty |w_j| < \infty$. Denote by $\mathcal{M}_L$ the set of sequences given by

$$\mathcal{M}_L = \{\mu; \mu = (\mu_1, \mu_2, \cdots, \mu_k),\ 1 \leq \mu_1 < \mu_2 < \cdots < \mu_k \leq L\} \quad \text{and} \quad \mathcal{M}_\infty = \cup_{L=1}^\infty \mathcal{M}_L.$$

We regard $\emptyset \in \mathcal{M}_L$ and for any $j \in \mathbb{N}$, we put $(j) \in \mathcal{M}_\infty$. For each $r \in \mathbb{N}$, we may correspond a member $\mu \in \mathcal{M}_\infty$ by using

$$r = \frac{1}{2}(2^{\mu_1} + 2^{\mu_2} + \cdots + 2^{\mu_k}). \tag{2.21}$$

Conversely, for each $\mu \in \mathcal{M}_\infty$, we define e_μ as $e_\mu = (\overbrace{0, \cdots, 0, 1}^{r}, 0, \cdots)$ where r and μ are related by (2.21). Then, $w = \sum_\mu w_\mu e_\mu$. Now, we introduce the multiplication by

$$\begin{cases} e_\mu e_\emptyset = e_\emptyset e_\mu = e_\mu \quad \text{for} \quad \mu \in \mathcal{M}_\infty, \\ e_{(i)} e_{(j)} = -e_{(j)} e_{(i)} \quad \text{for} \quad i, j \in \mathbb{N}, \\ e_\mu = e_{(\mu_1)} e_{(\mu_2)} \cdots e_{(\mu_k)} \quad \text{where} \quad \mu = (\mu_1, \mu_2, \cdots, \mu_k). \end{cases} \tag{2.22}$$

That is, we identify

$$w = (w_1, w_2, w_3, w_4, \cdots) = \sum_{j=1} w_j e_{(j)} \longleftrightarrow (w_{(1)}, w_{(2)}, w_{(1,2)}, w_{(3)}, \cdots) = \sum_\mu w_\mu e_\mu$$

where

$$e_{(j)} \leftrightarrow \sigma_j,\ e_{(1)} e_{(2)} = e_{(1,2)} \leftrightarrow \sigma_1 \sigma_2 = \sigma^I,\ I_{(1,2)} = (1, 1, 0, \cdots),$$

$$e_\mu = e_{(\mu_1)} e_{(\mu_2)} \cdots e_{(\mu_k)} \leftrightarrow \sigma_{\mu_1} \sigma_{\mu_2} \cdots \sigma_{\mu_k} = \sigma^I,\ I_\mu = (\underbrace{\overbrace{0, \cdots, 0, 1}^{\mu_1}, 0, \cdots, 0, 1}_{\mu_k}, 0, \cdots).$$

Therefore, the real Banach-Grassmann algebra introduced by Rogers consists of the absolutely convergent sequence

$$\|X\| = \sum_{I \in \mathcal{I}} |X_I| < \infty \quad \text{for } X = \textstyle\sum_{I \in \mathcal{I}} X_I \sigma^I \text{ with } X_I \in \mathbb{R}, \text{ and it satisfies } \|XY\| \leq \|X\| \|Y\|.$$

Proposition 2.1.8 (Roger) ℓ^1 *with the above multiplication forms a Banach-Grassmann algebra with countably infinite generators.*

2.1.5 The supernumber

$\mathfrak{C}$ is called the *(complex) supernumber algebra* over $\mathbb{C}$ and any element X of $\mathfrak{C}$ is called *(complex) supernumber.*

We introduce the parity in $\mathfrak{C}$ by setting

$$p(X) = \begin{cases} 0 & \text{if } X = \sum_{I\in\mathcal{I},|I|=\text{even}} X_I\sigma^I, \\ 1 & \text{if } X = \sum_{I\in\mathcal{I},|I|=\text{odd}} X_I\sigma^I. \end{cases} \tag{2.23}$$

$X \in \mathfrak{C}$ is called homogeneous if it satisfies $p(X) = 0$ or $= 1$. We put also

$$\begin{cases} \mathfrak{C}_{\text{ev}} = \oplus_{j=0}^{\infty}\mathfrak{C}_{[2j]} = \{X \in \mathfrak{C} \mid p(X) = 0\}, \\ \mathfrak{C}_{\text{od}} = \oplus_{j=0}^{\infty}\mathfrak{C}_{[2j+1]} = \{X \in \mathfrak{C} \mid p(X) = 1\}, \\ \mathfrak{C} \cong \mathfrak{C}_{\text{ev}} \oplus \mathfrak{C}_{\text{od}} \cong \mathfrak{C}_{\text{ev}} \times \mathfrak{C}_{\text{od}}. \end{cases} \tag{2.24}$$

Moreover, it splits into its even and odd parts, called *(complex) even number* and *(complex) odd number*, respectively :

$$X = X_{\text{ev}} + X_{\text{od}} = \sum_{|a|=\text{even}} X_a\sigma^a + \sum_{|a|=\text{odd}} X_a\sigma^a = \sum_{j=\text{even}} X_{[j]} + \sum_{j=\text{odd}} X_{[j]}. \tag{2.25}$$

Using (2.25), we decompose

$$X = X_{\text{B}} + X_{\text{S}} \qquad \text{where} \quad X_{\text{S}} = \sum_{1\le j<\infty} X_{[j]} \quad \text{and} \quad X_{\text{B}} = X_{\tilde{0}} = X_{[0]} \tag{2.26}$$

and the number X_{B} is called *the body (part)* of X and the remainder X_{S} is called *the soul (part)* of X, respectively. We define the map π_{B} from $\mathfrak{C}$ to $\mathbb{C}$ by $\pi_{\text{B}}(X) = X_{\text{B}}$, called the body projection (or called the augmentation map in [184]).

Important Remark. $\mathfrak{C}$ does not form a field because $X^2 = 0$ for any $X \in \mathfrak{C}_{\text{od}}$. But, it is easily proved that

(i) if X satisfies $XY = 0$ for any $Y \in \mathfrak{C}_{\text{od}}$, then $X = 0$, and

(ii) the decomposition of X with respect to degree in (2.12) is unique.

These properties are shared only if the number of Grassmann generators is infinite. For example, if the number of Grassmann generators is finite, say n, then the number $\sigma_1\sigma_2\cdots\sigma_n$ is recognized 0 for the multiplication of any odd number.

Example (the invertible elements). Let $X \in \mathfrak{C}$ with $X_{\text{B}} \neq 0$. Then there exists a unique element $Y \in \mathfrak{C}$ such that $XY = 1 = YX$. In fact, decomposing $X = X_{\text{B}} + X_{\text{S}}$ and $Y = Y_{\text{B}} + Y_{\text{S}}$, we should have

$$X_{\text{B}}Y_{\text{B}} = 1, \quad X_{\text{B}}Y_{\text{S}} + X_{\text{S}}Y_{\text{B}} + X_{\text{S}}Y_{\text{S}} = 0.$$

Therefore, putting $X_{\text{S}} = \sum_{|I|>0} X_I\sigma^I$ and $Y_{\text{S}} = \sum_{|J|>0} Y_J\sigma^J$ and noting that $\sigma^I\sigma^J = (-1)^{\tau(K;I,J)}\sigma^K$ for $K = I + J$, we have

$$Y_{\text{B}} = X_{\text{B}}^{-1}, \quad Y_K = -X_{\text{B}}^{-1}\sum_{K=I+J}(-1)^{\tau(K;I,J)}X_IY_J.$$

For example,

$$\text{for } |K| = 1, \text{ then} \quad Y_K = -X_{\text{B}}^{-1}X_KY_{\text{B}}, \cdots,$$
$$\text{for } |K| = \ell, \text{ then} \quad Y_K = -X_{\text{B}}^{-1}\sum_{K=I+J}(-1)^{\tau(K;I,J)}X_IY_J.$$

If $X_{\rm B}=0$, there exists no Y satisfying $XY=1$ or $YX=1$.

Now, we define our *(real) supernumber algebra* over $\mathbb{R}$ (but not over $\mathbb{C}$) by

$$\mathfrak{R}=\pi_{\rm B}^{-1}(\mathbb{R})\cap\mathfrak{C}=\left\{X=\sum_{I\in\mathcal{I}}X_I\sigma^I \;\middle|\; X_{\rm B}\in\mathbb{R}\quad\text{and}\quad X_I\in\mathbb{C}\quad\text{for}\quad |I|\neq 0\right\}. \tag{2.27}$$

Defining as same as before, we have

$$\mathfrak{R}=\mathfrak{R}_{\rm ev}\oplus\mathfrak{R}_{\rm od},\quad \mathfrak{R}=\oplus_{j=0}^{\infty}\mathfrak{R}_{[j]}. \tag{2.28}$$

Analogous to $\mathfrak{C}$, we put

$$\begin{cases}\mathfrak{R}=\{X\in\mathfrak{C}\mid \pi_{\rm B}X\in\mathbb{R}\}, & \mathfrak{R}_{[j]}=\mathfrak{R}\cap\mathfrak{C}_{[j]},\\ \mathfrak{R}_{\rm ev}=\mathfrak{R}\cap\mathfrak{C}_{\rm ev}, \quad \mathfrak{R}_{\rm od}=\mathfrak{R}\cap\mathfrak{C}_{\rm od}=\mathfrak{C}_{\rm od}, \\ \mathfrak{R}\cong\mathfrak{R}_{\rm ev}\oplus\mathfrak{R}_{\rm od}\cong\mathfrak{R}_{\rm ev}\times\mathfrak{R}_{\rm od}.\end{cases} \tag{2.29}$$

Here, we introduced the body (projection) map $\pi_{\rm B}$ by $\pi_{\rm B}X=\text{proj}_{\rm B}(X)=X_{\tilde{0}}=X_{\rm B}$.

$\mathfrak{R}_{(j)}$ and other terminologies are analogously introduced.

2.1.6 Conjugation

We define the operation * as follows: Denoting the complex conjugation of X_I by $\overline{X_I}$ and defining $\overline{\sigma^I}=\sigma_n^{i_n}\cdots\sigma_1^{i_1}$ for $I=(i_1,\cdots,i_n)$, we put

$$X^*=\sum_{I\in\mathcal{I}}\overline{X_I}\,\overline{\sigma^I}=\sum_{I\in\mathcal{I}}(-1)^{\frac{|I|(|I|-1)}{2}}\overline{X_I}\sigma^I. \tag{2.30}$$

Then,

Lemma 2.1.9 *For $X,Y\in\mathfrak{C}$ and $\lambda\in\mathbb{C}$, we have*

$$(X^*)^*=X,\quad (XY)^*=Y^*X^*,\quad (\lambda X)^*=\bar{\lambda}X^*. \tag{2.31}$$

Exercise 2.1.10 *Prove $\overline{\sigma^I\sigma^J}=\overline{\sigma^J}\,\overline{\sigma^I}$. (Hint: Use* (1.46)*)*

%%%

$$\overline{\sigma^I\sigma^J}=\overline{(-1)^{\tau(K;I,J)}\sigma^K}=(-1)^{\tau(K;I,J)}(-1)^{\frac{|K|(|K|-1)}{2}}\sigma^K,$$
$$\overline{\sigma^J}\,\overline{\sigma^I}=(-1)^{\frac{|I|(|I|-1)}{2}}(-1)^{\frac{|J|(|J|-1)}{2}}\sigma^J\sigma^I=(-1)^{\frac{|I|(|I|-1)}{2}}(-1)^{\frac{|J|(|J|-1)}{2}}(-1)^{\tau(K;J,I)}\sigma^K,$$
$$(-1)^{|J||K|}(-1)^{\tau(I;J,K)}=(-1)^{\tau(I;K,J)}.$$

Therefore, we get the desired result. Onthe other hand, we prove also, if $K=I+J$,

$$(-1)^{\tau(K;I,J)}(-1)^{\frac{|K|(|K|-1)}{2}}=(-1)^{\frac{|I|(|I|-1)}{2}}(-1)^{\frac{|J|(|J|-1)}{2}}(-1)^{\tau(K;J,I)}.$$

%%%

Remark 2.1.11 *We may introduce "real" as $X^*=X$ for $X\in\mathfrak{C}$, or from purely aethetical point of view, the set of "reals" may be defined by*

$$\mathfrak{R}^{\mathbb{R}}=\{X=\sum_{I\in\mathcal{I}}X_I\sigma^I\mid X_I\in\mathbb{R}\},$$

but we don't use this "real" in the sequel. Because the analysis is really done for the body part and the soul part is used only for reordering the places, therefore, we imagine that the set

$$\mathcal{R}_K = \left\{ x = \sum_{I\in\mathcal{I}} x_I\sigma^I \;\middle|\; x_{\rm B}\in\mathbb{R} \quad and \quad x_I \in K \right\}$$

would be more natural as our 'supernumber algebra'. Here, K should be an associative algebra such that we may define seminorms analogously as before. This point of view will be discussed if necessity occurs.

Remark 2.1.12 *There is another way of defining the conjugation: We define $\bar\sigma_j$ as a linear mapping from $\mathfrak{C}$ to $\mathbb{C}$ such that $\langle\bar\sigma_j, \sigma_k\rangle = \delta_{jk}$, and by this, we may introduce the duality $\langle\cdot,\cdot\rangle$ between $\mathfrak{C}$ and $\bar{\mathfrak{C}}$ which is the Grassmann algebra generated by $\{\bar\sigma_j\}$, and whose Fréchet topology is compatible with the duality above. In this case, putting $\overline{\sigma^I} = \bar\sigma_n^{i_n}\cdots\bar\sigma_1^{i_1}$ for $I=(i_1,\cdots,i_n)$ and*

$$X^* = \sum_{I\in\mathcal{I}} \overline{X_I}\,\overline{\sigma^I} = \sum_{I\in\mathcal{I}} (-1)^{\frac{|I|(|I|-1)}{2}} \overline{X_I}\bar\sigma^I,$$

we have also (2.31).

Example(absolute value for $\xi\in\mathfrak{R}^m_{\rm ev}$). For $\xi=(\xi_1,\cdots,\xi_m)\in\mathfrak{R}^{m|0}=\mathfrak{R}^m_{\rm ev}$, we define the "absolute value" $|\xi|\in\mathfrak{R}_{\rm ev}$ as follows:

$$|\xi| = |\xi|_{\rm B} + |\xi|_{\rm S}, \;\; \text{where } |\xi|_{\rm S} = \sum_{|I|=\text{even}\geq 2} |\xi|_I\sigma^I, \quad |\xi|_{\rm B}\geq 0,\; |\xi|_I\in\mathbb{R},$$

and $|\xi|^2$ satisfies

$$|\xi|^2 = \sum_{j=1}^m (\xi_{j,\rm B}+\xi_{j,\rm S})(\xi_{j,\rm B}+\overline{\xi_{j,\rm S}}) = \sum_{j=1}^m \xi_{j,\rm B}^2 + \sum_{j=1}^m \xi_{j,\rm B}(\xi_{j,\rm S}+\overline{\xi_{j,\rm S}}) + \sum_{j=1}^m \xi_{j,\rm S}\overline{\xi_{j,\rm S}},$$
$$\xi_{j,\rm S} = \sum_{|I|=\text{even}\geq 2} \xi_{j,I}\sigma^I,\; \overline{\xi_{j,\rm S}} = \sum_{|I|=\text{even}\geq 2} \overline{\xi_{j,I}}\sigma^I.$$

(Here, $\overline{\xi_{j,I}}$ is the complex conjugate of $\xi_{j,I}$). Therefore,

$$|\xi|_{\rm B} = \{\sum_{j=1}^m \xi_{j,\rm B}^2\}^{1/2},$$
$$2|\xi|_K|\xi|_{\rm B} + \sum_{I+J=K} |\xi|_I|\xi|_J(-1)^{\tau(K;I,J)} = 2\sum_{j=1}^m \xi_{j,\rm B}(\Re\xi_{j,K}) + \sum_{I+J=K}\sum_{j=1}^m \xi_{j,I}\overline{\xi_{j,J}}(-1)^{\tau(K;I,J)}.$$

Each $|\xi|_K$ is defined by the induction of the length $|K|$ of $K\in\mathcal{I}$. For example, when $|K|=2$, we have

$$|\xi|_K = |\xi|_{\rm B}^{-1}\sum_{j=1}^m \xi_{j,\rm B}(\Re\xi_{j,K}).$$

Using this, when $|K|=4$, we have

$$2|\xi|_K = |\xi|_{\rm B}^{-1}\big(2\sum_{j=1}^m \xi_{j,\rm B}(\Re\xi_{j,K}) + \sum_{I+J=K}\sum_{j=1}^m \xi_{j,I}\overline{\xi_{j,J}}(-1)^{\tau(K;I,J)} - \sum_{I+J=K}\sum_{j=1}^m |\xi|_I|\xi|_J(-1)^{\tau(K;I,J)}\big),$$

etc.

2.2 The superspace

Definition 2.2.1 *The super Euclidean space or (real) superspace* $\mathfrak{R}^{m|n}$ *of dimension* $m|n$ *is defined by*

$$\begin{gathered}\mathfrak{R}^{m|n}=\mathfrak{R}^m_{\mathrm{ev}}\times\mathfrak{R}^n_{\mathrm{od}}\ni X={}^t({}^tx,{}^t\theta),\\ \textit{where}\quad x={}^t(x_1,\cdots,x_m)\quad\textit{and}\quad \theta={}^t(\theta_1,\cdots,\theta_n)\quad\textit{with}\quad x_j\in\mathfrak{R}_{\mathrm{ev}},\,\theta_s\in\mathfrak{R}_{\mathrm{od}}.\end{gathered}\tag{2.32}$$

Notation: In the following, we abbreviate the symbol 'transposed' ${}^t(x_1,\cdots,x_m)$ and denote $x=(x_1,\cdots,x_m)$, etc. unless there occurs confusion.

The topology of $\mathfrak{R}^{m|n}$ is induced from the metric defined by $\mathrm{dist}_{m|n}(X,Y)=\mathrm{dist}_{m|n}(X-Y)$ for $X,Y\in\mathfrak{R}^{m|n}$, where we put

$$\mathrm{dist}_{m|n}(X)=\sum_{j=1}^m\bigg(\sum_{I\in\mathcal{I}}\frac{1}{2^{r(I)}}\frac{|\operatorname{proj}_I(x_j)|}{1+|\operatorname{proj}_I(x_j)|}\bigg)+\sum_{s=1}^n\bigg(\sum_{I\in\mathcal{I}}\frac{1}{2^{r(I)}}\frac{|\operatorname{proj}_I(\theta_s)|}{1+|\operatorname{proj}_I(\theta_s)|}\bigg).\tag{2.33}$$

Clearly, $\mathrm{dist}_{1|1}(X)=\mathrm{dist}(X)$ for $X\in\mathfrak{R}^{1|1}\cong\mathfrak{R}\subset\mathfrak{C}$. Analogously, the complex superspace of dimension $m|n$ is defined by

$$\mathfrak{C}^{m|n}=\mathfrak{C}^m_{\mathrm{ev}}\times\mathfrak{C}^n_{\mathrm{od}}.\tag{2.34}$$

We generalize the body map π_{B} from $\mathfrak{R}^{m|n}$ or $\mathfrak{R}^{m|0}$ to $\mathbb{R}^m$ by $\pi_{\mathrm{B}}X=\pi_{\mathrm{B}}x=(\pi_{\mathrm{B}}x_1,\cdots,\pi_{\mathrm{B}}x_m)\in\mathbb{R}^m$ for $X=(x,\theta)\in\mathfrak{R}^{m|n}$. The (complex) superspace $\mathfrak{C}^{m|n}$ is defined analogously.

Dual superspace. We denote the superspace $\mathfrak{R}^{m|n}$ by $\mathfrak{R}^{m|n}_X$ whose point is presented by $X=(x,\theta)=(x_1,\cdots,x_m,\theta_1,\cdots,\theta_n)$. We prepare another superspace $\mathfrak{R}^{m|n}_\Xi$ whose point is denoted by $\Xi=(\xi,\pi)=(\xi_1,\cdots,\xi_m,\pi_1,\cdots,\pi_n)$, such that they are "dual" each other by

$$\langle X|\Xi\rangle_{m|n}=\sum_{j=1}^m\langle x_j|\xi_j\rangle+\sum_{k=1}^n\langle\theta_k|\pi_k\rangle\in\mathfrak{R}_{\mathrm{ev}}.\tag{2.35}$$

We abbreviate $\langle\cdot|\cdot\rangle_{m|n}$ above by $\langle\cdot|\cdot\rangle$ unless there occurs confusion.

Remark 2.2.2 *(0) Defining* $\mathfrak{R}$ *in* (2.27)*, we used both* $\mathbb{R}$ *and* $\mathbb{C}$*. The reason of this definition is explained in §4 where we solve a certain Hamiltonian equation stemming from the Pauli equation.*
(1) De Witt [60] introduces his space $\mathbb{R}^{m|n}_{dW}=(\Lambda^{\mathbb{R}}_{\mathrm{ev}})^m\times(\Lambda^{\mathbb{R}}_{\mathrm{od}})^n$*. Here,* $\Lambda^{\mathbb{R}}_{\mathrm{ev}}=\lim_{L\to\infty}\Lambda^{\mathbb{R}}_{\mathrm{ev}}(\mathbb{R}^L)$ *and* $\Lambda^{\mathbb{R}}_{\mathrm{ev}}(\mathbb{R}^L)$ *is isomorphic to the exterior algebra of even forms on* $\mathbb{R}^L$ *with real coefficients.* $\Lambda^{\mathbb{R}}_{\mathrm{od}}$ *and* $\Lambda^{\mathbb{R}}=\Lambda^{\mathbb{R}}_{\mathrm{ev}}+\Lambda^{\mathbb{R}}_{\mathrm{od}}$ *are 'defined' analogously. In the above, the meaning of '*$\lim_{L\to\infty}$*' is not so clear. And his topology in* $\mathbb{R}^{m|n}_{dW}$ *is the weakest topology which makes continuous the projection* π_{B} *from* $\mathbb{R}^{m|n}_{dW}$ *to* $\mathbb{R}^m$*. This does not give the Hausdorff topology in* $\mathbb{R}^{m|n}_{dW}$ *but he claims that non-Hausdorff property of his space is not serious in his analysis.*
(2) Rogers [184] defines her space $\mathbb{R}^{m|n}_R$ *based on the real Banach-Grassmann algebra* ℓ^1 *in order to develop her theory of superanalysis, using the known differential calculus for functions on Banach spaces. But we are not sure whether such a strong topology is really necessary. Or rather, we claim in the following that though generally speaking, the differential calculus on locally convex spaces are rather troublesome, see for example, Keller [129], Yamamuro [225], but we may carry out almost the same procedures as she done in [184] using the ring structure directly in our Fréchet-Grassmann algebra.*
(3) Matsumoto and Kakazu [158], Yagi [222] and Bryant [41], in order to refine the idea of De Witt, defined a Fréchet space which is the projective limit of the Banach space modelled on the exterior algebra of forms on $\mathbb{R}^L$ *with real coefficients, though the grading and the ring structure of it is obscured by their construction.*

(4) See also the papers, [188], Jadczyk and Pilch [122] and Hoyos et al. [100] who use the Banach-Grassmann algebra.

Chapter 3

Elements of the linear algebra on the superspace

3.1 Matrix algebras on the superspace

Definition 3.1.1 *A rectangular array M, whose cells are indexed by pairs consisting of a row number and a column number, is called a supermatrix and denoted by $M \in \mathrm{Mat}\,((m|n)\times(r|s):\mathfrak{C})$, if it satisfies the following:*

1. *A $(m+n)\times(r+s)$ matrix M is decomposed blockwisely as $M=\begin{pmatrix} A & C \\ D & B\end{pmatrix}$ where A, B, C and D are $m\times r$, $n\times s$, $m\times s$ and $n\times r$ matrices with elements in $\mathfrak{C}$, respectively.*
2. *One of the following conditions is satisfied: Either*
   - *$p(M)=0$, that is, $p(A_{jk})=0=p(B_{uv})$ and $p(C_{jv})=1=p(D_{uk})$ or*
   - *$p(M)=1$, that is, $p(A_{jk})=1=p(B_{uv})$ and $p(C_{jv})=0=p(D_{uk})$.*

We call M is even denoted by $\mathrm{Mat}_{\,\mathrm{ev}}((m|n)\times(r|s):\mathfrak{C})$ (resp. odd denoted by $\mathrm{Mat}_{\,\mathrm{od}}((m|n)\times(r|s):\mathfrak{C})$) if $p(M)=0$ (resp. $p(M)=1$). Therefore, we have

$$\mathrm{Mat}\,((m|n)\times(r|s):\mathfrak{C})=\mathrm{Mat}_{\,\mathrm{ev}}((m|n)\times(r|s):\mathfrak{C})\oplus\mathrm{Mat}_{\,\mathrm{od}}((m|n)\times(r|s):\mathfrak{C}).$$

Moreover, we may decompose M as $M=M_{\mathrm{B}}+M_{\mathrm{S}}$ where

$$M_{\mathrm{B}}=\begin{cases}\begin{pmatrix} A_{\mathrm{B}} & 0 \\ 0 & B_{\mathrm{B}}\end{pmatrix} & \textit{when}\quad p(M)=0,\\[2ex] \begin{pmatrix} 0 & C_{\mathrm{B}} \\ D_{\mathrm{B}} & 0\end{pmatrix} & \textit{when}\quad p(M)=1.\end{cases}$$

The summation of two matrices in $\mathrm{Mat}_{\,\mathrm{ev}}((m|n)\times(r|s):\mathfrak{C})$ or in $\mathrm{Mat}_{\,\mathrm{od}}((m|n)\times(r|s):\mathfrak{C})$ is defined as usual, but the sum of $\mathrm{Mat}_{\,\mathrm{ev}}((m|n)\times(r|s):\mathfrak{C})$ and $\mathrm{Mat}_{\,\mathrm{od}}((m|n)\times(r|s):\mathfrak{C})$ is not defined except at least one of them being zero matrix.

It is clear that if M is the $(m+n)\times(r+s)$ matrix and N is the $(r+s)\times(p+q)$ matrix, then we may define the product MN and its parity $p(MN)$ as

$$(MN)_{ij}=\sum_k M_{ik}N_{kj},\quad p(MN)=p(M)+p(N)\mod 2.$$

Moreover, we define $\mathrm{Mat}\,[m|n:\mathfrak{C}]$ as the algebra of $(m+n)\times(m+n)$ supermatrices.

Matrices as Linear Transformations:

$$\mathrm{Mat}_{\,\mathrm{ev}}((m|n)\times(r|s):\mathfrak{C})\ni M=\begin{pmatrix}A & C\\ D & B\end{pmatrix}:\mathfrak{R}^{r|s}\to\mathfrak{R}^{m|n},$$

$$\mathrm{Mat}_{\,\mathrm{od}}((m|n)\times(r|s):\mathfrak{C})\ni M=\begin{pmatrix}A & C\\ D & B\end{pmatrix}:\mathfrak{R}^{r|s}\to\mathfrak{R}^m_{\mathrm{od}}\times\mathfrak{R}^n_{\mathrm{ev}},$$

$$\mathrm{Mat}_{\,\mathrm{od}}((n|m)\times(m|n):\mathfrak{C})\ni\Lambda_{n,m}=\begin{pmatrix}0 & \mathbb{I}_n\\ \mathbb{I}_m & 0\end{pmatrix}:\mathfrak{R}^m_{\mathrm{od}}\times\mathfrak{R}^n_{\mathrm{ev}}\to\mathfrak{R}^n_{\mathrm{ev}}\times\mathfrak{R}^m_{\mathrm{od}}=\mathfrak{R}^{n|m}.$$

If we introduce the duality between $\mathfrak{R}^{m|n}$ as in (2.35), we may define the transposed operator as

$$\langle MX|\Xi\rangle_{m|n}=\langle X|M^t\,\Xi\rangle_{r|s}\quad\text{for any}\quad M\in\mathrm{Mat}_{\,\mathrm{ev}}((m|n)\times(r|s):\mathfrak{C}),$$

for $X=(x,\theta)\in\mathfrak{R}^{r|s}$ and $\Xi=(\xi,\omega)\in\mathfrak{R}^{m|n}$. More precisely, we have

$$M^t=\begin{pmatrix}A & C\\ D & B\end{pmatrix}^t=\begin{pmatrix}A^t & D^t\\ -C^t & B^t\end{pmatrix}\quad\text{and}\quad M^{tttt}=M.$$

Analogously, if we define the duality between $\mathfrak{C}^{m|n}_Z$ and $\mathfrak{C}^{m|n}_\Upsilon$ by

$$\langle Z|\Upsilon\rangle_{m|n}=\sum_{j=1}^m\langle z_j|\overline{\eta_j}\rangle+\sum_{k=1}^n\langle\theta_k|\overline{\rho_k}\rangle,\quad\text{or}\quad=\sum_{j=1}^m\langle\overline{z_j}|\eta_j\rangle+\sum_{k=1}^n\langle\overline{\theta_k}|\rho_k\rangle,$$

for $Z=(z,\theta)\in\mathfrak{C}^{r|s}$, $\Upsilon=(\eta,\rho)\in\mathfrak{C}^{m|n}$, we may introduce M^*, the adjoint of matrix M, by

$$\langle MZ|\Upsilon\rangle_{m|n}=\langle Z|M^*\Upsilon\rangle_{r|s}.$$

Therefore, we have

$$M^*=\begin{pmatrix}A & C\\ D & B\end{pmatrix}^*=\begin{pmatrix}A^* & D^*\\ -C^* & B^*\end{pmatrix}\quad\text{and}\quad M^{****}=M.$$

The conjugation is defined by

$$\overline{MZ}=\bar M\bar Z\quad\text{for any}\quad M\in\mathrm{Mat}\,((m|n)\times(r|s):\mathfrak{C})\quad\text{and}\quad Z\in\mathfrak{C}^{r|s}.$$

That is, using (2.31), we have

$$\bar M=\overline{\begin{pmatrix}A & C\\ D & B\end{pmatrix}}=\begin{pmatrix}\bar A & -\bar C\\ \bar D & \bar B\end{pmatrix}.$$

Lemma 3.1.2 *For $M\in\mathrm{Mat}\,((m|n)\times(r|s):\mathfrak{C})$ and $N\in\mathrm{Mat}\,((r|s)\times(p|q):\mathfrak{C})$, we have*

$$(MN)^t=N^t\,M^t,\quad(MN)^*=N^*M^*,\quad\overline{MN}=\bar M\bar N,$$

$$(M^t)^t=\Lambda M\Lambda,\quad(\bar M)^t=M^*,\quad\textit{where}\quad\Lambda=\begin{pmatrix}\mathbb{I}_m & 0\\ 0 & -\mathbb{I}_n\end{pmatrix}.$$

If $M \in \mathrm{Mat}\,[m|n:\mathfrak{C}]$ is even, denoted by $M \in \mathrm{Mat}_{\mathrm{ev}}[m|n:\mathfrak{C}]$, then M acts on $\mathfrak{R}^{m|n}$ linearly. Denoting this by T_M, we call it super linear transformation on $\mathfrak{R}^{m|n}$ and M is called the representative matrix of T_M.

Proposition 3.1.3 *Let* $M \in \mathrm{Mat}_{\mathrm{ev}}[m|n:\mathfrak{C}]$ *and assume* $\det M_{\mathrm{B}} \neq 0$. *Then, for given* $Y \in \mathfrak{R}^{m|n}$,

$$T_M X = Y \tag{3.1}$$

has the unique solution $X \in \mathfrak{R}^{m|n}$, *which is denoted by* $X = M^{-1}Y$.

Proof. Since M_{B} has the inverse matrix M_{B}^{-1}, (3.1) is reduced to

$$X + N_{\mathrm{S}}X = Y', \quad Y' = M_{\mathrm{B}}^{-1}Y$$

where $N_{\mathrm{S}} = M_{\mathrm{B}}^{-1}M_{\mathrm{S}}$. Remark that $N_{\mathrm{S}}X_{[j]} \in \sum_{k\geq j+1}^{\infty}\mathfrak{C}_{[k]}$ for $j \geq 0$. Decomposing by degree, we get

$$X_{[j]} = Y'_{[j]} - (N_{\mathrm{S}}X_{(j-1)})_{[j]} \quad \text{for} \quad j = 1, 2, \ldots.$$

As $X_{(0)} = X_{[0]} = Y'_{[0]}$, we get $X_{[j]}$ from $X_{(j-1)}$ for $j \geq 1$ by induction. □

Exercise 3.1.4 *How about* $M \in \mathrm{Mat}_{\mathrm{od}}((m|n)\times(n|m):\mathfrak{C})$ *?*

Definition 3.1.5 $M \in \mathrm{Mat}_{\mathrm{ev}}[m|n:\mathfrak{C}]$ *is called invertible or non-singular if* M_{B} *is invertible, i.e.* $\det A_{\mathrm{B}} \det B_{\mathrm{B}} \neq 0$, *and denoted by* $M \in \mathrm{GL}_{\mathrm{ev}}[m|n:\mathfrak{C}]$.

3.2 Supertrace, superdeterminant and Paffian

Lemma 3.2.1 *Let* V, W *be two rectangular matrices with odd elements,* $m\times n$, $n \times m$, *respectively. We have*
(1) $\mathrm{tr}\,(VW)^k = -\mathrm{tr}\,(WV)^k$ *for any* $k = 1, 2, \cdots$.
(2) $\det(\mathbb{I}_m + VW) = \det(\mathbb{I}_n + WV)^{-1}$.

Proof. Let $V = (v_{ij})$, $W = (w_{jk})$ with $v_{ij}, w_{jk} \in \mathfrak{C}_{\mathrm{od}}$.

$$\begin{aligned}\mathrm{tr}\,(VW)^k &= \sum v_{ij_1}w_{j_1j_2}v_{j_2j_3}\cdots v_{j_{k-1}j_\ell}w_{j_ki}\\ &= -\sum w_{j_1j_2}v_{j_2j_3}\cdots v_{j_{\ell-1}j_k}w_{j_ki}v_{ij_1} = -\mathrm{tr}\,(WV)^k.\end{aligned}$$

Using this, we have $\mathrm{tr}\,((WV)^{\ell-1}WV) = -\mathrm{tr}\,(V(WV)^{\ell-1}W)$ which yields

$$\begin{aligned}\log\det(\mathbb{I}_n + WV) = \mathrm{tr}\,\log(\mathbb{I}_n + WV) &= \sum_\ell \frac{(-1)^{\ell+1}}{\ell}\mathrm{tr}\,((WV)^{\ell-1}WV)\\ &= \sum_\ell \frac{(-1)^{\ell+1}}{\ell}\big[-\mathrm{tr}\,(V(WV)^{\ell-1}W)\big] = -\sum_\ell \frac{(-1)^{\ell+1}}{\mathrm{tr}}(VW)^\ell\\ &= -\log\det(\mathbb{I}_m + VW). \qquad \square\end{aligned}$$

Comparison 3.2.2 *If* $A = (a_{ij}) \in \mathrm{Mat}\,(m\times n:\mathfrak{C}_{\mathrm{ev}})$, $B = (b_{jk}) \in \mathrm{Mat}\,(n\times m:\mathfrak{C}_{\mathrm{ev}})$, *then we have*
(1) $\mathrm{tr}\,(AB)^k = \mathrm{tr}\,(BA)^k$,
(2) $\det(\mathbb{I}_m + AB) = \det(\mathbb{I}_n + BA)$.

Definition 3.2.3 *Let* $M = \begin{pmatrix} A & C \\ D & B \end{pmatrix} \in \mathrm{Mat}\,[m|n : \mathfrak{C}]$. *We define the supertrace of* M *by*

$$\mathrm{str}\ M = \mathrm{tr}\ A - (-1)^{p(M)}\,\mathrm{tr}\ B.$$

Using Lemma 3.2.1, we get readily

Proposition 3.2.4 *(a) Let* $M, N \in \mathrm{Mat}\,[m|n : \mathfrak{C}]$ *such that* $p(M) + p(N) \equiv 0 \bmod 2$. *Then, we have*

$$\mathrm{str}\,(M + N) = \mathrm{str}\ M + \mathrm{str}\ N.$$

(b) M *is a matrix of size* $(m+n) \times (r+s)$ *and* N *is a matrix of size* $(r+s) \times (m+n)$. *Then,*

$$\mathrm{str}\,(MN) = (-1)^{p(M)p(N)}\,\mathrm{str}\,(NM).$$

Definition 3.2.5 *Let* $B = (B_{jk})$ *be* $(\ell \times \ell)$*-matrix with elements in* $\mathfrak{C}_{\mathrm{ev}}$, *denoted by,* $B \in \mathrm{Mat}\,[\ell : \mathfrak{C}_{\mathrm{ev}}]$. *As* $\mathfrak{C}_{\mathrm{ev}}$ *is a commutative ring, we may define* $\det B$ *as usual:*

$$\det B = \sum_{\rho \in \wp_\ell} \mathrm{sgn}\,(\rho) B_{1\,\rho(1)} \cdots B_{\ell\,\rho(\ell)}.$$

Then, we have, as ordinary case,

$$\det(AB) = \det A \det B, \quad \det(\exp\ A) = \exp\,(\mathrm{tr}\ A) \quad \text{for} \quad A, B \in \mathrm{Mat}\,[\ell : \mathfrak{C}_{\mathrm{ev}}]. \tag{3.2}$$

Definition 3.2.6 *Let* M *be a supermatrix. When* $\det B_{\mathrm{B}} \neq 0$, *we put*

$$\mathrm{sdet}\, M = (\det(A - CB^{-1}D))(\det B)^{-1}$$

and call it superdeterminant or Berezinian of M.

Comparison 3.2.7 *Let*

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad M = \begin{pmatrix} \mathbb{I}_m & 0 \\ -A_{22}^{-1} A_{21} & \mathbb{I}_n \end{pmatrix},$$

be block matrices of even elements. Then, we have

$$\det A = \det AM = \det \begin{pmatrix} A_{11} - A_{12}A_{22}^{-1}A_{21} & A_{12} \\ 0 & A_{22} \end{pmatrix} = \det(A_{11} - A_{12}A_{22}^{-1}A_{21}) \det A_{22}.$$

Corollary 3.2.8 *When* $\det B_{\mathrm{B}} \neq 0$ *and* $\mathrm{sdet}\, M \neq 0$, *then* $\det A_{\mathrm{B}} \neq 0$.

Exercise 3.2.9 *Prove the above corollary.*

Remark 3.2.10 *It seems meaningful to cite here the result of Dyson [68]:*

> **Theorem 3.2.11 (Dyson)** *Let* R *be a ring with a unit element and without divisors of zero. Assume that on the matrix ring* A *with* $n > 1$, *a mapping* D *exists satisfying the following axioms:*
>
> *Axiom 1. For any* $a \in A$, $D(a) = 0$ *if and only if there is a non-zero* $w \in W$ *with* $aw = 0$. *Here,* W *is the set of single-column matrices with elements in* R.
>
> *Axiom 2.* $D(a)D(b) = D(ab)$.

Axiom 3. Let the elements of a be a_{ij} $i,j=1,\cdots,n$, and similarly for b and c. If for some row-index k we have

$$\begin{cases} a_{ij}=b_{ij}=c_{ij}, & i\neq k \\ a_{ij}+b_{ij}=c_{ij}, & i=k, \end{cases}$$

then

$$D(a)+D(b)=D(c).$$

Then, R is commutative.

This states that if the elements of matrix are taken from non-commutative algebra, then it is impossible to define the determinant having above three properties. But, he claims a certain 'determinant' is defined for some class of matrices with elements in 'quarternion' requiring only one or two properties above (By the way, Moore's point of view, explaind in [68], should be reconsidered significantly).
In fact, we may define "superdeterminant" for "supermatrix" as above which staisfies the properties below.

Lemma 3.2.12 *(1) Let $L\in \mathrm{Mat}_{\mathrm{ev}}[\ell:\mathfrak{C}_{\mathrm{ev}}]$ such that the product of any two entries of it is zero. Then*

$$(\mathbb{I}_\ell+L)^{-1}=\mathbb{I}_\ell-L, \quad \det(\mathbb{I}_\ell+L)=1+\mathrm{tr}\ L.$$

(2) Let $M\in \mathrm{Mat}_{\mathrm{ev}}[m|n:\mathfrak{C}]$ such that the product of any two entries of it is zero. Then

$$\mathrm{sdet}\,(\mathbb{I}_{m+n}+M)=1+\mathrm{str}\ M.$$

Proof. (1) Remarking

$$(\mathbb{I}_\ell+L)^{-1}=\mathbb{I}_\ell-L+L^2-L^3+\cdots \quad \text{and} \quad \det(e^L)=e^{\mathrm{tr}\ L},$$

we get the result readily.
(2) For $M=\begin{pmatrix} A & C \\ D & B\end{pmatrix}$, satisfying $C(\mathbb{I}_n+B)^{-1}D=0$ and $\mathrm{tr}\ A\,\mathrm{tr}\ B=0$ garanteed by the product of any two entries of M being zero,

$$\begin{aligned}\mathrm{sdet}\,(\mathbb{I}_{m+n}+M)&=\det(\mathbb{I}_m+A-C(\mathbb{I}_n+B)^{-1}D)\det(\mathbb{I}_n+B)^{-1}\\ &=\det(\mathbb{I}_m+A)\det(\mathbb{I}_n-B)=1+\mathrm{tr}\ A-\mathrm{tr}\ B=1+\mathrm{str}\ M. \qquad \square\end{aligned}$$

Theorem 3.2.13 *Let $M,N\in \mathrm{Mat}\,[m|n:\mathfrak{C}]$.*

(1) If M is invertible, then we have $\mathrm{sdet}\,M\neq 0$. Moreover, if A is nonsingular, then

$$(\mathrm{sdet}\,M)^{-1}=(\det A)^{-1}(\det(B-DA^{-1}C)). \tag{3.3}$$

(2) Multiplicativity of sdet:

$$\mathrm{sdet}\,(MN)=\mathrm{sdet}\,M\mathrm{sdet}\,N. \tag{3.4}$$

(3) str and sdet are matrix invariants. That is, if N is invertible, then

$$\mathrm{str}\ M=(-1)^{p(M)+p(N)}\,\mathrm{str}\ NMN^{-1}, \quad \mathrm{sdet}\,M=\mathrm{sdet}\,NMN^{-1}. \tag{3.5}$$

Proof (due to Leites [144]). (1) By

$$\begin{pmatrix} A & C \\ D & B \end{pmatrix} = \begin{pmatrix} \mathbb{I}_m & 0 \\ DA^{-1} & \mathbb{I}_n \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & B - DA^{-1}C \end{pmatrix} \begin{pmatrix} \mathbb{I}_m & A^{-1}C \\ 0 & \mathbb{I}_n \end{pmatrix} \quad \text{if } \det A_{\mathrm{B}} \neq 0, \tag{3.6}$$

we have readily by definition, $\operatorname{sdet} M = \det A (\det(B - DA^{-1}C))^{-1}$, which yields (3.3).

(2) [Step 1]: Let $\mathcal{G}_+$, $\mathcal{G}_0$ and $\mathcal{G}_-$ be subgroups of $\mathrm{GL}\,[m|n : \mathfrak{C}]$, given by

$$\mathcal{G}_+ = \left\{ \begin{pmatrix} \mathbb{I}_m & C \\ 0 & \mathbb{I}_n \end{pmatrix} \right\}, \quad \mathcal{G}_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \right\}, \quad \mathcal{G}_- = \left\{ \begin{pmatrix} \mathbb{I}_m & 0 \\ D & \mathbb{I}_n \end{pmatrix} \right\}.$$

Then, we have, $M = M_+ M_0 M_-$ with $M_+ \in \mathcal{G}_+$, $M_0 \in \mathcal{G}_0$ and $M_- \in \mathcal{G}_-$. i.e., for any $M \in \mathrm{GL}\,[m|n : \mathfrak{C}]$,

$$M = \begin{pmatrix} A & C \\ D & B \end{pmatrix} = \begin{pmatrix} \mathbb{I}_m & CB^{-1} \\ 0 & \mathbb{I}_n \end{pmatrix} \begin{pmatrix} A - CB^{-1}D & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} \mathbb{I}_m & 0 \\ B^{-1}D & \mathbb{I}_n \end{pmatrix} \quad \text{if } \det B_{\mathrm{B}} \neq 0. \tag{3.7}$$

Remarking that

$$\begin{pmatrix} \mathbb{I}_m & C \\ 0 & \mathbb{I}_n \end{pmatrix} \times \begin{pmatrix} \mathbb{I}_m & C' \\ 0 & \mathbb{I}_n \end{pmatrix} = \begin{pmatrix} \mathbb{I}_m & C + C' \\ 0 & \mathbb{I}_n \end{pmatrix},$$

we introduce the notion of elemantary matrices having the form

$$\begin{pmatrix} \mathbb{I}_m & E \\ 0 & \mathbb{I}_n \end{pmatrix}$$

where E has only one non-zero entry.

[Step 2]: We claim $\operatorname{sdet}(MN) = \operatorname{sdet} M \operatorname{sdet} N$ whenever $M \in \mathcal{G}_+$ or $M \in \mathcal{G}_0$, and similarly, whenever $N \in \mathcal{G}_0$ or $N \in \mathcal{G}_-$. For example, when

$$M = \begin{pmatrix} \mathbb{I}_m & C' \\ 0 & \mathbb{I}_n \end{pmatrix} \in \mathcal{G}_+ \quad N = \begin{pmatrix} A & C \\ D & B \end{pmatrix},$$

we have

$$\begin{aligned} \operatorname{sdet}(MN) &= \operatorname{sdet} \begin{pmatrix} \mathbb{I}_m & C' \\ 0 & \mathbb{I}_n \end{pmatrix} \begin{pmatrix} A & C \\ D & B \end{pmatrix} = \operatorname{sdet} \begin{pmatrix} A + C'D & C + C'B \\ D & B \end{pmatrix} \\ &= \det(A + C'D - (C + C'B)B^{-1}D)(\det D)^{-1} = \det(A - CB^{-1}D)(\det D)^{-1} \\ &= \operatorname{sdet} M \operatorname{sdet} N. \end{aligned}$$

Exercise 3.2.14 *Check other cases analogously.*

[Step 3]: We claim that $\operatorname{sdet}(MN) = \operatorname{sdet} M \operatorname{sdet} N$ for any elementary matrix N

$$N = \begin{pmatrix} \mathbb{I}_m & E \\ 0 & \mathbb{I}_n \end{pmatrix} \in \mathcal{G}_+.$$

Since we have

$$\begin{aligned} &\operatorname{sdet}(MN) = \operatorname{sdet}(M_+ M_0 (M_- N)) = \operatorname{sdet} M_0 \operatorname{sdet}(M_- N), \\ &\operatorname{sdet} M \operatorname{sdet} N = \operatorname{sdet} M_0 \operatorname{sdet} M_- \operatorname{sdet} N, \end{aligned}$$

by Step 1 and Step 2, we need to prove

$$\operatorname{sdet}(M_- N) = \operatorname{sdet} M_- \operatorname{sdet} N = 1$$

when N is an elementary matrix. By definition,

$$\mathrm{sdet}\begin{pmatrix}\mathbb{I}_m & 0\\ D & \mathbb{I}_n\end{pmatrix}\begin{pmatrix}\mathbb{I}_m & E\\ 0 & \mathbb{I}_n\end{pmatrix}=\mathrm{sdet}\begin{pmatrix}\mathbb{I}_m & E\\ D & \mathbb{I}_n+DE\end{pmatrix}=\det(1-E(1+DE)^{-1}D)\det(1+DE)^{-1}.$$

As E has only one non-zero entry, the product of any two of the matrices E, DE, $E(1+DE)^{-1}D$ is zero. Applying Lemma, we get, by $(1+DE)^{-1}=1-DE$ and $E\cdot DE=0$,

$$\mathrm{sdet}\,(M_-N)=\det(1-DE)(\det(1+DE))^{-1}=(1-\mathrm{tr}\ DE)(1+\mathrm{tr}\ DE)^{-1}.$$

As $\mathrm{tr}\ DE=-\,\mathrm{tr}\ ED$, we have

$$\mathrm{sdet}\,(M_-N)=1=\mathrm{sdet}\,M_-\mathrm{sdet}\,N.$$

[Step 4]: Put

$$\mathcal{G}=\Big\{N\in\mathrm{GL}\,[m|n:\mathfrak{R}]\mid\mathrm{sdet}\,(MN)=\mathrm{sdet}\,M\mathrm{sdet}\,N\quad\text{for any}\quad M\in\mathrm{GL}\,[m|n:\mathfrak{R}]\Big\}.$$

For $N_1,N_2\in\mathcal{G}$, we have

$$\begin{aligned}\mathrm{sdet}\,(M\cdot N_1N_2)&=\mathrm{sdet}\,((MN_1)N_2)=\mathrm{sdet}\,(MN_1)\mathrm{sdet}\,N_2\\&=\mathrm{sdet}\,M\mathrm{sdet}\,N_1\mathrm{sdet}\,N_2=\mathrm{sdet}\,M\mathrm{sdet}\,(N_1N_2),\end{aligned}\tag{3.8}$$

which implies $\mathcal{G}$ froms a group. BY Steps 2 and 3, $\mathcal{G}$ contains $\mathcal{G}_-$ and $\mathcal{G}_0$ and all elementary matrices $N\in\mathcal{G}_+$. By Step1, $\mathrm{GL}\,[m|n:\mathfrak{C}]$ is generated by these matrices, we have $\mathcal{G}=\mathrm{GL}\,[m|n:\mathfrak{C}]$, that is, $\mathrm{sdet}\,(MN)=\mathrm{sdet}\,M\mathrm{sdet}\,N$.

(3) Let N,M be given. Then, using (3.8), we get

$$\mathrm{str}\ NMN^{-1}=(-1)^{p(N)p(MN^{-1})}\,\mathrm{str}\ MN^{-1}N=(-1)^{p(N)+p(M)}\,\mathrm{str}\ M,$$

since $p(MN^{-1})=p(M)+p(N^{-1})\mod 2$ and $0=p(NN^{-1})=p(N)+p(N^{-1})\mod 2$, we have $p(N)p(MN^{-1})=p(N)+p(M)\mod 2$.

Using (3.8), we have $\mathrm{sdet}\,MN=\mathrm{sdet}\,NM$ which implies $\mathrm{sdet}\,NMN^{-1}=\mathrm{sdet}\,N^{-1}NM=\mathrm{sdet}\,M$. □

By simple calculation, we have

Lemma 3.2.15

$$\begin{aligned}\begin{pmatrix}\tilde A & \tilde C\\ \tilde D & \tilde B\end{pmatrix}=\begin{pmatrix}A & C\\ D & B\end{pmatrix}^{-1}&=\begin{pmatrix}(A-CB^{-1}D)^{-1} & -A^{-1}C(B-DA^{-1}C)^{-1}\\ -B^{-1}D(A-CB^{-1}D)^{-1} & (B-DA^{-1}C)^{-1}\end{pmatrix}\\&=\begin{pmatrix}(\mathbb{I}_m-A^{-1}CB^{-1}D)^{-1}A^{-1} & -(\mathbb{I}_m-A^{-1}CB^{-1}D)^{-1}A^{-1}CB^{-1}\\ -(\mathbb{I}_n-B^{-1}DA^{-1}C)^{-1}B^{-1}DA^{-1} & (\mathbb{I}_n-B^{-1}DA^{-1}C)^{-1}B^{-1}\end{pmatrix}\\&=\begin{pmatrix}A^{-1}(\mathbb{I}_m-CB^{-1}DA^{-1})^{-1} & -A^{-1}CB^{-1}(\mathbb{I}_n-DA^{-1}CB^{-1})^{-1}\\ -B^{-1}DA^{-1}(\mathbb{I}_m-CB^{-1}DA^{-1})^{-1} & B^{-1}(\mathbb{I}_n-DA^{-1}CB^{-1})^{-1}\end{pmatrix}.\end{aligned}$$

$$\begin{aligned}\mathrm{sdet}\begin{pmatrix}A & C\\ D & B\end{pmatrix}&=(\det A)(\det B)^{-1}\det(\mathbb{I}_m-A^{-1}CB^{-1}D)\\&=(\det A)(\det B)^{-1}\det(\mathbb{I}_m-CB^{-1}DA^{-1})=(\det\tilde A)^{-1}(\det B)^{-1}\\&=(\det A)(\det B)^{-1}\det(\mathbb{I}_n-B^{-1}DA^{-1}C)\\&=(\det A)(\det B)^{-1}\det(\mathbb{I}_n-DA^{-1}CB^{-1})=(\det A)(\det\tilde B).\end{aligned}$$

Proof. The invertibility of matrices appeared (3.2.15) is garanteed by

$$\begin{pmatrix} A & C \\ D & B \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & B^{-1} \end{pmatrix} \begin{pmatrix} \mathbb{I}_m & 0 \\ 0 & -\mathbb{I}_n \end{pmatrix} \begin{pmatrix} A & C \\ D & B \end{pmatrix} \begin{pmatrix} \mathbb{I}_m & 0 \\ 0 & -\mathbb{I}_n \end{pmatrix} = \begin{pmatrix} A - CB^{-1}D & 0 \\ 0 & B - DA^{-1}C \end{pmatrix}. \qquad \square \tag{3.9}$$

Theorem 3.2.16 (Liouville's theorem) *Let $M(t) \in \mathrm{Mat}\,[m|n : \mathfrak{C}]$ with a real parameter t. Let $X(t) \in \mathrm{Mat}\,[m|n : \mathfrak{C}]$ satisfy*

$$\frac{d}{dt} X(t) = M(t)X(t), \quad X(0) = \mathbb{I}_{m+n}. \tag{3.10}$$

Then $X(t) \in \mathrm{GL}\,[m|n : \mathfrak{C}]$, and

$$\mathrm{sdet}\, X(t) = \exp \left\{ \int_0^t ds\, \mathrm{str}\ M(s) \right\}. \tag{3.11}$$

Proof (due to Berezin [20]). Let $\tilde{X}(t)$ be a solution of

$$\frac{d}{dt} \tilde{X}(t) = -\tilde{X}(t)M(t), \quad \tilde{X}(0) = \mathbb{I}_{m+n}.$$

Then, since

$$\frac{d}{dt} (\tilde{X}(t)X(t)) = 0 \quad \text{with} \quad \tilde{X}(0)X(0) = \mathbb{I}_{m+n},$$

we have $\tilde{X}(t)X(t) = \mathbb{I}_{m+n}$ which implies $X(t) \in \mathrm{GL}\,[m|n : \mathfrak{C}]$.

Let

$$M(t) = \begin{pmatrix} A(t) & C(t) \\ D(t) & B(t) \end{pmatrix}, \quad X(t) = \begin{pmatrix} X_{11}(t) & X_{12}(t) \\ X_{21}(t) & X_{22}(t) \end{pmatrix}.$$

Then, putting $Y(t) = X_{11}(t) - X_{12}(t)X_{22}^{-1}(t)X_{21}(t)$ and $Z = X_{22}^{-1}(t)$, we have, by simple calculation from (3.10),

$$\frac{d}{dt} Y = (A - X_{12}X_{22}^{-1}D)Y, \quad \frac{d}{dt} Z = -Z(DX_{12}X_{22}^{-1} + B).$$

As all elements appeared in the above equations are even, we may apply the classical Liouville theorem to have

$$\frac{d}{dt} \det Y = \mathrm{tr}\,(A - X_{12}X_{22}^{-1}D) \det Y, \quad \frac{d}{dt} \det Z = -\,\mathrm{tr}\,(DX_{12}X_{22}^{-1} + B) \det Z.$$

As $\mathrm{tr}\,(A - X_{12}X_{22}^{-1}D) = \mathrm{tr}\,(A + DX_{12}X_{22}^{-1})$, we have

$$\frac{d}{dt} \mathrm{sdet}\, X = \frac{d}{dt} (\det Y \det Z) = \mathrm{tr}\,(A - B) \det Y \det Z = \mathrm{str}\ M \mathrm{sdet}\, X \quad \text{with} \quad \mathrm{sdet}\, X(0) = 1.$$

This yields the desired result. $\square$

Corollary 3.2.17 *For $M, N \in \mathrm{Mat}_{\mathrm{ev}}[m|n : \mathfrak{C}]$ we have*

$$\begin{aligned} \mathrm{sdet}\,(MN) &= \mathrm{sdet}\, M \mathrm{sdet}\, N, \\ \exp\,(\mathrm{str}\ M) &= \mathrm{sdet}\,(\exp\ M). \end{aligned} \tag{3.12}$$

Proof. (1) Put $X(t) = (1-t)\mathbb{I}_{m+n} + tM$ and $Y(t) = (1-t)\mathbb{I}_{m+n} + tN$. As $X(t)$ and $Y(t)$ are differentiable in t and invertible except at most one t, we my define

$$A(t) = \frac{dX(t)}{dt} X(t)^{-1}, \quad B(t) = \frac{dX(t)}{dt} Y(t)^{-1}.$$

Then
$$\frac{d}{dt}(X(t)Y(t)) = (A(t)+B_1(t))X(t)Y(t) \quad \text{where} \quad B_1(t) = X(t)B(t)X(t)^{-1}.$$
Applying above theorem, we have
$$\begin{aligned}\operatorname{sdet}(MN) = \operatorname{sdet}(X(1)Y(1)) &= \exp\left\{\int_0^1 ds\, \operatorname{str}(A(t)+B_1(t))\right\} = \exp\left\{\int_0^1 ds(\operatorname{str}\ A(t) + \operatorname{str}\ B(t))\right\}\\ &= \operatorname{sdet} X(1)\operatorname{sdet} Y(1) = \operatorname{sdet} M \operatorname{sdet} N.\end{aligned}$$

(2) Putting $M(t) = M$, $X(t) = e^{tM}$ and $t = 1$ in theorem above, we get the desired result. □

Definition 3.2.18 *For a skew-symmetric $n\times n$ matrix $\tilde{B} = (\tilde{B}_{jk})$ with even elements, we define Paffian of B as*
$$\operatorname{Pff}(\tilde{B}) = \frac{1}{(n/2)!}\sum_{\rho\in\wp_n} \operatorname{sgn}(\rho)\tilde{B}_{\rho(1)\,\rho(2)}\cdots\tilde{B}_{\rho(n-1)\,\rho(n)} \tag{3.13}$$
where $\wp_n$ is the permutation group of order n and $\operatorname{sgn}(\rho)$ stands for the signature of $\rho\in\wp_n$.

Or classical result is given in [**?**]:

Definition 3.2.19 *Let $n = 2\ell$. Let $A = (A_{jk})$ be a real skew-symmetric $n\times n$-matrix. We define the Paffian of A, $\operatorname{Pff}(A)$, as*
$$\begin{aligned}\operatorname{Pff}(A) &= (2^\ell \ell!)^{-1}\sum_{\rho\in\wp_n}\operatorname{sgn}(\rho)A_{\rho(1)\,\rho(2)}\cdots A_{\rho(n-1)\,\rho(n)}\\ &= \sum_{\substack{i_1<\cdots<i_\ell,\\ i_1<j_1,\cdots,i_\ell<j_\ell}}\operatorname{sgn}(\rho)A_{i_1j_1}\cdots A_{i_\ell j_\ell}.\end{aligned}$$

Proposition 3.2.20 *(a) If B is an arbitrary matrix and*
$$\bar{A}_{ij} = \sum_{k,l} B_{ik}B_{jl}A_{kl} \Longrightarrow \operatorname{Pff}(\bar{A}) = \det(B)\operatorname{Pff}(A).$$

(b) $\operatorname{Pff}(A)^2 = \det A$.

Proof. (a) Let $e_1,\cdots,e_n$ be the canonical basis for $\mathbb{C}^n$ and view e_i as elements of $\Lambda^*(\mathbb{C}^n)$. Let
$$a = \sum_{i,j} A_{ij}e_i\wedge e_j.$$

Then the definition of $\operatorname{Pff}(A)$ yields
$$\overbrace{a\wedge\cdots\wedge a}^{\ell\text{-times}} = (2^\ell\ell!)\operatorname{Pff}(A)e_1\wedge\cdots\wedge e_n.$$

For $B = (B_{ij})$, we have $Be_i = \sum_j B_{ji}e_j$ and
$$\bar{a} = \sum_{i,j}\bar{A}_{ij}e_i\wedge e_j = \sum_{i,j}A_{ij}Be_i\wedge Be_j = \Lambda^*(B)a,$$

so we get
$$\overbrace{\bar{a}\wedge\cdots\wedge\bar{a}}^{\ell\text{-times}} = \Lambda^n(B)(\overbrace{a\wedge\cdots\wedge a}^{\ell\text{-times}}).$$

Since $\Lambda^n(B)$ on the n-dimensional space $\Lambda^n(\mathbb{C}^n)$ is just multiplication by $\det B$, we get the result.

(b) iA is a self-adjoint matrix whose eigenvalues are pure imaginary. If $f \in \mathbb{C}^n$ is an eigenvector for iA, with $iAf = \lambda f$ ($\lambda \in \mathbb{R}$), then $(iA)\bar{f} = \overline{-iAf} = -\lambda \bar{f}$ so A has eigenvalues $\pm i\lambda_1, \cdots, \pm i\lambda_\ell$ with orthonormal eigenvectors $f_2 = \bar{f}_1, f_4 = \bar{f}_3, \cdots$. Letting $g_{2j-1} = (f_{2j} + f_{2j-1})/\sqrt{2}, g_{2j} = (if_{2j} - if_{2j-1})/\sqrt{2}$ we obtain real orthonormal vectors g_j with

$$Ag_{2j} = \lambda_j g_{2j-1}, \quad Ag_{2j-1} = -\lambda_j g_{2j};$$

so there exists an orthonormal matrix B so that

$$BAB^{-1} = \begin{pmatrix} 0 & \lambda_1 & & & \\ -\lambda_1 & 0 & & & \\ & & 0 & \lambda_2 & \\ & & -\lambda_2 & 0 & \\ & & & & \ddots \end{pmatrix} = \bar{A}.$$

Since B is orthonormal, $\det(B) = \pm 1$ and $(B^{-1})_{ij} = B_{ji}$, so $\bar{A}$ and A are related as $\bar{A}_{ij} = \sum_{k,l} B_{ik} B_{jl} A_{kl}$, Thus, $\mathrm{Pff}\,(A) = \pm \mathrm{Pff}\,(\bar{A}) = \pm \lambda_1 \cdots \lambda_\ell$. Thus, $\mathrm{Pff}\,(A)^2 = \prod_{j=1}^{\ell} \lambda_j^2 = \prod_{j=1}^{\ell} (i\lambda_j)(-i\lambda_j) = \det(A)$. □

3.3 Diagonalization

Let

$$M = \begin{pmatrix} A & C \\ D & B \end{pmatrix} \in \mathrm{Mat}\,[m|n : \mathfrak{C}] \quad \text{with} \quad M_{\mathrm{B}} = \begin{pmatrix} A_{\mathrm{B}} & 0 \\ 0 & B_{\mathrm{B}} \end{pmatrix}.$$

Definition 3.3.1 *A matrix $M \in \mathrm{Mat}\,[m|n : \mathfrak{C}]$ is called generic if all eigenvalues of M_{B} as $\mathrm{Mat}\,[m+n : \mathbb{C}]$ are different each others.*

Theorem 3.3.2 (Berezin[20]) *Let $M \in \mathrm{Mat}\,[m|n : \mathfrak{C}]$ be generic. Then, there exists a matrix $X \in \mathrm{GL}\,[m|n : \mathfrak{C}]$ such that $E = XMX^{-1}$ is diagonal.*

Proof. Decomposing the equality $EX = XM$ with respect to the degree, we have

$$(EX)_{[k]} = \sum_{j=0}^{k} E_{[j]} X_{[k-j]} = \sum_{j=0}^{k} X_{[j]} M_{[k-j]} = (XM)_{[k]}. \tag{3.14}$$

From this, we want to construct $X_{[k]}$ and $E_{[k]}$: For $k = 0$, we have

$$E_{[0]} X_{[0]} = X_{[0]} M_{[0]}. \tag{3.15}$$

By the assumption, there exist $X_{[0]11}$, $X_{[0]22}$, $E_{[0]11} = \mathrm{diag}(\cdots, \lambda_i^{[0]}, \cdots)(1 \le i \le m)$ and $E_{[0]22} = \mathrm{diag}(\cdots, \lambda_k^{[0]}, \cdots)(m+1 \le k \le m+n)$ such that

$$X_{[0]11} A_{\mathrm{B}} = E_{[0]11} X_{[0]11} \quad \text{and} \quad X_{[0]22} B_{\mathrm{B}} = E_{[0]22} X_{[0]22}.$$

Defining

$$X_{[0]} = \begin{pmatrix} X_{[0]11} & 0 \\ 0 & X_{[0]22} \end{pmatrix}, \ E_{[0]} = \begin{pmatrix} E_{[0]11} & 0 \\ 0 & E_{[0]22} \end{pmatrix},$$

we have the desired one satisfying (3.15).

Assume that there exist $X_{[j]}$ and $E_{[j]}$ for $0 \le j \le k-1$ satisfying (3.14). Multiplying $X_{[0]}^{-1}$ from the right to (3.14) for k, we have

$$E_{[0]} X_{[k]} X_{[0]}^{-1} - X_{[k]} X_{[0]}^{-1} E_{[0]} + E_{[k]} = K_{[k]} \tag{3.16}$$

where

$$K_{[k]} = (\sum_{j=0}^{k-1} X_{[j]} M_{[k-j]}) X_{[0]}^{-1} - (\sum_{j=1}^{k-1} E_{[j]} X_{[k-j]}) X_{[0]}^{-1}.$$

By inductive assumption, the matrix $K_{[k]}$ is known and belongs to $\mathrm{Mat}\,[m|n:\mathfrak{C}]$. From (3.16), we have

$$(\lambda_i^{[0]} - \lambda_j^{[0]})(X_{[k]} X_{[0]}^{-1})_{ij} + \lambda_i^{[k]} \delta_{ij} = (K_{[k]})_{ij}. \tag{3.17}$$

This equation is uniquely solvable since $\lambda_i^{[0]} \neq \lambda_j^{[0]}$ and

$$\begin{cases} \lambda_i^{[k]} = (K_{[k]})_{ii}, \\ (X_{[k]} X_{[0]}^{-1})_{ij} = \frac{(K_{[k]})_{ij}}{\lambda_i^{[0]} - \lambda_j^{[0]}}, \qquad \text{for} \quad i \neq j. \end{cases}$$

Therefore, we define $X_{[j]}$ and $E_{[j]}$ for any $j \geq 0$. Since $X_{[0]}$ is invertible, $X \in \mathrm{GL}\,[m|n:\mathfrak{C}]$. This implies X and E are defined as desired. □

3.4 An example

Let

$$Q = \begin{pmatrix} x_1 & \theta_1 \\ \theta_2 & ix_2 \end{pmatrix} \quad \text{with} \quad x_1,\, x_2 \in \mathfrak{R}_{\mathrm{ev}},\ \theta_1,\, \theta_2 \in \mathfrak{R}_{\mathrm{od}},$$

which maps $\mathfrak{R}^{1|1}$ to $\mathfrak{R}^{1|1}$ or $\mathfrak{R}_{\mathrm{od}} \times i\mathfrak{R}_{\mathrm{ev}}$ to $\mathfrak{R}_{\mathrm{od}} \times i\mathfrak{R}_{\mathrm{ev}}$. This supermatrix appears in Efetov's calculation in Random Matrix Theory, which will be explained in Part II.

3.4.1 Invertibility of Q.

Find Y for a given V such that

$$QY = V \quad \text{with} \quad Y = \begin{pmatrix} y_1 \\ \omega_2 \end{pmatrix},\ V = \begin{pmatrix} v_1 \\ \rho_2 \end{pmatrix} \in \mathfrak{R}^{1|1},$$

$$x_1 y_1 + \theta_1 \omega_2 = v_1,\ \theta_2 y_1 + ix_2 \omega_2 = \rho_2.$$

If $(x_1 x_2)_{\mathrm{B}} \neq 0$, we have readily

$$y_1 = \frac{ix_2 v_1 - \theta_1 \rho_2}{D_-},\ \omega_2 = \frac{x_1 \rho_2 - \theta_2 v_1}{D_+} \quad \text{with} \quad D_\pm = ix_1 x_2 \pm \theta_1 \theta_2.$$

Analogously, for

$$\tilde{Y} = \begin{pmatrix} \omega_1 \\ iy_2 \end{pmatrix} \in \mathfrak{R}_{\mathrm{od}} \times i\mathfrak{R}_{\mathrm{ev}},\ \tilde{V} = \begin{pmatrix} \rho_1 \\ v_2 \end{pmatrix} \in \mathfrak{R}_{\mathrm{od}} \times \mathfrak{R}_{\mathrm{ev}},$$

satisfying $Q\tilde{Y} = \tilde{V}$, we have

$$\omega_1 = \frac{ix_2 \rho_1 - \theta_1 v_2}{D_-},\ iy_2 = \frac{x_1 v_2 - \theta_2 \rho_1}{D_+}.$$

To relate the above quantity with the sdet Q, we proceed as follows: Let

$$Y = \begin{pmatrix} y_1 & \omega_1 \\ \omega_2 & iy_2 \end{pmatrix} \quad \text{with} \quad QY = YQ = I_2.$$

Then, from $QY = I_2$, we have

$$x_1y_1 + \theta_1\omega_2 = 1, \quad x_1\omega_1 + iy_2\theta_1 = 0,$$
$$\theta_2y_1 + ix_2\omega_2 = 0, \quad \theta_2\omega_1 - x_2y_2 = 1.$$

Therefore, we have

$$Y = \begin{pmatrix} \frac{ix_2}{D_-} & -\frac{\theta_1}{D_-} \\ -\frac{\theta_2}{D_+} & \frac{x_1}{D_+} \end{pmatrix} = (\text{sdet } Q)^{-1} \begin{pmatrix} \frac{1}{ix_2} & \frac{\theta_1}{x_2^2} \\ \frac{\theta_2}{x_2^2} & -\frac{x_1x_2+2i\theta_1\theta_2}{x_2^3} \end{pmatrix},$$

which yields $YQ = I_2$ also. Here, we used

$$\text{sdet } Q = \det(x_1 - \theta_1(ix_2)^{-1}\theta_2)(\det(ix_2))^{-1} = \frac{ix_1x_2 - \theta_1\theta_2}{(ix_2)^2}, \ (\text{sdet } Q)^{-1} = \frac{ix_1x_2 + \theta_1\theta_2}{x_1^2}.$$

Therefore,

$$\begin{pmatrix} y_1 \\ \omega_2 \end{pmatrix} = \frac{D_+}{x_1^2} \begin{pmatrix} \frac{1}{ix_2} & \frac{-\theta_1}{(ix_2)^2} \\ \frac{-\theta_2}{(ix_2)^2} & \frac{ix_1x_2-2\theta_1\theta_2}{(ix_2)^3} \end{pmatrix} \begin{pmatrix} v_1 \\ \rho_2 \end{pmatrix} = \frac{D_+}{x_1^2} \begin{pmatrix} \frac{ix_2v_1-\theta_1\rho_2}{(ix_2)^2} \\ \frac{-ix_2\theta_2v_1+(ix_1x_2-2\theta_1\theta_2)\rho_2}{(ix_2)^3} \end{pmatrix},$$
$$\begin{pmatrix} \omega_1 \\ iy_2 \end{pmatrix} = \frac{D_+}{x_1^2} \begin{pmatrix} \frac{1}{ix_2} & \frac{-\theta_1}{(ix_2)^2} \\ \frac{-\theta_2}{(ix_2)^2} & \frac{ix_1x_2-2\theta_1\theta_2}{(ix_2)^3} \end{pmatrix} \begin{pmatrix} \rho_1 \\ v_2 \end{pmatrix} = \frac{D_+}{x_1^2} \begin{pmatrix} \frac{ix_2\rho_1-i\theta_1v_2}{(ix_2)^2} \\ \frac{-ix_2\theta_2\rho_1+(ix_1x_2-2\theta_1\theta_2)v_2}{(ix_2)^3} \end{pmatrix}.$$

3.4.2 Eigenvalues.

Let

$$QU = \lambda U \quad \text{with} \quad U = \begin{pmatrix} u \\ \omega \end{pmatrix}, \quad u \in \mathfrak{R}_{\text{ev}},\ \omega \in \mathfrak{R}_{\text{od}},\ \lambda \in \mathfrak{R}_{\text{ev}}.$$

Then,

$$(x_1 - \lambda)u + \theta_1\omega = 0,\ \theta_2u + (ix_2 - \lambda)\omega = 0.$$

Putting

$$D_+(\lambda) = (x_1 - \lambda)(ix_2 - \lambda) + \theta_1\theta_2,\ D_-(\lambda) = (x_1 - \lambda)(ix_2 - \lambda) - \theta_1\theta_2,$$

we have

$$D_-(\lambda)u = 0,\ D_+(\lambda)\omega = 0.$$

To garantee the existence of $u_{\rm B} \neq 0$ satisfying above, we take λ satisfying

$$D_-(\lambda) = \lambda^2 - (x_1 + ix_2)\lambda + ix_1x_2 - \theta_1\theta_2 = 0.$$

This yields

$$\lambda = x_1 + \frac{\theta_1\theta_2}{x_1 - ix_2} \quad (\text{or } \lambda = ix_2 - \tfrac{\theta_1\theta_2}{x_1-ix_2}, \text{ but this is not fitted because} \notin \mathfrak{R}_{\text{ev}})$$

and

$$U = \begin{pmatrix} 1 \\ \frac{\theta_2}{x_1-ix_2} \end{pmatrix}, \quad QU = (x_1 + \frac{\theta_1\theta_2}{x_1 - ix_2})U.$$

Analogously, we seek $\tilde\lambda \in i\mathfrak{R}_{\text{ev}}$, $\tilde U \in \mathfrak{R}_{\text{od}} \times \mathfrak{R}_{\text{ev}}$ satisfying $Q\tilde U = \tilde\lambda\tilde U$ which is given

$$\tilde U = \begin{pmatrix} \frac{-\theta_1}{x_1-ix_2} \\ 1 \end{pmatrix}, \quad Q\tilde U = (ix_2 + \frac{\theta_1\theta_2}{x_1 - ix_2})\tilde U.$$

Therefore,

$$Q\begin{pmatrix} 1 & -\frac{\theta_1}{x_1-ix_2} \\ \frac{\theta_2}{x_1-ix_2} & 1 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{\theta_1}{x_1-ix_2} \\ \frac{\theta_2}{x_1-ix_2} & 1 \end{pmatrix}\begin{pmatrix} x_1 + \frac{\theta_1\theta_2}{x_1-ix_2} & 0 \\ 0 & ix_2 + \frac{\theta_1\theta_2}{x_1-ix_2} \end{pmatrix}.$$

3.4.3 Diagonalization.

We may diagonalize the matrix Q by using the change of variables

$$\begin{cases} y_1 = x_1 + \dfrac{\theta_1\theta_2}{x_1 - ix_2},\ y_2 = x_2 - \dfrac{i\theta_1\theta_2}{x_1 - ix_2}, \\ \rho_1 = \dfrac{\theta_1}{x_1 - ix_2},\ \rho_2 = -\dfrac{\theta_2}{x_1 - ix_2}, \end{cases} \tag{3.18}$$

or

$$\begin{cases} x_1 = y_1 + \rho_1\rho_2(y_1 - iy_2),\ x_2 = y_2 - i\rho_1\rho_2(y_1 - iy_2), \\ \theta_1 = \rho_1(y_1 - iy_2),\ \theta_2 = -\rho_2(y_1 - iy_2), \end{cases} \tag{3.19}$$

such that

$$GQG^{-1} = \begin{pmatrix} y_1 & 0 \\ 0 & iy_2 \end{pmatrix}, \quad GQ^2G^{-1} = \begin{pmatrix} y_1^2 & 0 \\ 0 & -y_2^2 \end{pmatrix} \tag{3.20}$$

where

$$G = \begin{pmatrix} 1 + 2^{-1}\rho_1\rho_2 & \rho_1 \\ \rho_2 & 1 - 2^{-1}\rho_1\rho_2 \end{pmatrix}, \quad G^{-1} = \begin{pmatrix} 1 + 2^{-1}\rho_1\rho_2 & -\rho_1 \\ -\rho_2 & 1 - 2^{-1}\rho_1\rho_2 \end{pmatrix}.$$

It is clear that

$$\operatorname{str} Q = x_1 - ix_2 = y_1 - iy_2 = \operatorname{str} GQG^{-1}, \quad \text{and}$$
$$\operatorname{str} Q^2 = x_1^2 + x_2^2 + 2\theta_1\theta_2 = y_1^2 + y_2^2 = \operatorname{str} (GQG^{-1})^2.$$

Chapter 4

Supersmooth functions and their basic properties

4.1 The definition of supersmooth functions

Let $\phi(q)$ be a $\mathfrak{C}$-valued function on an open set $\Omega \subset \mathbb{R}^m$, that is,

$$\phi(q) = \sum_{I \in \mathcal{I}} \phi_I(q)\sigma^I \quad \text{with} \quad \phi_I : \Omega \ni q \to \phi_I(q) \in \mathbb{C}.$$

By the definition of the topology of $\mathfrak{C}$, we have

$$\lim_{q \to q_0} \phi(q) = \sum_{I \in \mathcal{I}} \big(\lim_{q \to q_0} \phi_I(q) \big) \sigma^I.$$

The differentiation and integration of such $\phi(q)$ are defined by

$$\frac{\partial}{\partial q_j} \phi(q) = \sum_{I \in \mathcal{I}} \frac{\partial}{\partial q_j} \phi_I(q)\sigma^I,$$

$$\int_\Omega dq\, \phi(q) = \sum_{I \in \mathcal{I}} \bigg(\int_\Omega dq\, \phi_I(q) \bigg) \sigma^I.$$

We say $\phi \in C^\infty(\Omega : \mathfrak{C})$ if $\phi_I \in C^\infty(\Omega : \mathbb{C})$ for each $I \in \mathcal{I}$.

Remark 4.1.1 *If we use Banach-Grassmann algebra instead of Fréchet-Grassmann algebra, we need to check whether $\sum_{I \in \mathcal{I}} |\phi_I(q)| < \infty$, etc., which seems rather cumbersome.*

Lemma 4.1.2 *Let $\phi(t)$ and $\Phi(t)$ be continuous $\mathfrak{C}$-valued functions on an interval $[a, b] \subset \mathbb{R}$. Then,*
(1) $\int_a^b dt\, \phi(t)$ exists,
(2) if $\Phi'(t) = \phi(t)$ on $[a, b]$, then $\displaystyle\int_a^b dt\, \phi(t) = \Phi(b) - \Phi(a)$,
(3) if $\lambda \in \mathfrak{C}$ is a constant, then $\displaystyle\int_a^b dt\, (\phi(t) \cdot \lambda) = \bigg(\int_a^b dt\, \phi(t) \bigg) \cdot \lambda$ and $\displaystyle\int_a^b dt\, (\lambda \cdot \phi(t)) = \lambda \cdot \int_a^b dt\, \phi(t)$.

Moreover, we may generalize above lemma for a $\mathfrak{C}$-valued function $\phi(q)$ on an open set $\Omega \subset \mathbb{R}^m$.

Definition 4.1.3 *A set $U_{\rm ev} \subset \mathfrak{R}^{m|0} = \mathfrak{R}^m_{\rm ev}$ is called an even superdomain if $U_{\rm ev,B} = \pi_{\rm B}(U_{\rm ev}) \subset \mathbb{R}^m$ is open and connected and $\pi_{\rm B}^{-1}(\pi_{\rm B}(U_{\rm ev})) = U_{\rm ev}$. When $U \subset \mathfrak{R}^{m|n}$ is represented by $U = U_{\rm ev} \times \mathfrak{R}^n_{\rm od}$ with a even superdomain $U_{\rm ev} \subset \mathfrak{R}^{m|0}$, U is called a superdomain in $\mathfrak{R}^{m|n}$.*

Remark 4.1.4 *This definition of superdomain corresponds to the "saturated domain" which appears in Jadczyk and Pilch [122] and Hoyos et al [100]. That is, a subset U in $\mathfrak{R}^{m|n}$ is called saturated if it is G-connected (or convex), that is, for any $X \in U$, $\{\tilde{X}\} \cap U$ is connected (or convex) where $\tilde{X} = \{(\pi_{\mathrm{B}}(X), \theta) \,|\, \theta \in \mathfrak{R}^m_{\mathrm{od}}\}$.*

Not only this saturated domain but also the superdomain above doesn't seem suitable to construct "supermanifolds with non-trivial fermion sectors", though its existence is desired by Manin [152].

Proposition 4.1.5 *Let $U_{\mathrm{ev}} \subset \mathfrak{R}^{m|0}$ be a even superdomain. Assume that f is a smooth mapping from $U_{\mathrm{ev,B}} = \pi_{\mathrm{B}}(U_{\mathrm{ev}})$ into $\mathfrak{C}$, denoted simply by $f \in C^\infty(U_{\mathrm{ev,B}} : \mathfrak{C})$. That is, we have the expression*

$$f(q) = \sum_{J \in \mathcal{I}} f_J(q)\, \sigma^J \quad \text{with} \quad f_J(q) \in C^\infty(U_{\mathrm{ev,B}} : \mathbb{C}) \quad \text{for each } J \in \mathcal{I}. \tag{4.1}$$

Then, we may define a mapping $\tilde{f}$ of U_{ev} into $\mathfrak{C}$, called the Grassmann continuation of f, by

$$\tilde{f}(x) = \sum_{|\alpha| \geq 0} \frac{1}{\alpha!} \partial_q^\alpha f(x_{\mathrm{B}})\, x_{\mathrm{S}}^\alpha \quad \text{where} \quad \partial_q^\alpha f(x_{\mathrm{B}}) = \sum_J \partial_q^\alpha f_J(x_{\mathrm{B}})\, \sigma^J. \tag{4.2}$$

Here, we put $x = (x_1, \cdots, x_m)$, $x = x_{\mathrm{B}} + x_{\mathrm{S}}$ with $x_{\mathrm{B}} = (x_{1,\mathrm{B}}, \cdots, x_{m,\mathrm{B}}) = (q_1, \cdots, q_m) = q \in U_{ev,\mathrm{B}}$, $x_{\mathrm{S}} = (x_{1,\mathrm{S}}, \cdots, x_{m,\mathrm{S}})$ and $x^\alpha = x_1^{\alpha_1} \cdots x_m^{\alpha_m}$.

Proof. [Main point of this proposition is to see whether this mapping (4.2) is well-defined. Therefore, by using the degree argument, we need to define $\tilde{f}_{[k]}$, the k-th degree component of $\tilde{f}$.]

Denoting by $x_{1,\mathrm{S},[k_1]}$, the k_1-th degree component of $x_{1,\mathrm{S}}$, we get

$$(x_{1,\mathrm{S}}^{\alpha_1})_{[k_1]} = \sum (x_{1,\mathrm{S},[r_1]})^{p_{1,1}} \cdots (x_{1,\mathrm{S},[r_\ell]})^{p_{1,\ell}}.$$

Here, the summation is taken for all partitions of an integer α_1 into $\alpha_1 = p_{1,1} + \cdots + p_{1,\ell}$ satisfying $\sum_{i=1}^\ell r_i p_{1,i} = k_1$, $r_i \geq 0$. Using these notations, we put

$$\tilde{f}_{[k]}(x) = \sum \frac{1}{\alpha!} (\partial_q^\alpha f)_{[k_0]}(x_{\mathrm{B}})\, (x_{1,\mathrm{S}}^{\alpha_1})_{[k_1]} \cdots (x_{m,\mathrm{S}}^{\alpha_m})_{[k_m]} \tag{4.3}$$

where

$$(\partial_q^\alpha f)_{[k_0]}(x_{\mathrm{B}}) = \sum_{|J| = k_0} \partial_q^\alpha f_J(x_{\mathrm{B}})\, \sigma^J.$$

Or more precisely, we have

$$\begin{aligned}
\tilde{f}_{[0]}(x) &= f_{[0]}(x_{\mathrm{B}}), \\
\tilde{f}_{[1]}(x) &= f_{[1]}(x_{\mathrm{B}}), \\
\tilde{f}_{[2]}(x) &= f_{[2]}(x_{\mathrm{B}}) + \sum_{j=1}^m (\partial_{q_j} f)_{[0]}(x_{\mathrm{B}})(x_{j,\mathrm{S}})_{[2]}, \\
\tilde{f}_{[3]}(x) &= f_{[3]}(x_{\mathrm{B}}) + \sum_{j=1}^m (\partial_{q_j} f)_{[1]}(x_{\mathrm{B}})(x_{j,\mathrm{S}})_{[2]}, \\
\tilde{f}_{[4]}(x) &= f_{[4]}(x_{\mathrm{B}}) + \sum_{j=1}^m (\partial_{q_j} f)_{[2]}(x_{\mathrm{B}})(x_{j,\mathrm{S}})_{[2]} \\
&\quad + \frac{1}{2} \sum_{j=1}^m (\partial_{q_j}^2 f)_{[0]}(x_{\mathrm{B}})(x_{j,\mathrm{S}}^2)_{[4]} + \sum_{j \neq k} (\partial_{q_j\, q_k}^2 f)_{[0]}(x_{\mathrm{B}})(x_{j,\mathrm{S}})_{[2]}(x_{k,\mathrm{S}})_{[2]}, \quad \textit{etc.}
\end{aligned}$$

Since $\tilde{f}_{[j]}(x) \neq \tilde{f}_{[k]}(x)$ $(j \neq k)$ in $\mathfrak{C}$, we may take the sum $\sum_{j=0}^\infty \tilde{f}_{[j]}(x) \in \mathfrak{C} = \oplus_{k=0}^\infty \mathfrak{C}_{[k]}$, which is denoted by $\tilde{f}(x)$. Therefore, rearranging the above 'summation', we get rather the 'familiar' expression as in (4.2).

□

Remark 4.1.6 *(1) More primitively, we may represent* $\tilde f(x) = \sum_H \tilde f_H(x)\sigma^H$ *where*

$$\tilde f_H(x) = \sum_{\substack{H=J+I_1^{(1)}+\cdots+I_m^{(\alpha_m)}\\ \alpha=(\alpha_1,\cdots,\alpha_m)}} (-1)^{\tau(*)}\frac{1}{\alpha!}\partial_q^\alpha f_J(x_{\mathrm{B}})x_{1,I_1^{(1)}}\cdots x_{m,I_m^{(\alpha_m)}}$$

but this representation obscures the form of $\tilde f$ *comparing with* (4.2).
(2) Defining H^∞*-functions, Rogers [188] used* C^∞*-functions with values in* $\mathbb{R}$ *defined on an open connected set* U *in her* ℓ^1*-topology.*
(3) Even if $f(q) \in \mathcal{D}'(\mathbb{R})$*, we may define* $\tilde f$ *rather easily. This point will be re-explained after defining integration and suitable function spaces.*

Corollary 4.1.7 *If* f *and* $\tilde f$ *be given as above, then (i)* $\tilde f$ *is continuous and (ii)* $\tilde f(x) = 0$ *in* U_{ev} *implies* $f(x_{\mathrm{B}}) = 0$ *in* $U_{\mathrm{ev,B}}$*. Moreover, if we define the partial derivatives of* $\tilde f$ *by*

$$\partial_{x_j}\tilde f(x) = \frac{d}{dt}\tilde f(x+te_{(j)})\Big|_{t=0} \quad \textit{where} \quad e_{(j)} = (\overbrace{0,\cdots,0,1}^{j},0,\cdots,0) \in \mathfrak{R}^{m|0}, \tag{4.4}$$

then we get

$$\partial_{x_j}\tilde f(x) = \widetilde{\partial_{q_j} f}(x) \quad \textit{for} \quad j=1,\cdots,m. \tag{4.5}$$

Proof. Let $y_j = y_{j,\mathrm{B}} + y_{j,\mathrm{S}} \in \mathfrak{R}_{\mathrm{ev}}$. For $y_{(j)} = y_j e_{(j)} = y_{j,\mathrm{B}}e_{(j)} + y_{j,\mathrm{S}}e_{(j)} = y_{(j),\mathrm{B}} + y_{(j),\mathrm{S}} \in \mathfrak{R}^{m|0}$, as

$$\frac{d}{dt}\tilde f(x+ty_{(j)}) = \frac{d}{dt}\left\{\sum_\alpha \frac{1}{\alpha!}\left(\sum_J \partial_q^\alpha f_J(x_{\mathrm{B}}+ty_{(j),\mathrm{B}})\sigma^J\right)(x_{\mathrm{S}}+ty_{(j),\mathrm{S}})^\alpha\right\},$$

we get easily,

$$\begin{aligned}\frac{d}{dt}\tilde f(x+ty_{(j)})\Big|_{t=0} &= y_{(j),\mathrm{B}}\sum_\alpha \frac{1}{\alpha!}\left(\sum_J \partial_q^\alpha f_J(x_{\mathrm{B}})\sigma^J\right)x_{\mathrm{S}}^\alpha + y_{(j),\mathrm{S}}\sum_{\breve\alpha}\frac{1}{\breve\alpha!}\left(\sum_J \partial_q^{\breve\alpha}\partial_{q_j} f_J(x_{\mathrm{B}})\sigma^J\right)x_{\mathrm{S}}^{\breve\alpha}\\ &= y_j\sum_\alpha \frac{1}{\alpha!}\partial_q^\alpha\partial_{q_j} f(x_{\mathrm{B}})\,x_{\mathrm{S}}^\alpha = y_j\widetilde{\partial_{q_j} f}(x).\end{aligned}$$

Here $\breve\alpha = (\alpha_1,\cdots,\alpha_j-1,\cdots,\alpha_m)$. Putting $y_j = 1$ in the above, we have (4.5). □

Remark 4.1.8 *By the same argument as above, we get, for* $y = (y_1,\cdots,y_m) \in \mathfrak{R}^{m|0}$,

$$\frac{d}{dt}\tilde f(x+ty)\Big|_{t=0} = \sum_{j=1}^m y_j\sum_\alpha \frac{1}{\alpha!}\partial_q^\alpha\partial_{q_j} f(x_{\mathrm{B}})\,x_{\mathrm{S}}^\alpha = \sum_{j=1}^m y_j\partial_{x_j}\tilde f(x). \tag{4.6}$$

Remark 4.1.9 *Let* $(X, \|\cdot\|_X)$ *and* $(Y, \|\cdot\|_Y)$ *be two Banach spaces. A function* $\Phi : X \to Y$ *is called* **Gâteaux-differentiable** *at* $x \in X$ *in the direction* $h \in X$ *if there exists an element* $\Phi'_G(x;h) \in Y$ *such that*

$$\|\Phi(x+th) - \Phi(x) - t\Phi'_F(x;h)\|_Y \to 0 \quad \textit{when} \quad t \to 0.$$

$\Phi'_F(x;h)$ *is also denoted by* $\Phi'_G(x)(h)$, $(d\Phi(x))(h)$*. Moreover,* $\Phi : X \to Y$ *is called* **Fréchet-differentiable** *at* $x \in X$ *if there exist a linear bounded operator* $\Phi'_F(x) : X \to Y$ *and an element* $\tau(x,h) \in Y$ *such that*

$$\Phi(x+h) - \Phi(x) - \Phi'_F(x)h = \tau(x,h) \quad \textit{with} \quad \frac{\|\tau(x,h)\|}{\|h\|} \to 0 \quad \textit{when} \quad 0 \neq \|h\| \to 0.$$

Problem: How to extend these notion of differentiability for two Fréchet-Grassmann spaces? Replace the norm by the distance in superspaces!

Therefore, in other word, (4.6) implies that $\tilde{f}$, the Grassmann extension of f, is super Fréchet-differentiable at x in the direction y, that is, there exists $F_j(x) \in \mathfrak{C}$ such that for each y

$$\tilde{f}(x+ty) - \tilde{f}(x) - t\sum_{j=1}^{m} y_j F_j(x) \to 0 \quad \text{in } \mathfrak{C} \quad \text{when} \quad t \to 0. \tag{4.7}$$

That is, by super Fréchet-differentiability at x in the direction y, there exists a linear operator $\tilde{f}'_F(x,\cdot) : \mathfrak{C}^m_{\text{ev}} \to \mathfrak{C}^m_{\text{ev}}$ such that when $t \to 0$,

$$\tilde{f}(x+ty) - \tilde{f}(x) - t\tilde{f}'_F(x,y) \to 0 \quad \text{in } \mathfrak{C} \quad \text{with} \quad \tilde{f}'_F(x,y) = \sum_{j=1}^{m} y_j F_j(x).$$

Moreover, $\tilde{f}$, the Grassmann extension of f, satisfies the following (see Matsumoto & Kakazu [158]):

Lemma 4.1.10 *Let f be real analytic on $\mathbb{R}^m$. Then, its Grassmann extension $\tilde{f}$ is super G-differentiable at x, i.e. there exist $F_j(x) \in \mathfrak{C}$ and $\epsilon_j(x,y) \in \mathfrak{C}$ such that*

$$\tilde{f}(x+y) = \tilde{f}(x) + \sum_{j=1}^{m} y_j F_j(x) + \sum_{j=1}^{m} \epsilon_j(x,y) y_j, \quad \textit{with} \quad \epsilon_j(x,y) \to 0 \quad \textit{in } \mathfrak{C} \quad \textit{when} \quad y \to 0 \quad \textit{in } \mathfrak{C}.$$

Proof. For the sake of simplicity, we consider only the case $m = 1$. Then, we have

$$\begin{aligned}
\tilde{f}(x+y) &= \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x_{\mathrm{B}} + y_{\mathrm{B}})(x_{\mathrm{S}} + y_{\mathrm{S}})^n \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \Big(\sum_{\ell=0}^{\infty} \frac{1}{\ell!} f^{(\ell+n)}(x_{\mathrm{B}}) y_{\mathrm{B}}^{\ell} \Big) \Big(\sum_{k=0}^{n} \frac{n!}{k!(n-k)!} x_{\mathrm{S}}^{n-k} y_{\mathrm{S}}^{k} \Big) \quad \text{by real analyticity of } f(q), \\
&= \sum_{n=0}^{\infty} \Big(\sum_{\ell=0}^{\infty} \frac{1}{\ell!} f^{(\ell+j+k)}(x_{\mathrm{B}}) y_{\mathrm{B}}^{\ell} \Big) \Big(\sum \frac{1}{k!j!} x_{\mathrm{S}}^{j} y_{\mathrm{S}}^{k} \Big) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \Big(\sum_{j=0}^{\infty} \frac{1}{j!} f^{(n+j)}(x_{\mathrm{B}}) x_{\mathrm{S}}^{j} \Big) \Big(\sum \frac{n!}{\ell!k!} y_{\mathrm{B}}^{\ell} y_{\mathrm{S}}^{k} \Big) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \Big(\sum_{j=0}^{\infty} \frac{1}{j!} f^{(n+j)}(x_{\mathrm{B}}) x_{\mathrm{S}}^{j} \Big) (y_{\mathrm{B}} + y_{\mathrm{S}})^n \quad \text{by putting } n = \ell + k, \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \tilde{f}^{(n)}(x) y^n.
\end{aligned}$$

Therefore,

$$\tilde{f}(x+y) - \tilde{f}(x) = \sum_{n=1}^{\infty} \frac{1}{n!} \tilde{f}^{(n)}(x) y^n = \tilde{f}^{(1)}(x) y + [\sum_{n=2}^{\infty} \frac{1}{n!} \tilde{f}^{(n)}(x) y^{n-1}] y,$$

with

$$\epsilon(x,y) = \sum_{n=2}^{\infty} \frac{1}{n!} \tilde{f}^{(n)}(x) y^{n-1} \to 0 \quad \text{in } \mathfrak{C} \quad \text{when} \quad y \to 0. \qquad \square$$

More generally,

Lemma 4.1.11 *Let $f(q) \in C^\infty(\mathbb{R}^m)$, we have the Taylor expansion for $\tilde{f}$: For any N, there exists $\tilde{\tau}_N(x,y) \in \mathfrak{C}$ such that*

$$\tilde{f}(x+y) = \sum_{|\alpha|=0}^{N} \frac{1}{\alpha!} \tilde{f}^{\alpha}(x) y^{\alpha} + \tilde{\tau}_N(x,y), \tag{4.8}$$

with

$$\tilde{\tau}_N(x,y) = \sum_{|\alpha|=N+1} (x-y)^\alpha \int_0^1 dt \frac{1}{N!}(1-t)^N \partial_x^\alpha \tilde{f}(y+t(x-y)).$$

Proof. Put $q = x_{\rm B}$, $q' = y_{\rm B}$. For any N, we have

$$\begin{aligned}
\sum_{|\alpha|=N+1} (q-q')^\alpha \int_0^1 dt \frac{1}{N!}(1-t)^N \partial_q^\alpha f(q'+t(q-q')) \\
&= \int_0^1 dt \frac{(1-t)^N}{N!} (\frac{d}{dt})^{N+1} f(q'+t(q-q')) \\
&= -\frac{1}{N!} \sum_{|\alpha|=N} (q-q')^\alpha \partial_q^\alpha f(q) + \int_0^1 dt \frac{(1-t)^{N-1}}{(N-1)!} (\frac{d}{dt})^N f(q'+t(q-q')) \\
&= f(q) - \sum_{|\alpha|=0}^N \frac{1}{\alpha!}(q-q')^\alpha \partial_q^\alpha f(q').
\end{aligned}$$

Extending both sides above, we have the desired result. □

Notation: Hereafter, for the sake of notational simplicity, $\tilde{f}$ **is denoted simply by** f unless there occurs confusion.

Definition 4.1.12 *(1) For a given even superdomain* $U_{\rm ev} \subset \mathfrak{R}^{m|0}$*, a mapping* $\tilde{f}$ *from* $U_{\rm ev}$ *into* $\mathfrak{C}$ *is called a supersmooth function if* $\tilde{f}$ *is the Grassmann continuation of a smooth mapping* f *from* $U_{\rm ev,B} = \pi_{\rm B}(U_{\rm ev})$ *into* $\mathfrak{C}$*. We denote by* $\mathcal{C}_{\rm SS}(U_{\rm ev} : \mathfrak{C})$*, the set of supersmooth functions on* $U_{\rm ev}$*.*

(2) A mapping f *from a superdomain* $U \subset \mathfrak{R}^{m|n}$ *to* $\mathfrak{C}$ *is called supersmooth, if it has the following form:*

$$f(x,\theta) = \sum_{|a|\le n} f_a(x)\theta^a \tag{4.9}$$

with $a = (a_1, \cdots a_n) \in \{0,1\}^n$*,* $\theta^a = \theta_1^{a_1}\cdots\theta_n^{a_n}$ *and* $f_a(x) \in \mathcal{C}_{\rm SS}(U_{\rm ev} : \mathfrak{C})$*. In the following, supersmooth functions are assumed to be homogeneous (i.e.,* $f_a(x)$ *is homogeneous for each* a*), unless otherwise mentioned and we denote the set of them by* $\mathcal{C}_{\rm SS}(U : \mathfrak{C})$*. Moreover, we put*

$$\not{\mathcal{C}}_{\rm SS} = \{f(x,\theta) \in \mathcal{C}_{\rm SS}(U : \mathfrak{C}) \mid f_a(x) \in \mathbb{C}\}.$$

(3) For $f \in \mathcal{C}_{\rm SS}(U : \mathfrak{C})$*,* $j = 1,2,\cdots,m$ *and* $s = 1,2,\cdots,n$*, we put*

$$\begin{cases} F_j(X) = \displaystyle\sum_{|a|\le n} \partial_{x_j} f_a(x)\theta^a, \\ F_{s+m}(X) = \displaystyle\sum_{|a|\le n} (-1)^{l(a)+p(f_a(x))} f_a(x)\theta_1^{a_1}\cdots\theta_s^{a_s-1}\cdots\theta_n^{a_n} \end{cases} \tag{4.10}$$

where $l(a) = \sum_{j=1}^{s-1} a_j$ *and* $\theta_s^{-1} = 0$*.* $F_\kappa(X)$ *are called the partial derivatives of* f *with respect to* X_κ *at* $X = (x,\theta)$ *and are denoted by*

$$\begin{cases} F_j(X) = \dfrac{\partial}{\partial x_j} f(x,\theta) = \partial_{x_j} f(x,\theta) = f_{x_j}(x,\theta) \quad \text{for} \quad j = 1,2,\cdots,m, \\ F_{m+s}(X) = \dfrac{\partial}{\partial \theta_s} f(x,\theta) = \partial_{\theta_s} f(x,\theta) = f_{\theta_s}(x,\theta) \quad \text{for} \quad s = 1,2,\cdots,n \end{cases} \tag{4.11}$$

or simply by

$$F_\kappa(X) = \partial_{X_\kappa} f(X) = f_{X_\kappa}(X) \quad \text{for} \quad \kappa = 1,\cdots,m+n. \tag{4.12}$$

Remark 4.1.13 *(1) We only use the derivatives defined above which are called the left derivatives with respect to odd variables. Because, after bringing the variable θ_k to the left in each monomial, we replace it with* 1*. (Some people call these as right derivatives, cf. Vladimirov and Volovivh [214]etc.) Similarly, we define the right derivatives with respect to odd variables as follows: For $f \in \mathcal{C}_{\rm SS}(U:\mathfrak{C})$, $j=1,2,\cdots,m$ and $s=1,2,\cdots,n$, we put*

$$\begin{cases} F_j^{(r)}(X) = \sum_{|a|\le n} \partial_{x_j} f_a(x)\theta^a, \\ F_{s+m}^{(r)}(X) = \sum_{|a|\le n} (-1)^{r(a)} f_a(x)\theta_1^{a_1}\cdots\theta_s^{a_s-1}\cdots\theta_n^{a_n} \end{cases}$$

where $r(a) = \sum_{j=s+1}^n a_j$. $F_\kappa^{(r)}(X)$ are called the (right) partial derivatives of f with respect to X_κ at $X=(x,\theta)$ and are denoted by

$$F_j^{(r)}(X) = \frac{\partial}{\partial x_j} f(x,\theta) = \partial_{x_j} f(x,\theta), \quad F_{m+s}^{(r)}(X) = f(x,\theta)\frac{\overleftarrow{\partial}}{\partial\theta_s} = f(x,\theta)\overleftarrow{\partial}_{\theta_s}$$

for $j=1,2,\cdots,m$ and $s=1,2,\cdots,n$.
(2) As we use the infinite dimensional Grassmann algebras, the expression (4.10) *is unique. In fact, $\sum_a f_a(x)\theta^a \equiv 0$ on U implies $f_a(x)\equiv 0$ (see, p 322 in Vladimirov and Volovich [214].)*
(3) The higher derivatives are defined analogously and we use the following notations.

$$\partial_x^\alpha = \partial_{x_1}^{\alpha_1}\cdots\partial_{x_m}^{\alpha_m} \quad and \quad \partial_\theta^a = \partial_{\theta_1}^{a_1}\cdots\partial_{\theta_n}^{a_n},$$

for multi-indeces $\alpha=(\alpha_1,\cdots,\alpha_m)\in(\mathbb{N}\cup\{0\})^m$ and $a=(a_1,\cdots,a_n)\in\{0,1\}^n$.

Repeating the argument in proving Corollary 4.1.7, we get the following formula for $f\in\mathcal{C}_{\rm SS}(U:\mathfrak{C})$:

$$\frac{d}{dt}f(X+tY)\bigg|_{t=0} = \sum_{j=1}^m y_j\frac{\partial}{\partial x_j}f(X) + \sum_{s=1}^m \omega_s\frac{\partial}{\partial\theta_s}f(X) \tag{4.13}$$

where $X=(x,\theta), Y=(y,\omega)\in\mathfrak{R}^{m|n}$ such that $X+tY\in U$ for any $t\in[0,1]$.

Definition 4.1.14 *A function f from a superdomain $U\subset\mathfrak{R}^{m|n}$ to $\mathfrak{C}$ is called G-differentiable at $X=(x,\theta)$ if*

$$f(x+y,\theta+\omega)-f(x,\theta) = \sum(y_iF_i+\omega_sF_s)+\sum(y_iR_i+\omega_sR_s),$$
$$d(R_i,0)\to 0, \quad d(R_s,0)\to 0, \quad d_{m|n}((y,\omega),0)\to 0.$$

To understand the meaning of supersmoothness, we consider the dependence with respect to the "coordinate" more precisely.

Proposition 4.1.15 *Let $f(X)=\sum_I f_I(X)\sigma^I \in \mathcal{C}_{\rm SS}(U:\mathfrak{C})$ where U is a superdomain in $\mathfrak{R}^{m|n}$. Let $X=(X_\kappa)$ be represented by $X_\kappa=\sum_I X_{\kappa,I}\sigma^I$ where $\kappa=1,\cdots,m+n$, $X_{\kappa,I}\in\mathbb{C}$ for $|I|\ne 0$ and $X_{\kappa,\tilde{0}}\in\mathbb{R}$. Then, $f(X)$, considered as a function of countably many variables $\{X_{\kappa,I}\}$ with values in $\mathfrak{C}$, satisfies the following (Cauchy-Riemann type) equations.*

$$\begin{cases} \dfrac{\partial}{\partial X_{\kappa,I}}f(X) = \sigma^I\dfrac{\partial}{\partial X_{\kappa,\tilde{0}}}f(X) \quad for \quad 1\le\kappa\le m,\ |I| = even, \\ \sigma^K\dfrac{\partial}{\partial X_{\kappa,J}}f(X) + \sigma^J\dfrac{\partial}{\partial X_{\kappa,K}}f(X) = 0 \\ \qquad\qquad\qquad\qquad for \quad m+1\le\kappa\le m+n,\ |J| = odd = |K|. \end{cases} \tag{4.14}$$

Here, we define

$$\frac{\partial}{\partial X_{\kappa,I}} f(X) = \frac{d}{dt} f(X + tY_{(\kappa,I)})\Big|_{t=0}, \tag{4.15}$$

with $Y_{(\kappa,I)} = (\overbrace{0,\cdots,0,\sigma^I}^{\kappa},0,\cdots,0) \in \mathfrak{R}^{m|n}$.

Conversely, let a function $f(X) = \sum_I f_I(X)\sigma^I$ *be given such that* $f_I(X+tY) \in C^\infty([0,1]:\mathbb{C})$ *for each fixed* $X, Y \in U$ *and* $f(X)$ *satisfies the following equations:*

$$\begin{cases} \frac{d}{dt} f(X+tY_{(\kappa,J)})\Big|_{t=0} = \frac{d}{dt} f(X+tY_{(\kappa,\tilde{0})})\Big|_{t=0} \sigma^J \\ \qquad\qquad \text{for} \quad 1 \le \kappa \le m, \quad \text{and} \quad |J| = even \\ \sigma^K \frac{d}{dt} f(X+tY_{(\kappa,J)})\Big|_{t=0} + \sigma^J \frac{d}{dt} f(X+tY_{(\kappa,K)})\Big|_{t=0} = 0 \\ \qquad\qquad \text{for} \quad \kappa = m+1,\cdots,m+n, \quad \text{and} \quad |J| = odd = |K| \end{cases} \tag{4.16}$$

where we put $Y_{(\kappa,J)} = (\overbrace{0,\cdots,0,\sigma^J}^{\kappa},0,\cdots,0) \in \mathfrak{R}^{m|n}$ *and* $Y_{(\kappa,0)} = (\overbrace{0,\cdots,0,1}^{\kappa},0,\cdots,0) \in \mathfrak{R}^{m|n}$. *Then,* $f \in \mathcal{C}_{\mathrm{SS}}(U:\mathfrak{C})$.

Proof. Replacing Y with $Y_{(\kappa,J)}$ with $1 \le \kappa \le m$ and $|J|$ =even in (4.13), we get readily the first equation of (4.14). Here, we have used (4.5). Considering $Y_{(\kappa,J)}$ or $Y_{(\kappa,K)}$ for $m+1 \le \kappa \le m+n$ and $|J| = \text{odd} = |K|$ in (4.13) and multiplying σ^K or σ^J from left, respectively, we have the second equality in (4.14) readily.

To prove the converse statement, we have to construct functions $F_i (1 \le i \le m+n)$ such that

$$\frac{d}{dt} f(X+tH)\big|_{t=0} = \sum_{i=1}^{m+n} F_i(X) H_i$$

for $X \in U$ and $H \in \mathfrak{R}^{m|n}$.

From the first equation of (4.16), we have

$$F_\kappa(X) = \frac{\partial}{\partial X_{\kappa,\tilde{0}}} f(X) \quad \text{for} \quad 1 \le \kappa \le m \; X \in U.$$

To define $F_\kappa(X)(m+1 \le \kappa \le m+n)$, we need to prepare the following lemma.

Lemma 4.1.16 *Let* $\{F_I \in \mathfrak{C} : I \in \mathcal{I}_1\}$ $(\mathcal{I}_1 = \{I \in \mathcal{I} \,|\, |I| = 1\})$ *be a set of supernumbers such that* $F_I \sigma^J + F_J \sigma^I = 0$ *for any* $I, J \in \mathcal{I}_1$. *Then there exists a unique supernumber* $F \in \mathfrak{C}$ *such that* $F_I = F\sigma^I$ *for each* $I \in \mathcal{I}_1$.

Taking this lemma in granted and from the second equation of (4.16), we have an element $F_\kappa(X)(m+1 \le \kappa \le m+n)$ such that

$$F_\kappa(X)\sigma^J = \frac{\partial}{\partial X_{\kappa,J}} f(X).$$

Using these, we claim that

$$\frac{d}{dt} f(X+tH)\big|_{t=0} = \sum_{\kappa=1}^{m+n} F_\kappa(X) H^\kappa.$$

Therefore, for $m+1 \le \kappa \le m+n$ and $|J| = odd$, putting $H = X_{\kappa,J}$, we get the desired result. □

Proof of Lemma. Put $\{F_{(i)} = \sum_{J \in \mathcal{I}_1} a^i_J \sigma_i\}$. If this set satisfies the assumption in the lemma, we have $F_{(i)}\sigma_i = 0$ which implies $\sum_{i \notin J \in \mathcal{I}_1} a^i_J \sigma_i = 0$. Therefore each $F_{(i)}$ can be written uniquely as

$F_{(i)} = (\sum_{i\notin J\in\mathcal{I}_1} b^i_J)\sigma_i$ with some $b^i_J \in \mathbb{C}$. Then the condition that $F_{(i)}\sigma_j = F_{(j)}\sigma_i$ holds for any i, j implies that $b^i_J = b^j_J$ for $i, j \notin J \in \mathcal{I}_1$. Letting $b_J = b^i_J$ for $i \notin J \in \mathcal{I}_1$, $F = \sum_{J\in\mathcal{I}_1} \sigma^J$ is well-defined and $F_{(i)} = F\sigma_i$. □

Remark 4.1.17 *In order to obtain the converse statement of Proposition 4.1.15 (see Vladimirov and Volovich [214], Yagi [222], it seems better to modify a general theory of differential calculus on locally convex spaces developped in Keller [129], Yamamuro [225] etc. For example, we may introduce "k-times super Fréchet or Gâteaux-differentiability" as similar as proposed in Hoyos et al. [100], Boyer and Gitler [35] or Jadczyk and Pilch [122].but this will not be pursued here.*

The reason why we start with the Grassmann continuation of smooth functions is here because we don't know the results of differential calculus on locally convex spaces. Concerning this and the definition of supermanifolds, we will study in our forthcoming works. This will clarify the relation with some notions, G^∞, H^∞ smoothness, developped by many others.

4.2 Elementary differential calculus

Definition 4.2.1 *For a supersmooth function f, we define df by*

$$df(X) = d_X f(X) = \sum_{\kappa=1}^{m+n} dX_\kappa \frac{\partial f(X)}{\partial X_\kappa},$$

or

$$df(x,\theta) = \sum_{j=1}^{m} dx_j \frac{\partial f(x,\theta)}{\partial x_j} + \sum_{s=1}^{n} d\theta_s \frac{\partial f(x,\theta)}{\partial \theta_s}.$$

From Definition 4.1.12, we get readily

Proposition 4.2.2 *Let U be a superdomain in $\mathfrak{R}^{m|n}$. For $f, g \in \mathcal{C}_{\rm SS}(U:\mathfrak{C})$, the product fg belongs to $\mathcal{C}_{\rm SS}(U:\mathfrak{C})$ and the differentials $d_X f(X)$ and $d_X g(X)$ may be regarded as continuous linear mappings from $\mathfrak{R}^{m|n}$ into $\mathfrak{C}^{m+n}$. Moreover, they satisfy the following:*

(1) For any homogeneous elements λ, $\mu \in \mathfrak{C}$, we have

$$d_X(\lambda f + \mu g)(X) = (-1)^{p(\lambda)p(X)}\lambda d_X f(X) + (-1)^{p(\mu)p(X)}\mu d_X g(X). \tag{4.17}$$

(2) (Leibnitz formula)

$$\partial_{X_\kappa}[f(X)g(X)] = (\partial_{X_\kappa} f(X))g(X) + (-1)^{p(X_\kappa)p(f(X))} f(X)(\partial_{X_\kappa} g(X)). \tag{4.18}$$

Proof. (4.17) is obvious. For the product, as we get

$$(fg)(x_{\rm B}) = \left(\sum_I f_I(x_{\rm B})\sigma^I\right)\left(\sum_J g_J(x_{\rm B})\sigma^J\right) = \sum_H h_H(x_{\rm B})\sigma^H$$

where $h_H(x_{\rm B}) = \sum_{H=I+J}(-1)^{\tau(H;I,J)} f_I(x_{\rm B}) g_J(x_{\rm B}) \in C^\infty(U_{\rm B}:\mathbb{C})$, so we have the desired result (4.18), by using the formula (2.19). □

Proposition 4.2.3 (Taylor's formula) *Let $X = (x,\theta), Y = (y,\omega) \in U \subset \mathfrak{R}^{m|n}$ satisfying $Y + t(X - Y) \in U$ for $0 \le t \le 1$. For $f \in \mathcal{C}_{\rm SS}(U:\mathfrak{C})$, Taylor's formula holds. That is, for any positive integer p, we have*

$$f(x,\theta) - \sum_{|\alpha|+|a|\le p,\ |a|\le n} \frac{1}{\alpha!}(x-y)^\alpha(\theta-\omega)^a \partial_x^\alpha \partial_\theta^a f(y,\omega) = \tau_p(X,Y) \tag{4.19}$$

where

$$\tau_p(X,Y) = \sum_{|\alpha|+|a|=p+1,\ |a|\le n} (x-y)^\alpha(\theta-\omega)^a \int_0^1 dt\, \frac{1}{p!}(1-t)^p \partial_x^\alpha \partial_\theta^a f(y+t(x-y),\omega+t(\theta-\omega)). \quad (4.20)$$

Proof. Use the following equality

$$\int_0^1 dt\, \frac{(1-t)^p}{p!}\left(\frac{d}{dt}\right)^{p+1} f(y+t(x-y),\omega+t(\theta-\omega))$$
$$= \sum_{|\alpha|+|a|=p+1} (x-y)^\alpha(\theta-\omega)^a \int_0^1 dt\, \frac{1}{p!}(1-t)^p \partial_x^\alpha \partial_\theta^a f(y+t(x-y),\omega+t(\theta-\omega)).$$

Using the integration by parts in the left hand side, we get that of (4.20). □

Definition 4.2.4 *Let $U \subset \mathfrak{R}^{m|n}$ and $U' \subset \mathfrak{R}^{m'|n'}$ be superdomains and let φ be a continuous mapping from U to U', denoted by $\varphi(X) = (\varphi_1(X),\cdots,\varphi_{m'}(X),\varphi_{m'+1}(X),\cdots,\varphi_{m'+n'}(X)) \in \mathfrak{R}^{m'|n'}$. φ is called a supersmooth mapping from U to U' if each $\varphi_\kappa(X) \in \mathcal{C}_{\rm SS}(U:\mathfrak{C})$ for $\kappa = 1,\cdots,m'+n'$ and $\varphi(U) \subset U'$.*

Proposition 4.2.5 (Composition of supersmooth mappings) *Let $U \subset \mathfrak{R}^{m|n}$ and $U' \subset \mathfrak{R}^{m'|n'}$ be superdomains and let $\Phi : U \to U'$ and $\Phi' : U' \to \mathfrak{R}^{m''|n''}$ be supersmooth mappings. Then, the composition $\Psi = \Phi' \circ \Phi : U \to \mathfrak{R}^{m''|n''}$ gives a supersmooth mapping and*

$$d_X\Psi(X) = [d_Y\Phi'(Y)]\big|_{Y=\Phi(X)}[d_X\Phi(X)]. \quad (4.21)$$

Proof. (1) First of all, we prove our assertion for the case m, m' are arbitrary, $n = n' = 0$ and $m'' = n'' = 1$: Let $U_{ev} \subset \mathfrak{R}^{m|0}$ and $U'_{ev} \subset \mathfrak{R}^{m'|0}$ be even superdomains and let $\varphi : U_{ev} \to U'_{ev}$ be a supersmooth mapping represented by $\varphi(x) = (\varphi_1(x),\cdots,\varphi_{m'}(x))$ with $\varphi_j(x) \in \mathcal{C}_{\rm SS}(U_{ev}:\mathfrak{C})$. For any $f \in \mathcal{C}_{\rm SS}(U'_{ev}:\mathfrak{C})$, we want to claim that $(\varphi^* f)(x) = (f\circ\varphi)(x) = f(\varphi(x))$, is well-defined and belongs to $\mathcal{C}_{\rm SS}(U_{ev}:\mathfrak{C})$. Putting

$$y = \varphi(x_{\rm B}) = \varphi_{\rm B}(x_{\rm B}) + \varphi_{\rm S}(x_{\rm B}) = y_{\rm B} + y_{\rm S} \quad \text{with} \quad \varphi_{\rm S}(x_{\rm B}) = \sum_{|J|\ge 1} \varphi_J(x_{\rm B})\sigma^J,$$

we define, by using the supersmoothness of f and φ,

$$f(\varphi(x_{\rm B}))_{[k]} = \sum_{\substack{|\alpha|\le k,\\ k_0+k_1+\cdots+k_m=k}} \frac{1}{\alpha!}(\partial_y^\alpha f)_{[k_0]}(y_{\rm B})(y_{1,{\rm S}}^{\alpha_1})_{[k_1]}\cdots(y_{m,{\rm S}}^{\alpha_m})_{[k_m]}\Big|_{y=\varphi(x_{\rm B})}. \quad (4.22)$$

By the same reasoning as in the proof of Proposition 4.1.5, $f(\varphi(x_{\rm B}))_{[k]}$ is well-defined and belongs to $C^\infty(U_{\rm B}:\mathfrak{C}_{[k]})$, so $f(\varphi(x_{\rm B})) = \sum_{k=0}^\infty f(\varphi(x_{\rm B}))_{[k]} \in C^\infty(U_{\rm B}:\mathfrak{C})$. Therefore, it has the Grassmann continuation which should be denoted by $(f\circ\varphi)(x)$. On the other hand, as we get from (4.5),

$$\begin{aligned}
&\partial_{x_{j,{\rm B}}}(f\circ\varphi)_{[k]}(x_{\rm B})\\
&= \sum_{\substack{\ell,\,|\alpha|\le k,\\ k_0+k_1+\cdots+k_m=k}} \frac{1}{\alpha!}(\partial_y^\alpha\partial_{y_\ell} f)_{[k_0]}(y_{\rm B})\frac{\partial\varphi_{\ell,{\rm B}}(x_{\rm B})}{\partial x_{j,{\rm B}}}(y_{1,{\rm S}}^{\alpha_1})_{[k_1]}\cdots(y_{m,{\rm S}}^{\alpha_m})_{[k_m]}\Big|_{y=\varphi(x_{\rm B})}\\
&= \sum_{\substack{\ell,\,|\alpha|\le k,\\ k_0+k_1+\cdots+\\ k'_\ell+k''_\ell+\cdots+k_m=k}} \frac{1}{\alpha!}(\partial_y^\alpha\partial_{y_\ell} f)_{[k_0]}(y_{\rm B})\\
&\qquad\times (y_{1,{\rm S}}^{\alpha_1})_{[k_1]}\cdots\alpha_\ell(y_{\ell,{\rm S}}^{\alpha_\ell-1})_{[k'_\ell]}\frac{\partial\varphi_{\ell,{\rm S}}(x_{\rm B})}{\partial x_{j,{\rm B}}}\Big)_{[k''_\ell]}\cdots(y_{m,{\rm S}}^{\alpha_m})_{[k_m]}\Big|_{y=\varphi(x_{\rm B})}\\
&= \sum_\ell\sum_{k_0=0}^k \Big(\partial_{y_\ell} f(\varphi(x_{\rm B}))\Big)_{[k_0]}\left(\frac{\partial\varphi_\ell(x_{\rm B})}{\partial x_{j,{\rm B}}}\right)_{[k-k_0]}.
\end{aligned} \quad (4.23)$$

This is the desired result (4.21) in the case of (1).
(2) Now, we treat the case $m,\ m',\ n,\ n'$ are arbitrary and $m''=n''=1$: Let $U\subset\mathfrak{R}^{m|n}$ and $U'\subset\mathfrak{R}^{m'|n'}$ be superdomains and let $\varphi:U\to U'$ and $f:U'\to\mathfrak{C}$ be supersmooth mappings. Put $\varphi(x,\theta)=(\varphi_\kappa(x,\theta))$, $1\le\kappa\le m'+n'$ where $\varphi_\kappa(x,\theta)=\sum_a\varphi_{\kappa,a}(x)\theta^a$ and $f(y,\omega)=\sum_b f_b(y)\omega^b$ with $b=(b_1,\cdots,b_{n'})\in\{0,1\}^{n'}$. We decompose

$$\varphi_j(x,\theta)=Y_j=Y_j^{(0)}+Y_j^{(1)}\quad\text{for}\quad 1\le j\le m'$$

where

$$\begin{cases}Y_j^{(0)}=\varphi_{j,\tilde{0}}(x)=Y_{j,\mathrm{B}}^{(0)}+Y_{j,\mathrm{S}}^{(0)}\quad\text{with}\quad Y_{j,\mathrm{B}}^{(0)}=\varphi_{j,\tilde{0},\mathrm{B}}(x),Y_{j,\mathrm{S}}^{(0)}=\varphi_{j,\tilde{0},\mathrm{S}}(x),\\ Y_j^{(1)}=\sum_{1\le|a|\le n}\varphi_{j,a}(x)\theta^a.\end{cases}$$

Then, we consider formally

$$\tilde{F}(x,\theta)=\sum_b f_b(Y_1,\cdots,Y_{m'})(Y_{m'+1})^{b_1}\cdots(Y_{m'+n'})^{b_{n'}}.$$

Remarking that $Y_j^{(1)}Y_j^{(1)}=0$, we apply Taylor's formula for $f_b(Y^{(0)}+Y^{(1)})$ at $Y=Y^{(0)}$ to get

$$\begin{aligned}f_b(Y^{(0)}+Y^{(1)})=f_b(Y^{(0)})&+\sum_{j=1}^{m'}\partial_{y_j}f_b(Y^{(0)})Y_j^{(1)}\\&+\cdots+\partial_{y_1}\cdots\partial_{y_{m'}}f_b(Y^{(0)})Y_1^{(1)}\cdots Y_{m'}^{(1)}.\end{aligned}\tag{4.24}$$

On the other hand, as

$$f_b(Y^{(0)})=\sum_\alpha\frac{1}{\alpha!}\partial_Y^\alpha f_b(Y_{\mathrm{B}}^{(0)})(Y_{\mathrm{S}}^{(0)})^\alpha,\tag{4.25}$$

we get easily

$$f_b(\varphi_1(x,\theta),\cdots,\varphi_{m'}(x,\theta))=\sum_c g_{b,c}(x)\theta^c\tag{4.26}$$

where $g_{b,c}(x)$ is a supersmooth function on U_{ev} composed by the products of supersmooth functions $\partial_y^\alpha f(\varphi_{\mathrm{B}}(x))$ and $\varphi_{\kappa,a}(x)$. Combining these, we get

$$\begin{aligned}\tilde{F}(x,\theta)&=\sum_b(\sum_c g_{b,c}(x)\theta^c)(\sum_{\tilde{a}_1}\varphi_{m'+1,\tilde{a}_1}(x)\theta^{\tilde{a}_1})^{b_1}(\sum_{\tilde{a}_{n'}}\varphi_{m'+n',\tilde{a}_{n'}}(x)\theta^{\tilde{a}_{n'}})^{b_{n'}}\\&=\sum_d\tilde{F}_d(x)\theta^d,\end{aligned}\tag{4.27}$$

where $d=(d_s)$, $c=(c_s)$, $\tilde{a}_s=(\tilde{a}_{s,r})$, $d_s=c_s+b_1\tilde{a}_{1,s}+\cdots+b_{n'}\tilde{a}_{n',s}$ with $1\le s\le n$ and $1\le r\le n'$. Therefore, we get $\tilde{F}_d(x)\in\mathcal{C}_{\mathrm{SS}}(U_{ev}:\mathfrak{C})$, that is, $\tilde{F}(x,\theta)=f(\varphi(x,\theta))\in\mathcal{C}_{\mathrm{SS}}(U;\mathfrak{C})$. To get (4.21) in case (2), we differentiate (4.27) with respect to x_k,

$$\begin{aligned}\partial_{x_k}\tilde{F}(x,\theta)=&\sum_{j=1}^{m'}\sum_b\partial_{y_j}f_b(\varphi_{ev}(x,\theta))\frac{\partial\varphi_j(x,\theta)}{\partial x_k}(\varphi_{od}(x,\theta))^b\\&+\sum_b f_b(\varphi_{ev}(x,\theta))\sum_{s=m'+1}^{m'+n'}(-1)^{b_1+\cdots+b_{s-1}}b_s\frac{\partial\varphi_s(x,\theta)}{\partial x_k}\prod_{\ell=1}^{(s,n')}\varphi_{m'+\ell}(x,\theta)^{b_\ell}.\end{aligned}$$

Here, $\prod_{\ell=1}^{(s,n')}\varphi_{m'+\ell}(x,\theta)^{b_\ell}=\overbrace{\varphi_{m'+1}(x,\theta)^{b_1}\cdots 1}^{s}\cdots\varphi_{m'+n'}(x,\theta)^{b_{n'}}$, $\varphi_{ev}(x,\theta)=(\varphi_j(x,\theta))_{j=1}^{m'}$ and $\varphi_{od}(x,\theta)=(\varphi_{m'+s}(x,\theta))_{s=1}^{n'}$.

Taking derivatives with respect to θ_r, we get the similar expression as above and combining these, we have

$$[\partial_{x_k}\tilde{F}(x,\theta), \partial_{\theta_r}\tilde{F}(x,\theta)] = \begin{bmatrix} \dfrac{\partial\varphi_j(x,\theta)}{\partial x_k}, \cdots, \dfrac{\partial\varphi_j(x,\theta)}{\partial\theta_r} \\ \dfrac{\partial\varphi_s(x,\theta)}{\partial x_k}, \cdots, \dfrac{\partial\varphi_s(x,\theta)}{\partial\theta_r} \end{bmatrix} \left[\frac{\partial f(y,\omega)}{\partial y_j}, \frac{\partial f(y,\omega)}{\partial\omega_s}\right],$$

that is, (4.21) in the case of (2).
(3) For the general situation mentioned above, using the arguments in (2) repeatedly, we get the result after tedious but straightfoward calculations. □

Definition 4.2.6 *Let $U \subset \mathfrak{R}^{m|n}$ and $U' \subset \mathfrak{R}^{m'|n'}$ be superdomains and let $\varphi : U \to U'$ be a supersmooth mapping represented by $\varphi(X) = (\varphi_1(X), \cdots, \varphi_{m'+n'}(X))$ with $\varphi_\kappa(X) \in \mathcal{C}_{\rm SS}(U;\mathfrak{C})$.*
(1) φ is called a supersmooth diffeomorphism if
(i) φ is a homeomorphism between U and U' and
(ii) φ and φ^{-1} are supersmooth mappings.
(2) For any $f \in \mathcal{C}_{\rm SS}(U';\mathfrak{C})$, $(\varphi^*f)(X) = (f\circ\varphi)(X) = f(\varphi(X))$, called the pull back of f, is well-defined and belongs to $\mathcal{C}_{\rm SS}(U;\mathfrak{C})$.

Remarks. (1) It is easy to see that if φ is a supersmooth diffeomorphism, then $\varphi_{\rm B} = \pi_{\rm B}\circ\varphi$ is an (ordinary) C^∞ diffeomorphism from $U_{\rm B}$ to $U'_{\rm B}$.
(2) If we introduce the topologies in $\mathcal{C}_{\rm SS}(U':\mathfrak{C})$ and $\mathcal{C}_{\rm SS}(U:\mathfrak{C})$ properly, φ^* gives a continuous linear mapping from $\mathcal{C}_{\rm SS}(U':\mathfrak{C})$ to $\mathcal{C}_{\rm SS}(U:\mathfrak{C})$. Moreover, if $\varphi : U \to U'$ is a supersmooth diffeomorphism, then φ^* defines an automorphism from $\mathcal{C}_{\rm SS}(U':\mathfrak{C})$ to $\mathcal{C}_{\rm SS}(U:\mathfrak{C})$.

Proposition 4.2.7 (Inverse function theorem) *Let U be a superdomain in $\mathfrak{R}^{m|n}$ and let $G(X) : U \subset \mathfrak{R}^{m|n} \to \mathfrak{R}^{m|n}$ be a supersmooth mapping. We assume the super matrix $[d_XG(X)]$ is invertible at $X = \tilde{X}_{\rm B} \in \pi_{\rm B}(U)$. Then, there exists a superdomain U', a neighbourhood of $\tilde{Y} = G(\tilde{X})$ and a unique supersmooth mapping F satisfying $F(G(X)) = X$ and we have*

$$d_YF(Y) = (d_XG(X))^{-1}\big|_{X=F(Y)} \quad in \quad U'. \tag{4.28}$$

Proof. (1) First of all, we treat the case $m=1$ and $n=0$, that is, $U_{ev}, U'_{ev} \subset \mathfrak{R}^{1|0}$. Let $g : U_{ev} \to U'_{ev}$ be a supersmooth function represented by

$$y = g(x_{\rm B}) = g_{\rm B}(x_{\rm B}) + \sum_{|J|=even\geq 2} g_J(x_{\rm B})\sigma^J = y_{\rm B} + y_{\rm S}.$$

Here, $g_{\rm B}(x_{\rm B}) \in C^\infty(U_{\rm B}:\mathbb{R})$ and $g_J(x_{\rm B}) \in C^\infty(U_{\rm B}:\mathbb{C})$. By assumption that $g'_{\rm B}(\tilde{x}_{\rm B}) \neq 0$, there exists a smooth function $f_{\rm B}$ such that $f_{\rm B}(g_{\rm B}(x_{\rm B})) = x_{\rm B}$ near $x_{\rm B} = \tilde{x}_{\rm B}$. We want to construct a family of functions $f_I \in C^\infty(U'_{\rm B}:\mathbb{C})$ such that $f(y_{\rm B}) = f_{\rm B}(y_{\rm B}) + f_{\rm S}(y_{\rm B})$, $f_{\rm S}(y_{\rm B}) = \sum_{|I|=even\geq 2} f_I(y_{\rm B})\sigma^I$ satisfying $f(g(x_{\rm B})) = x_{\rm B}$ near $x_{\rm B} = \tilde{x}_{\rm B}$. As we should have

$$\begin{aligned} x_{\rm B} &= f_{\rm B}(y_{\rm B}+y_{\rm S}) + f_{\rm S}(y_{\rm B}+y_{\rm S}) \\ &= f_{\rm B}(y_{\rm B}) + \sum_{k\geq 1}\frac{1}{k!}f_{\rm B}^{(k)}(y_{\rm B}){y_{\rm S}}^k + \sum_{\ell\geq 0}\frac{1}{\ell!}f_{\rm S}^{(\ell)}(y_{\rm B}){y_{\rm S}}^\ell, \end{aligned} \tag{4.29}$$

we get

$$f_{\rm S}(y_{\rm B}) = -\sum_{k\geq 1}\frac{1}{k!}f_{\rm B}^{(k)}(y_{\rm B}){y_{\rm S}}^k - \sum_{k\geq 1}\frac{1}{k!}f_{\rm S}^{(k)}(y_{\rm B}){y_{\rm S}}^k. \tag{4.30}$$

We prove our statement using the induction with respect to the degree. The degree 2 part of (4.30) is given by

$$f_{\rm S}(y_{\rm B})_{[2]} = -f'_{\rm B}(y_{\rm B})y_{{\rm S},[2]}. \tag{4.31}$$

In other word, for I such that $|I| = 2$, we may define functions $f_I(y_{\rm B})$ by

$$f_I(y_{\rm B}) = -f'_{\rm B}(y_{\rm B})\, g_I(f_{\rm B}(y_{\rm B}))(= -f'_{\rm B}(g_{\rm B}(x_{\rm B}))\, g_I(x_{\rm B})).$$

Assuming that $f_{\rm S}$ are defined for degrees less than $2i$, we put,

$$f_{\rm S}(y_{\rm B})_{[2i+2]} = -\sum_{k\geq 1}\frac{1}{k!}f_{\rm B}^{(k)}(y_{\rm B})(y_{\rm S}^k)_{[2i+2]} - \sum_{k\geq 1}\sum_{j=0}^{i}\frac{1}{k!}(f_{\rm S}^{(k)}(y_{\rm B}))_{[2j]}(y_{\rm S}^k)_{[2i+2-2j]}. \tag{4.32}$$

So, we may define $f(y_{\rm B}) = \sum_{j=0}^{\infty} f(y_{\rm B})_{[2j]} = f_{\rm B}(y_{\rm B}) + \sum_{j=1}^{\infty} f_{\rm S}(y_{\rm B})_{[2j]} \in C^\infty(U'_{\rm B} : \mathfrak{C})$. Taking the Grassmann continuation of $f(y_{\rm B})$ and remarking $\partial_x f(g(x)) = 1$, we get the desired result.
(2) We next consider the case $m = n = 1$, that is, $U, U' \subset \mathfrak{R}^{1|1}$. Let $G(x,\theta) = (g_{ev}(x,\theta), g_{od}(x,\theta)) : U \to U'$ be a supersmooth mapping given by

$$g_{ev}(x,\theta) = g_{ev,0}(x) + g_{ev,1}(x)\theta, \quad g_{od}(x,\theta) = g_{od,1}(x) + g_{od,0}(x)\theta. \tag{4.33}$$

For simplicity, we put

$$g_{ev}(x_{\rm B},\theta) = y_{\rm B} + y_{\rm S} + \bar{y}\theta \quad \text{where} \quad \begin{cases} y_{\rm B} = g_{ev,0,{\rm B}}(x_{\rm B}), \quad y_{\rm S} = \sum_{|I|=even\geq 2} g_{ev,0,I}(x_{\rm B})\sigma^I, \\ \bar{y} = \sum_{|\bar{I}|=odd\geq 1} g_{ev,1,\bar{I}}(x_{\rm B})\sigma^{\bar{I}}, \end{cases}$$

and

$$g_{od}(x_{\rm B},\theta) = \omega + \bar{\omega}\theta \quad \text{where} \quad \begin{cases} \omega = \sum_{|\bar{I}|=odd\geq 1} g_{od,1,\bar{I}}(x_{\rm B})\sigma^{\bar{I}}, \\ \bar{\omega} = \bar{\omega}_{\rm B} + \bar{\omega}_{\rm S} \\ \quad = g_{od,0,{\rm B}}(x_{\rm B}) + \sum_{|I|=even\geq 2} g_{od,0,I}(x_{\rm B})\sigma^I. \end{cases}$$

From $\tilde{Y} = G(\tilde{X})$ and the invertibility of $d_X G(X)\big|_{X=\tilde{X}}$, we get

$$g_{ev,0,{\rm B}}(\tilde{x}_{\rm B}) = \tilde{y}_{\rm B}, \quad g'_{ev,0,{\rm B}}(\tilde{x}_{\rm B})\, g_{od,0,{\rm B}}(\tilde{x}_{\rm B}) \neq 0. \tag{4.34}$$

Now, we seek a function $F(Y) = F(y,\omega) = (f_{ev}(y,\omega), f_{od}(y,\omega)) : U' \to U$ represented by

$$f_{ev}(y,\omega) = f_{ev,0}(y) + f_{ev,1}(y)\omega, \quad f_{od}(y,\omega) = f_{od,1}(y) + f_{od,0}(y)\omega$$

which satifies $F(G(X)) = X$ near $X = (x,\theta) = (\tilde{x},\tilde{\theta}) = \tilde{X}$. Here, we put

$$\begin{cases} f_{ev,0}(y_{\rm B}) = f_{ev,0,{\rm B}}(y_{\rm B}) + \sum_{|I|=even\geq 2} f_{ev,0,I}(y_{\rm B})\sigma^I, \\ f_{ev,1}(y_{\rm B}) = \sum_{|\bar{I}|=odd\geq 1} f_{ev,1,\bar{I}}(y_{\rm B})\sigma^{\bar{I}}, \\ f_{od,1}(y_{\rm B}) = \sum_{|\bar{I}|=odd\geq 1} f_{od,1,\bar{I}}(y_{\rm B})\sigma^{\bar{I}}, \\ f_{od,0}(y_{\rm B}) = f_{od,0,{\rm B}}(y_{\rm B}) + \sum_{|I|=even\geq 2} f_{od,0,I}(y_{\rm B})\sigma^I. \end{cases}$$

As $F(G(x_{\rm B},\theta)) = (x_{\rm B},\theta)$, we should have the relations

$$f_{ev}(g_{ev}(x_{\rm B},\theta), g_{od}(x_{\rm B},\theta)) = x_{\rm B}, \quad f_{od}(g_{ev}(x_{\rm B},\theta), g_{od}(x_{\rm B},\theta)) = \theta. \tag{4.35}$$

From the first equation in (4.29) and the supersmoothness, we have

$$
\begin{aligned}
x_{\mathrm{B}} &= f_{ev,0}(y_{\mathrm{B}} + y_{\mathrm{S}} + \bar{y}\theta) + f_{ev,1}(y_{\mathrm{B}} + y_{\mathrm{S}} + \bar{y}\theta)(\omega + \bar{\omega}\theta) \\
&= f_{ev,0}(y_{\mathrm{B}}) + \sum_{|k|\geq 1} \frac{1}{k!} f^{(k)}_{ev,0}(y_{\mathrm{B}})(y_{\mathrm{S}}^k + k y_{\mathrm{S}}^{k-1}\bar{y}\theta) \\
&\qquad + \sum_{|\ell|\geq 0} \frac{1}{\ell!} f^{(\ell)}_{ev,1}(y_{\mathrm{B}})(y_{\mathrm{S}}^\ell + \ell y_{\mathrm{S}}^{\ell-1}\bar{y}\theta)(\omega + \bar{\omega}\theta) \\
&= f_{ev,0}(y_{\mathrm{B}}) + \sum_{|k|\geq 1} \frac{1}{k!} f^{(k)}_{ev,0}(y_{\mathrm{B}})y_{\mathrm{S}}^k + \sum_{|\ell|\geq 0} \frac{1}{\ell!} f^{(\ell)}_{ev,1}(y_{\mathrm{B}})y_{\mathrm{S}}^\ell \omega \\
&\quad + \left\{ \sum_{|k|\geq 1} \frac{1}{(k-1)!}(f^{(k)}_{ev,0}(y_{\mathrm{B}}) + f^{(k)}_{ev,1}(y_{\mathrm{B}})\omega) y_{\mathrm{S}}^{k-1}\bar{y} + \sum_{|\ell|\geq 0} \frac{1}{\ell!} f^{(\ell)}_{ev,1}(y_{\mathrm{B}})y_{\mathrm{S}}^\ell \bar{\omega} \right\} \theta.
\end{aligned}
\tag{4.36}
$$

Therefore

$$
x_{\mathrm{B}} = f_{ev,0,\mathrm{B}}(y_{\mathrm{B}}) + f_{ev,0,\mathrm{S}}(y_{\mathrm{B}}) + \sum_{|k|\geq 1} \frac{1}{k!} f^{(k)}_{ev,0}(y_{\mathrm{B}})y_{\mathrm{S}}^k + \sum_{|\ell|\geq 0} \frac{1}{\ell!} f^{(\ell)}_{ev,1}(y_{\mathrm{B}})y_{\mathrm{S}}^\ell \omega \tag{4.37}
$$

and

$$
0 = \sum_{|k|\geq 1} \frac{1}{(k-1)!}(f^{(k)}_{ev,0}(y_{\mathrm{B}}) + f^{(k)}_{ev,1}(y_{\mathrm{B}})\omega) y_{\mathrm{S}}^{k-1}\bar{y} + \sum_{|\ell|\geq 0} \frac{1}{\ell!} f^{(\ell)}_{ev,1}(y_{\mathrm{B}})y_{\mathrm{S}}^\ell (\bar{\omega}_{\mathrm{B}} + \bar{\omega}_{\mathrm{S}}). \tag{4.38}
$$

As $g'_{ev,0,\mathrm{B}}(\tilde{x}_{\mathrm{B}}) \neq 0$ by (4.34), using the standard inverse function theorem, there exists a function $f_{ev,0,\mathrm{B}}(y_{\mathrm{B}})$ such that

$$
f_{ev,0,\mathrm{B}}(g_{ev,0,\mathrm{B}}(x_{\mathrm{B}})) = x_{\mathrm{B}} \tag{4.39}
$$

near $x_{\mathrm{B}} = \tilde{x}_{\mathrm{B}}$. Therefore, we get from (4.37),

$$
f_{ev,0,\mathrm{S}}(y_{\mathrm{B}}) + \sum_{|k|\geq 1} \frac{1}{k!} f^{(k)}_{ev,0}(y_{\mathrm{B}})y_{\mathrm{S}}^k + \left(f_{ev,1}(y_{\mathrm{B}}) + \sum_{|k|\geq 1} \frac{1}{k!} f^{(k)}_{ev,1}(y_{\mathrm{B}})y_{\mathrm{S}}^k \right) \omega = 0. \tag{4.40}
$$

For each I satisfying $|I| = 1$, we pick up the term of degree 1 from (4.38) to get

$$
f_{ev,1,I}(y_{\mathrm{B}})g_{od,0,\mathrm{B}}(x_{\mathrm{B}}) + f'_{ev,0,\mathrm{B}}(g_{ev,0,\mathrm{B}}(x_{\mathrm{B}}))g_{ev,1,I}(x_{\mathrm{B}}) = 0. \tag{4.41}
$$

As $g'_{ev,0,\mathrm{B}}(x_{\mathrm{B}})g_{od,0,\mathrm{B}}(x_{\mathrm{B}}) \neq 0$ by (4.34), there exists a function $f_{ev,1,I}(y_{\mathrm{B}})$ such that the above equation is satisfied when $y_{\mathrm{B}} = g_{ev,0,\mathrm{B}}(x_{\mathrm{B}})$. Equations (4.39) and (4.40) correspond to the degree 0 and 1 part of (4.37) and (4.38), respectively.

Using these, we may solve the degree 2 part of (4.37) and then the degree 3 part of (4.38). Doing recursively, we may construct functions $f_{ev,0}$ and $f_{ev,1}$.

From the second equation of (4.36), we get

$$
\begin{aligned}
\theta &= f_{od,1}(y_{\mathrm{B}} + y_{\mathrm{S}} + \bar{y}\theta) + f_{od,0}(y_{\mathrm{B}} + y_{\mathrm{S}} + \bar{y}\theta)(\omega + \bar{\omega}\theta) \\
&= \sum_{|\ell|\geq 0} \frac{1}{\ell!} f^{(\ell)}_{od,1}(y_{\mathrm{B}})y_{\mathrm{S}}^\ell + \sum_{|\ell|\geq 0} \frac{1}{\ell!} f^{(\ell)}_{od,0}(y_{\mathrm{B}})y_{\mathrm{S}}^\ell \omega \\
&\quad + \left\{ \sum_{|k|\geq 1} \frac{1}{(k-1)!}(f^{(k)}_{od,1}(y_{\mathrm{B}}) + f^{(k)}_{od,0}(y_{\mathrm{B}})\omega) y_{\mathrm{S}}^{k-1}\bar{y} + \sum_{|k|\geq 1} \frac{1}{k!} f^{(k)}_{od,0}(y_{\mathrm{B}})y_{\mathrm{S}}^k \bar{\omega} \right\} \theta.
\end{aligned}
\tag{4.42}
$$

That is,

$$
0 = f_{od,1,\mathrm{S}}(y_{\mathrm{B}}) + \sum_{|k|\geq 1} \frac{1}{k!} f^{(k)}_{od,1}(y_{\mathrm{B}})y_{\mathrm{S}}^k + \sum_{|\ell|\geq 0} \frac{1}{\ell!} f^{(\ell)}_{od,0}(y_{\mathrm{B}})y_{\mathrm{S}}^\ell \omega \tag{4.43}
$$

and

$$1=\sum_{|k|\ge 1}\frac{1}{(k-1)!}(f^{(k)}_{od,1}(y_{\rm B})+f^{(k)}_{od,0}(y_{\rm B})\omega)y_{\rm S}^{k-1}\bar{y}+\sum_{|k|\ge 1}\frac{1}{k!}f^{(k)}_{od,0}(y_{\rm B})y_{\rm S}^{k}\bar{\omega}. \tag{4.44}$$

By the same arguments as above, we may construct functions $f_{od,1}(y_{\rm B})$ and $f_{od,0}(y_{\rm B})$ which satisfy the desired properties. □

Exercise 4.2.8 *Do analogously as above for general m, n but with more patience.*

Moreover, we have

Proposition 4.2.9 (Implicit function theorem) *Let $\Phi(X,Y) : U\times U' \to \mathfrak{C}^{m'|n'}$ be a supersmooth mapping and $(\tilde X,\tilde Y)\in U\times U'$, where U and U' are superdomains of $\mathfrak{R}^{m|n}$ and $\mathfrak{R}^{m'|n'}$, respectively. Suppose $\Phi(\tilde X,\tilde Y)=0$ and $\partial_Y\Phi=[\partial_{y_j}\Phi,\partial_{\omega_r}\Phi]$ is a continuous and invertible supermatrix at $(\tilde X_{\rm B},\tilde Y_{\rm B})\in\pi_{\rm B}(U)\times\pi_{\rm B}(U')$. Then, there exist a superdomain $V\subset U$ satisfying $\tilde X_{\rm B}\in\pi_{\rm B}(V)$ and a unique supersmooth mapping $Y=f(X)$ on V such that $\tilde Y=f(\tilde X)$ and $\Phi(X,f(X))=0$ in V. Moreover, we have*

$$\partial_X f(X) = -\left[\partial_Y\Phi(X,Y)\right]^{-1}[\partial_X\Phi(X,Y)]\,\big|_{Y=f(X)}. \tag{4.45}$$

Proof. (4.45) is easily obtained by

$$0=\partial_X\Phi(X,f(X)) = \left(\partial_X\Phi(X,Y)+\partial_Y\Phi(X,Y)\partial_X f(X)\right)\big|_{Y=f(X)}.$$

The existence proof is omitted here because the arguments in proving Proposition 4.2.7 work well in this situation. □

4.3 Global Inverse Mapping Theorem

It is known that

Theorem 4.3.1 (Global Inverse Mapping Theorem for Banach spaces) *Let X, Y be two Banach spaces. ϕ is a continuously Fréchet differentiable mapping from $X\to Y$ such that ϕ' is invertible for any $x\in X$ and $|[\phi'(x)]^{-1}|\le K<\infty$ uniformly in X. Then, ϕ is homeomorphic from X onto Y.*

Here, we want to have an anologous theorem in our case:

Theorem 4.3.2 (Global Inverse Mapping Theorem for Superspaces) *Let $\Phi=(\Phi_A)_{A=1,\cdots,m+n}:\mathfrak{R}^{m|n}_X\to\mathfrak{R}^{m|n}_Y$ be a supersmooth mapping satisfying*

$$|\pi_{\rm B}(\mathrm{sdet}\left(\frac{\partial\Phi_A(X)}{\partial X_C}\right))|\ge\delta>0 \quad \text{for any}\quad X\in\mathfrak{R}^{m|n}_X, \tag{4.46}$$

and for $I\in\mathcal{I}$ with $|I|\ge 1$,

$$|\,\mathrm{proj}_I\left(\frac{\partial\Phi_A(X)}{\partial X_C}\right)|\le\gamma_I<\infty \quad \text{for any}\quad X\in\mathfrak{R}^{m|n}_X \quad \text{and}\quad A,C=1,\cdots,m+n. \tag{4.47}$$

Then, Φ gives a supersmooth diffeomorphism from $\mathfrak{R}^{m|n}_X$ **onto** *$\mathfrak{R}^{m|n}_Y$.*

To prove this, we prepare

Lemma 4.3.3 *Let Φ be as above. Put $\Delta=[0,1]\times[0,1]\ni(s,t)$. Let a continuous function $F(s,t)$ from Δ to $\mathfrak{R}^{m|n}_Y$ be given which satisfies; (i) For each $s\in[0,1]$, $F(s,t)$ is C^∞- differentiable in t. (ii) There exist points $Y^0,Y^1\in\mathfrak{R}^{m|n}_Y$ such that $F(s,0)=Y^0$ and $F(s,1)=Y^1$ for any $s\in[0,1]$. Then, there exists a continuous function $G(s,t)$ from Δ to $\mathfrak{R}^{m|n}_X$ which is C^∞- differentiable in t for each $s\in[0,1]$ and satisfies $\Phi(G(s,t))=F(s,t)$ for every $(s,t)\in\Delta$.*

Proof. (I) By (4.46) and Proposition 4.2.7, for $X^0 \in \mathfrak{R}_X^{m|n}$ and $Y^0 \in \mathfrak{R}_Y^{m|n}$ satisfying $\Phi(X^0) = Y^0$, there exist a neighbourhood U of X^0 and a neighbourhood V of Y^0 such that $\Phi : U \to V$ gives an onto supersmooth diffeomorphism. Therefore, there exists a positive number ϵ such that a function $G(s,t) = \Phi^{-1}(F(s,t))$ is well-defined on $[0,\epsilon] \times [0,1]$.

(II) Let a be the largest values such that the function $G(s,t)$ is defined on $[0,a) \times [0,1]$. Now, we want to claim $a = 1$. Assume $a < 1$.

(II.1) We assume that $G(s,t)$ is extendable at $t = a$ for any $s \in [0,1]$. Then by Proposition 4.2.7, for each s, there exist a neighbourhood U_s of $G(s,a)$ and a neighbourhood V_s of $F(s,a)$ such that $\Phi : U_s \to V_s$ gives an onto supersmooth diffeomorphism. As the set $\cup_{s\in[0,1]} G(s,a)$ is compact in $\mathfrak{R}_X^{m|n}$, there exists a finite covering such that $\cup_{s\in[0,1]} G(s,a) \subset \cup_{i=1}^k G(s,t_i)$. Applying once more Proposition 4.2.7, there exists a positive number ϵ_i such that $G(s,a)$ is extendable in U_i for $0 \le t < a+\epsilon_i$. This contradicts the definition of a because $G(s,t)$ is extendable on $[0, a + \min_i \epsilon_i) \times [0,1]$.

(II.2) Put $F(s,t) = (F_A(s,t))_{A=1,\cdots,m+n}$ and $G(s,t) = (G_C(s,t))_{C=1,\cdots,m+n}$. As $\Phi(G(s,t)) = F(s,t)$ on $[0,a) \times [0,1]$, we may differentiate this with respect to t which yields

$$\frac{\partial}{\partial t} G_C(s,t) \frac{\partial \Phi_A}{\partial X_C}(G(s,t)) = \frac{\partial}{\partial t} F_A(s,t). \tag{4.48}$$

On the other hand, we get, for $t_0, t_1 \in [0,a)$,

$$G_C(s,t_1) - G_C(s,t_0) = \int_{t_0}^{t_1} dt \frac{\partial}{\partial t} G_C(s,t). \tag{4.49}$$

We claim that $G(s,t)$ is a Lipshitz continuous function with respect to $t \in [0,a)$ for fixed s. That is, representing $G(s,t) = (G_C(s,t))_{C=1,\cdots,m+n}$, we have

$$|\operatorname{proj}_I(G_C(s,t_1) - G_C(s,t_0))| \le C_{\Phi,I}|t_1 - t_0| \quad \text{for any} \quad I \in \mathcal{I}. \tag{4.50}$$

For notational simplicity, we put $\tilde{F} = (\frac{\partial}{\partial t} F_A(s,t))$, $\tilde{G} = (\frac{\partial}{\partial t} G_C(s,t))$ and $\tilde{M} = (\frac{\partial \Phi_A}{\partial X_C}(G(s,t)))$. Then, rewriting (4.48) by

$$\tilde{F}_I = \sum_{I=J+K} \tilde{G}_J \tilde{M}_K \quad \text{for any} \quad I \in \mathcal{I}, \tag{4.51}$$

we get, for $I \in \mathcal{I}$ with $|I| = 0$,

$$\tilde{F}_{\mathrm{B}} = \tilde{G}_{\mathrm{B}} \tilde{M}_{\mathrm{B}} \quad \text{on} \quad [0,a) \times [0,1]. \tag{4.52}$$

As $\tilde{M}_{\mathrm{B}}$ has a bounded inverse for any $(s,t) \in \Delta$ by (4.46), and $\tilde{F}_{\mathrm{B}}$ is bounded in $(s,t) \in \Delta$ by (i), so $\tilde{G}_{\mathrm{B}}(s,t)$ is bounded in $(s,t) \in [0,a) \times [0,1]$. For $I \in \mathcal{I}$ with $|I| = 1$, we get

$$\tilde{F}_I = \tilde{G}_I \tilde{M}_{\mathrm{B}} + \tilde{G}_{\mathrm{B}} \tilde{M}_I \quad \text{on} \quad [0,a) \times [0,1]. \tag{4.53}$$

Therefore, using (4.46), (4.47) and (4.52), $\tilde{G}_I(s,t)$ is bounded in $(s,t) \in [0,a) \times [0,1]$. Proceeding inductively with respect to the degree $|I|$, $\tilde{G}_I(s,t)$ is bounded in $(s,t) \in [0,a) \times [0,1]$ for any $I \in \mathcal{I}$. Using (4.48) and (4.49), we get (4.50). □

Proof of Theorem 4.3.2. (I) Put $Y^0 = \Phi(0)$. For any $Y \in \mathfrak{R}_Y^{m|n}$, define $Y(t) = (1-t)Y^0 + tY$. As the special case of Lemma A.2, there exists a curve $X(t) \in \mathfrak{R}_X^{m|n}$ such that $\Phi(X(t)) = Y(t)$. Therefore, $\Phi(X(1)) = Y$, and Φ is onto.

(II) Put $Y^0 = \Phi(0)$. We assume that there exist two different points X^0 and X^1 in $\mathfrak{R}_X^{m|n}$ satisfying $\Phi(X^0) = \Phi(X^1) = Y^1$. Let $\{X^j(t)\}_{j=1,2}$ be curves connecting 0 at $t = 0$ to X^j at $t = 1$. Put $Y^j(t) = \Phi(X^j(t))$. As $\mathfrak{R}_Y^{m|n}$ is simply-connected, there exists a continuous function $F(s,t)$ from $\Delta \to \mathfrak{R}_Y^{m|n}$ which satisfies (i) for each $s \in [0,1]$, $F(s,t)$ is C^∞- differentiable in t, and (ii) there exist points $Y^0, Y^1 \in \mathfrak{R}_Y^{m|n}$

such that $F(s,0)=Y^0$ and $F(s,1)=Y^1$ for any $s\in[0,1]$. By Lemma A.2, there exists a continuous function $G(s,t)$ from $\Delta\to\mathfrak{R}_X^{m|n}$ satisfying $\Phi(G(s,t))=F(s,t)$. Therefore, a curve $G(s,1)$ which connects X^0 and X^1 is mapped onto Y^1 and this contradicts Proposition 4.2.7 unless $X^0=X^1$. This implies that Φ is one-to-one.
(III) Supersmoothness of Φ^{-1} has been proved in Proposition 4.2.7. □

Corollary 4.3.4 *Let $\phi\in\mathcal{C}_{SS,ev}(\mathfrak{R}^{m|n}:\mathfrak{R})$ satisfying that (i) for any multi-index $\mathfrak{a}$ with $|\mathfrak{a}|\geq 2$, $D_Y^{\mathfrak{a}}\phi(Y_{\rm B})\in\mathcal{B}_{\rm SS}(\mathfrak{R}^{m|n};\mathfrak{C})$, i.e. $\mathrm{proj}_I(D_Y^{\mathfrak{a}}\phi(Y_{\rm B}))$ is a bounded function of $Y_{\rm B}\in\mathbb{R}^m$ and for any $I\in\mathcal{I}$ and (ii) $|\pi_{\rm B}(\mathrm{sdet}\,((\mathrm{Hess}\,\phi)(X_{\rm B})))|\geq\delta>0$ for any $X\in\mathfrak{R}^{m|n}$. Then, we have*

$$\sum_{A=1}^m|\pi_{\rm B}(\frac{\partial\phi}{\partial X_A}(X_{\rm B})-\frac{\partial\phi}{\partial X_A}(X'_{\rm B}))|^2\geq C|X_{\rm B}-X'_{\rm B}|^2.$$

Proof. Define $\Phi(X)=(\partial_{X_A}\phi(X))_{A=1,\cdots,m+n}$. Then, applying Theorem 4.46, we get $X=X(Y)=\Phi^{-1}(Y)$. Each entry of Jacobian matrix $(\partial X/\partial Y)$ has an 'upper bound' by (i) and (ii), so we get

$$|\pi_{\rm B}(X(Y)-X(Y'))|^2\leq C|Y_{\rm B}-Y'_{\rm B}|^2.$$

Put $X'=X(Y')$, then $Y'=\Phi(X')$. This implies the desired inequality. □

Proposition 4.3.5 *Let $\Phi=(\Phi_A)_{i=A,\cdots,m+n}:\mathfrak{R}^{m|n}\to\mathfrak{R}^{m|n}$ be a diffeomorphism of class $\mathcal{B}_{\rm SS}$, that is, $\Phi_i\in\mathcal{B}_{\rm SS}(\mathfrak{R}^{m|n}:\mathfrak{R})$ for $i=1,\cdots,m+n$. Then, Φ^* sends a function $f\in\mathcal{B}_{\rm SS}(\mathfrak{R}^{m|n}:\mathfrak{C})$ to $\Phi^*f=f\circ\Phi\in\mathcal{B}_{\rm SS}(\mathfrak{R}^{m|n}:\mathfrak{C})$.*

Proposition 4.3.6 *Let $\Phi=(\Phi_A)_{A=1,\cdots,m+n}:\mathfrak{R}_X^{m|n}\to\mathfrak{R}_Y^{m|n}$ be a diffeomorphism of class $\mathcal{B}_{\rm SS}$. Assume*

$$|\pi_{\rm B}(\mathrm{sdet}\,(\frac{\partial\Phi_A(X)}{\partial X_C})|\geq\delta>0\quad\text{for any}\quad X\in\mathfrak{R}_X^{m|n},$$

then, Φ^{-1} gives a diffeomorphism of class $\mathcal{B}_{\rm SS}$.

Corollary 4.3.7 *Let Φ be given as above. Then, Φ^* gives an isomorphism between $\mathcal{B}_{\rm SS}(\mathfrak{R}_X^{m|n}:\mathfrak{C})$ to $\mathcal{B}_{\rm SS}(\mathfrak{R}_Y^{m|n}:\mathfrak{C})$.*

Remarks. (1) Proof of Theorem 4.3.2 using Lemma 4.3.3 is essentially due to Schwartz [200] except (II.2) in proof of Lemma 4.3.3.
(2) Propositions and Corolaries 4.3.4-4.3.7 are proved as same as in Asada and Fujiwara [8] with slight modifications.

4.4 Elementary Integral Calculus

4.4.1 Integration (even case)

Now, we define the integration of a supersmooth function $u(x)$ on an even superdomain $U_{ev}\subset\mathfrak{R}^{m|0}$, which is similar to the integral of holomorphic functions on a complex domain. (See, Rogers [186, 187, 189, 190].)

Definition 4.4.1 *Let $u(x)$ be a supersmooth function defined on a even super domain $U_{ev}\subset\mathfrak{R}^{1|0}$. Let $\lambda=\lambda_{\rm B}+\lambda_{\rm S}$, $\mu=\mu_{\rm B}+\mu_{\rm S}\in U_{ev}$ and let a continuous and piecewise C^1-curve $c:[\lambda_{\rm B},\mu_{\rm B}]\to U_{ev}$ be given such that $c(\lambda_{\rm B})=\lambda$, $c(\mu_{\rm B})=\mu$. We define*

$$\int_c dx\,u(x)=\int_{\lambda_{\rm B}}^{\mu_{\rm B}}dt\,u(c(t))\dot{c}(t)\in\mathfrak{C}\tag{4.54}$$

and call it the integral of u *along the curve c.*

Using the integration by parts, we get the following fundamental result (see de Witt [61]).

Proposition 4.4.2 *Let $u(t) \in C^\infty([\lambda_B,\mu_B]:\mathfrak{C})$ and let $u(x)$ be the Grassmann continuation of $u(t)$. Suppose that there exists a function $U(t) \in C^\infty([\lambda_B,\mu_B]:\mathfrak{C})$ satisfying $U'(t)=u(t)$ on $[\lambda_B,\mu_B]$. Then, for any continuous and piecewise C^1-curve $c:[\lambda_B,\mu_B]\to U_{ev}\subset\mathfrak{R}^{1|0}$ such that $c(\lambda_B)=\lambda$, $c(\mu_B)=\mu$, we have*

$$\int_c dx\, u(x) = U(\lambda)-U(\mu). \tag{4.55}$$

Proof. By definition, we get

$$\begin{aligned}
\int_{\lambda_B}^{\mu_B} dt\, u(c(t))\dot c(t) &= \int_{\lambda_B}^{\mu_B} dt \sum_{\ell\ge0}\frac{1}{\ell!}u^{(\ell)}(c_B(t))c_S(t)^\ell(\dot c_B(t)+\dot c_S(t))\\
&= \int_{\lambda_B}^{\mu_B} dt\, u(c_B(t))\dot c_B(t) + \int_{\lambda_B}^{\mu_B} dt \sum_{k\ge1}\frac{1}{k!}u^{(k)}(c_B(t))\dot c_B(t)c_S(t)^k\\
&\qquad + \int_{\lambda_B}^{\mu_B} dt \sum_{\ell\ge0}\frac{1}{\ell!}u^{(\ell)}(c_B(t))c_S(t)^\ell\dot c_S(t)\\
&= U(\mu_B)-U(\lambda_B)+\sum_{\ell\ge0}\frac{1}{(\ell+1)!}\Big\{U^{(\ell+1)}(\mu_B)(\mu_S)^{\ell+1}-U^{(\ell+1)}(\lambda_B)(\lambda_S)^{\ell+1}\Big\}\\
&= U(\mu)-U(\lambda). \qquad \square
\end{aligned}$$

Corollary 4.4.3 *Let $u(x)$ be a supersmooth function defined on a even superdomain $U_{ev}\subset\mathfrak{R}^{1|0}$ into $\mathfrak{C}$. Let c_1,c_2 be continuous and piecewise C^1-curves from $[\lambda_B,\mu_B]\to U_{ev}$ such that $\lambda=c_1(\lambda_B)=c_2(\lambda_B)$ and $\mu=c_1(\mu_B)=c_2(\mu_B)$. If c_1 is homotopic to c_2, then*

$$\int_{c_1} dx\, u(x) = \int_{c_2} dx\, u(x). \tag{4.56}$$

Thus, if $[\lambda_B,\mu_B]\subset\pi_B(U_{ev})$, we have

$$\int_\lambda^\mu dx\, u(x) = \int_{\lambda_B}^{\mu_B} dt\, u(t). \tag{4.57}$$

Because of (4.57), we have

Definition 4.4.4 *(1) Let I_{ev} be a even superdomain in $\mathfrak{R}^{m|0}$ such that $\pi_B(I_{ev})=\prod_{j=1}^m(a_j,b_j)\subset\mathbb{R}^m$ with $-\infty<a_j<b_j<\infty$, which is called a even supercube. For $u\in\mathcal{C}_{SS}(I_{ev}:\mathfrak{C})$, we define*

$$\int_{I_{ev}} dx\, u(x) = \int_{a_1}^{b_1} dq_1\cdots\int_{a_m}^{b_m} dq_m\, u(q_1,\cdots,q_m) = \int_{\pi_B(I_{ev})} dx_B\, u(x_B). \tag{4.58}$$

(2) For any even superdomain $U_{ev}\subset\mathfrak{R}^{m|0}$ such that $\pi_B(U_{ev})$ is of definite area, we may put

$$\int_{U_{ev}} dx\, u(x) = \int_{\pi_B(U_{ev})} dx_B\, u(x_B) \tag{4.59}$$

for $u\in\mathcal{C}_{SS}(U_{ev}:\mathfrak{C})$.

Remarks. (1) The formula (4.59) stemms easily from the well-known procedures to define multiple integrals in Riemannian integration.
(2) The reason why we should use 'contour integration' is explained precisely in Rogers [189]. As we treat only even superdomains here, her arguments there are simplified considerably. But we should change the

role of the 'body' in our treatment, if we need to catch up all arguments of Rogers, which is noted in the remark after Propositon ??.

Super p-forms ω: In order to define "super"-form ω of degree p over a p-dimensional singular manifold L, Volvich-Vladimirov [215] introduced the following:

Let M be a p-dimensional oriented manifold of the class C^1 and the mapping $\varphi : M \to U \subset \mathbb{R}^m$ be of the class C^1. The set $L = \varphi(M)$, or, more precisely, the pair (M, φ), is called a p-dimensional singular manifold. That is, every p-dimensional manifold $M \subset \mathbb{R}^m$ is a p-dimensional singular manifold with respect to the identity mapping.

The pairs (M, φ) and (M_1, φ_1) are said to be equivalent if $L = \varphi(M) = \varphi_1(M_1)$ and there exists a diffeomorphism $f : M \to M_1$ such that $\varphi = \varphi_1 \circ f$.

$$\int_M \varphi^*\omega = \int_{M_1} \varphi_1^*\omega. \tag{4.60}$$

We may interprete the equation (4.60) as a change of variables as follows:

4.4.2 Integration (odd case)

It seems natural to put formally

$$d\theta_j = \sum_{I\in\mathcal{I},|I|=odd} d\theta_{j,I}\sigma^I \quad \text{for} \quad \theta_j = \sum_{I\in\mathcal{I},|I|=odd} \theta_{j,I}\sigma^I.$$

Then, we have

$$d\theta_j \wedge d\theta_k = d\theta_k \wedge d\theta_j.$$

This make us imagine that even if there exists the notion of integration, it differs much from the ordinary one.

Let v be a polynomial of odd variables $\theta = (\theta_1, \cdots, \theta_n) \in \mathfrak{R}^n_{\text{od}}$ such that

$$v(\theta_1, \cdots, \theta_n) = \sum_{|b|\le n} v_b\theta^b \quad \text{with homogeneous } v_b\theta^b \in \mathfrak{C} \text{ for each } b.$$

Denote by $P_n(\mathfrak{C})$ the set of all v as above.

Definition 4.4.5 *For $v \in P_n(\mathfrak{C})$, we put*

$$\int_{\mathfrak{R}^{0|n}} d\theta\, v(\theta) = \int_{\mathfrak{R}^{0|n}} d\theta_n \cdots d\theta_1\, v(\theta_1, \cdots, \theta_n) = (\partial_{\theta_n} \cdots \partial_{\theta_1} v)(0)$$

and we call it the integral of v on $\mathfrak{R}^{0|n}$.

Above definition yields readily that

$$\int_{\mathfrak{R}^{0|n}} d\theta_n \cdots d\theta_1\, \theta_1 \cdots \theta_n = 1.$$

Moreover, we have

Proposition 4.4.6 *Given v, $w \in P_n(\mathfrak{C})$, we have the following:*

(1) ($\mathfrak{C}$-linearity) For any homogeneous λ, $\mu \in \mathfrak{C}$,

$$\int_{\mathfrak{R}^{0|n}} d\theta(\lambda v + \mu w)(\theta) = (-1)^{np(\lambda)}\lambda \int_{\mathfrak{R}^{0|n}} d\theta\, v(\theta) + (-1)^{np(\mu)}\mu \int_{\mathfrak{R}^{0|n}} d\theta\, w(\theta). \tag{4.61}$$

(2) (Translational invariance) For any $\rho \in \mathfrak{R}^{0|n}$, we have

$$\int_{\mathfrak{R}^{0|n}} d\theta\, v(\theta+\rho) = \int_{\mathfrak{R}^{0|n}} d\theta\, v(\theta). \tag{4.62}$$

(3) (Integration by parts) For $v \in P_n(\mathfrak{C})$ such that $p(v)=1$ or 0, we have

$$\int_{\mathfrak{R}^{0|n}} d\theta\, v(\theta)\partial_{\theta_s} w(\theta) = -(-1)^{p(v)} \int_{\mathfrak{R}^{0|n}} d\theta\, (\partial_{\theta_s} v(\theta)) w(\theta). \tag{4.63}$$

(4) (Linear change of variables) Let $A=(A_{jk})$ with $A_{jk} \in \mathfrak{R}_{\rm ev}$ be invertible. Then,

$$\int_{\mathfrak{R}^{0|n}} d\theta\, v(\theta) = (\det A)^{-1} \int_{\mathfrak{R}^{0|n}} d\omega\, v(A\cdot\omega). \tag{4.64}$$

(5) (Iteration of integrals)

$$\int_{\mathfrak{R}^{0|n}} d\theta\, v(\theta) = \int_{\mathfrak{R}^{0|n-k}} d\theta_n \cdots d\theta_{k+1} \left(\int_{\mathfrak{R}^{0|k}} d\theta_k \cdots d\theta_1\, v(\theta_1,\cdots,\theta_k,\theta_{k+1},\cdots,\theta_n) \right). \tag{4.65}$$

(6) (Odd change of variables) Let $\theta=\theta(\omega)$ be an odd change of variables such that $\theta(0)=0$ and $\det \left.\dfrac{\partial\theta(\omega)}{\partial\omega}\right|_{\omega=0} \neq 0$*. Then, for any $v \in P_n(\mathfrak{C})$,*

$$\int_{\mathfrak{R}^{0|n}} d\theta\, v(\theta) = \int_{\mathfrak{R}^{0|n}} d\omega\, v(\theta(\omega)) \det{}^{-1} \frac{\partial\theta(\omega)}{\partial\omega}. \tag{4.66}$$

(7) For $v \in P_n(\mathfrak{C})$ and $\omega \in \mathfrak{R}^{0|n}$,

$$\int_{\mathfrak{R}^{0|n}} d\theta\, (\theta_1-\omega_1)\cdots(\theta_n-\omega_n) v(\theta) = v(\omega). \tag{4.67}$$

Remarks. (1) All above assertions are easily obtained by following the arguments in pp.755-757 of Vladimirov and Volovivh [215], so proofs are omitted here.
(2) (4.67) allows us to put $\delta(\theta-\omega)$= $(\theta_1-\omega_1)\cdots(\theta_n-\omega_n)$, though $\delta(-\theta)=(-1)^n\delta(\theta)$.

4.4.3 Integration (mixed case)

Finally, we define

Definition 4.4.7 *Let $U = U_{ev} \times \mathfrak{R}^n_{\rm od} \subset \mathfrak{R}^{m|n}$ be a superdomain and let $u \in \mathcal{C}_{\rm SS}(U:\mathfrak{C})$, that is, $u(x,\theta)=\sum u_a(x)\theta^a$ with $u_a(x) \in \mathcal{C}_{\rm SS}(U_{ev}:\mathfrak{C})$. Then, we define*

$$\begin{aligned}\int_U dx d\theta\, u(x,\theta) &= \int_{U_{ev}} dx \left\{ \int_{\mathfrak{R}^{0|n}} d\theta\, u(x,\theta) \right\} \\ &= \int_{\pi_{\rm B}(U_{ev})} dx_{\rm B}\, u_{\tilde 1}(x_{\rm B}) \quad \textit{with} \quad \tilde 1 = (1,\cdots,1) \\ &= \int_{\mathfrak{R}^{0|n}} d\theta \left\{ \int_{U_{ev}} dx\, u(x,\theta) \right\}.\end{aligned} \tag{4.68}$$

More generally, we consider the following situation.

Definition 4.4.8 *Let Ω be a domain in $\mathbb{R}^m$.*

(1) Let $\varphi \in C^\infty(\Omega:\mathfrak{R}^m_{\rm ev})$ with image $L=\varphi(\Omega)$, $\varphi(q)=\sum_{I\in\mathcal{I},|I|=even} \varphi_I(q)\sigma^I$, $\varphi_I(q) \in C^\infty(\Omega:\mathbb{R}^m)$. For $f(x,\theta)=\sum_{|a|\le n} f_a(x)\theta^a$ with $f_a(x)$ being the Grassmann extension of $f_a(x_{\rm B})$ such that $f(x,\theta)$ is

integrable in L. In this case, we call the following expression "the integral of the function $f(x,\theta)$ over the manifold $S = L \times \mathfrak{R}^n_{\mathrm{od}}$" where

$$\int_S dx d\theta\, f(x,\theta) = \int d\theta \left[\int_L dx\, f(x,\theta) \right].$$

Since we define, for each θ,

$$\int_L dx\, f(x,\theta) = \int_\Omega dq\, f(\varphi(q)) \det\left(\frac{\partial \varphi(q)}{\partial q}\right),$$

we have

$$\int_S dx d\theta\, f(x,\theta) = \int_\Omega dq\, \frac{\partial}{\partial \theta_n} \cdots \frac{\partial}{\partial \theta_1} f(\varphi(q),0) \det\left(\frac{\partial \varphi(q)}{\partial q}\right).$$

In this case, it is clear that

$$\int_S dx d\theta\, f(x,\theta) = \int_L dx \left[\int d\theta\, f(x,\theta) \right].$$

(2) For $\theta \in \mathfrak{R}^n_{\mathrm{od}}$, we put a set $\varphi(\Omega,\theta)$ (called, a singular manifold) in $\mathfrak{R}^m_{\mathrm{ev}}$ by

$$\varphi(\Omega,\theta) = \{x \in \mathfrak{R}^m_{\mathrm{ev}} \mid x = \varphi(q,\theta),\, q \in \Omega \subset \mathbb{R}^m\} \quad \textit{for} \quad \theta \in \mathfrak{R}^n_{\mathrm{od}}.$$

Here,

$$\varphi(q,\theta) = \varphi(q) + \sum_{2 \le |a| = even \le n} \varphi_a(q)\theta^a \quad \textit{with} \quad \varphi(q), \varphi_a(q) \in C^\infty(\Omega : \mathbb{R}).$$

We call a set in $\mathfrak{R}^{m|n}$ of the form

$$S = S(\varphi,\Omega) = \{(x,\theta) \in \mathfrak{R}^{m|n} \mid \theta \in \mathfrak{R}^n_{\mathrm{od}},\, x \in \varphi(\Omega,\theta)\}$$

a foliated singular manifold. In this case, we define

$$\int_S dx d\theta\, f(x,\theta) = \int d\theta \left[\int_{L(\theta)} dx\, f(x,\theta) \right].$$

Let $f(x,\theta)$ be a function with values in $\mathfrak{C}$ defined on S. We define the integral of f on S as

$$\int_S dx d\theta\, f(x,\theta) = \int_{\mathfrak{R}^{0|n}} d\theta \left[\int_{\varphi(\Omega,\theta)} dx\, f(x,\theta) \right]$$

Here, the inner integral is understood in the following sense:

$$\int_{\varphi(\Omega,\theta)} dx\, f(x,\theta) = \int_\Omega dq\, f(\varphi(q,\theta),\theta) \det\left(\frac{\partial \varphi(q,\theta)}{\partial q}\right).$$

4.4.4 Change of variables under integral sign (Berezin case)

If an integrand has compact support, we have the following:

Theorem 4.4.9 *Let*

$$x = x(y,\omega), \quad \theta = \theta(y,\omega)$$

be a supersmooth diffeomorphism from $\mathfrak{R}^{m|n}_Y$ to $\mathfrak{R}^{m|n}_X$. Put

$$M = \begin{pmatrix} A & C \\ D & B \end{pmatrix}, \quad \begin{cases} A = \dfrac{\partial x}{\partial y}, & C = \dfrac{\partial x}{\partial \omega}, \\ D = \dfrac{\partial \theta}{\partial y}, & B = \dfrac{\partial \theta}{\partial \omega}. \end{cases}$$

Then, for any function $f \in \mathcal{C}_{\mathrm{SS}}(\mathfrak{R}_X^{m|n} : \mathfrak{C})$ with compact support, we have the change of variables formula

$$\int_{\mathfrak{R}_X^{m|n}} dxd\theta\, f(x,\theta) = \int_{\mathfrak{R}_Y^{m|n}} dyd\omega\, f(x(y,\omega),\theta(y,\omega))(\mathrm{sdet}\, M)(y,\omega).$$

Proof. [The proof borrowed from Berezin [20]]. First of all, we consider two simple cases:
(i) Let a linear coordinate change be given by

$$x_i = \sum_{k=1}^m A_{ik} y_k,\ \theta_j = \sum_{\ell=1}^n B_{j\ell}\omega_\ell$$

with $A_{ik}, B_{j\ell} \in \mathfrak{C}_{\mathrm{ev}}$. Then, $C = D = 0$ and

$$\mathrm{sdet}\left(\frac{\partial(x,\theta)}{\partial(y,\omega)}\right) = \det A \det{}^{-1} B.$$

In this case, we get our result easily.
(ii) For more general linear transformation,

$$x_i = \sum_{k=1}^m A_{ik} y_k + \sum_{\ell=1}^n C_{i\ell}\omega_\ell = x_i(y,\omega), \quad \theta_j = \sum_{k=1}^m D_{jk} y_k + \sum_{\ell=1}^n B_{j\ell}\omega_\ell = \theta_j(y,\omega)$$

with $A_{ik}, B_{j\ell} \in \mathfrak{C}_{\mathrm{ev}}$ and $C_{i\ell}, D_{jk} \in \mathfrak{C}_{\mathrm{od}}$, we have

$$\mathrm{sdet}\left(\frac{\partial(x,\theta)}{\partial(y,\omega)}\right) = \det A \det{}^{-1}(B - DA^{-1}C).$$

$$\begin{aligned}
&\int dyd\omega\, f(Ay + C\omega, Dy + B\omega)\ (y \to Ay,\ \omega \to B\omega) \\
&\quad = \det{}^{-1} A\ \det B \int dyd\omega\, f(y + CB^{-1}\omega, DA^{-1}y + \omega)\ ((y,\omega) \to (y, \omega + DA^{-1}y)) \\
&\quad = \det{}^{-1} A\ \det B \int dyd\omega\, f(y + CB^{-1}(\omega - DA^{-1}y), \omega)\ ((y,\omega) \to (y - CB^{-1}DA^{-1}y, \omega)) \\
&\quad = \det{}^{-1} A\ \det B \det{}^{-1}(1 - CB^{-1}DA^{-1}) \int dyd\omega\, f(y + CB^{-1}\omega, \omega)\ ((y,\omega) \to (y + CB^{-1}\omega, \omega)) \\
&\quad = \det B \det{}^{-1}(A - CB^{-1}D) \int dyd\omega\, f(y,\omega).
\end{aligned}$$

(iii) We consider the change of variables of the forms

$$x = x(y),\ \theta = \omega, \tag{4.69}$$

or

$$x = y,\ \theta = \theta(\omega). \tag{4.70}$$

For (4.69), we may use the ordinary change of variable formula which yields our result.

In case (4.70), we consider a transformation T which can be included in a 1-parameter group T_t of transformations of form (4.70). Set

$$\begin{aligned}
\theta(t) &= T_t\omega, \\
g(t) &= \int d\omega\, f(x,\theta(t))\mathrm{sdet}\left(\frac{\partial\theta(t)}{\partial\omega}\right).
\end{aligned}$$

Since

$$\mathrm{sdet}\left(\frac{\partial\theta(t+s)}{\partial\omega}\right) = \mathrm{sdet}\left(\frac{\partial\theta(t+s)}{\partial\theta(t)}\right)\mathrm{sdet}\left(\frac{\partial\theta(t)}{\partial\omega}\right),$$

we have

$$g(t+s) = \int d\omega\, f(x, \theta(t+s))\mathrm{sdet}\left(\frac{\partial\theta(t+s)}{\partial\omega}\right) = \int d\omega\, f(x, \theta(t+s))\mathrm{sdet}\left(\frac{\partial\theta(t+s)}{\partial\theta(t)}\right)\mathrm{sdet}\left(\frac{\partial\theta(t)}{\partial\omega}\right).$$

Putting

$$\Delta(s) = \mathrm{sdet}\left(\frac{\partial\theta(t+s)}{\partial\theta(t)}\right),$$

we get

$$g'(t) = \frac{d}{ds}g(t+s)\Big|_{s=0} = \int d\omega\left[\sum \frac{d\theta_j(t+s)}{ds}\left(\frac{\partial f}{\partial\theta_j}\right)\Delta(s) + f\frac{d}{ds}\Delta(s)\right]_{s=0}\mathrm{sdet}\,(\frac{\partial\theta(t)}{\partial\omega}). \tag{4.71}$$

Noting that

$$\Delta(s) = \det J(s)^{-1} = \exp\left(-\operatorname{tr}\log J(s)\right) \quad \text{where} \quad J(s) = \left(\frac{\partial\theta_j(t+s)}{\partial\theta_k(t)}\right),$$

we get

$$\frac{d}{ds}\Delta(s)\Big|_{s=0} = -\operatorname{tr}\left(J'(s)J^{-1}(s)\right)\exp\left(-\operatorname{tr}\log J(s)\right)\big|_{s=0} = -\operatorname{tr}\, J'(0) = -\sum\frac{d}{ds}\frac{\partial\theta_j(t+s)}{\partial\theta_j(t)}\Big|_{s=0}.$$

The expression in the square brackets in the right-hand side of (4.73) transforms into

$$\sum_j\left(\theta'_j\frac{\partial f}{\partial\theta_j} - f\frac{\partial\theta'_j}{\partial\theta_j}\right) = -\sum_j\frac{\partial}{\partial\theta_j}(\theta'_j f).$$

Since $T_t\omega_j = \theta_j(t)$, we have

$$\theta'_j(t) = -\Phi_j(\theta).$$

Therefore,

$$g'(t) = \int d\omega\sum_j\frac{\partial}{\partial\theta_j}(\Phi_j f)\mathrm{sdet}\left(\frac{\partial\theta(t)}{\partial\omega}\right).$$

We should remark that (i) $g'(t)$ has the same form as $g(t)$ by replacing f in $g(t)$ with $\sum_j\frac{\partial}{\partial\theta_j}(\Phi_j f)$, and (ii) $g'(0) = 0$ because for $t = 0$, $\theta_j(0) = \omega_j$ and $\mathrm{sdet}\,(\frac{\partial\theta(t)}{\partial\omega})|_{t=0} = 1$, therefore

$$g'(0) = \int d\omega\sum_j\frac{\partial}{\partial\omega_j}(\omega_j f(\omega)) = 0.$$

Repeating this process, we get $g^{(n)}(0) = 0$ for any $n > 0$.

(iv) The change of variables of the form

$$x = x(y,\omega),\ \theta = \omega, \tag{4.72}$$

or

$$x = y,\ \theta = \theta(y,\omega). \tag{4.73}$$

The case (4.73) can easily be reduced to (4.70).

For the case (4.72), we first consider the special type:

$$\begin{aligned} &x_i = y_i + tf_I\omega^I, \quad I = \{i_1,\cdots,i_{2k}\},\ = \omega_{i_1}\cdots\omega_{i_{2k}}, \\ &\theta_i = \omega_i \end{aligned} \tag{4.74}$$

$$x_i = y_i + \sum_k \sum f_I(y)\omega^I, \quad I = \{i_1, \cdots, i_k\}, = \omega_{i_1} \cdots \omega_{i_k},$$
$$\theta_i = \omega_i \tag{4.75}$$

(v) The change of variables of the forms

$$x = x(y, \omega),\ \theta = \theta(y, \omega) \tag{4.76}$$

Example 1 [The ambiguity in the Berezin integral]. It is known that if $U = \pi_{\mathrm{B}}^{-1}(U_{\mathrm{B}}) \times \mathfrak{R}_{\mathrm{od}}^2 \subset \mathfrak{R}^{1|2}$ with $U_{\mathrm{B}} = (0, 1)$, we have

$$\int_U D(x, \theta)(x + \theta_1\theta_2) = \int_0^1 dx \int d\theta_1\theta_2(x + \theta_1\theta_2) = 1.$$

But, if we use the coordinate change

$$y = x + \theta_1\theta_2,\ \omega_k = \theta_k : U \to U$$

whose Berezinian is

$$\mathrm{sdet}\,(\frac{\partial(x, \theta)}{\partial(y, \omega)}) = 1 \quad \text{where} \quad \frac{\partial(x, \theta)}{\partial(y, \omega)} = \begin{pmatrix} 1 & -\omega_2 & \omega_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

then we have

$$\int_U D(y, \omega)y = \int_0^1 dy \frac{\partial^2}{\partial\omega_2\partial\omega_1} y = 0.$$

This is a serious descrepancy! How to resolve this?

Example 2 [An ambiguity of Q-integration]. Let

$$\mathcal{Q} = \left\{ Q = \begin{pmatrix} x_1 & \theta_1 \\ \theta_2 & ix_2 \end{pmatrix} \mid x_1, x_2 \in \mathfrak{R}_{\mathrm{ev}},\ \theta_1, \theta_2 \in \mathfrak{R}_{\mathrm{od}} \right\} \cong \mathfrak{R}^{2|2}$$

with the volume element $dQ = \frac{dx_1 dx_2}{2\pi} d\theta_1 d\theta_2$. It is known that

$$\int_{\mathfrak{Q}} dQ\, e^{-\mathrm{str}\, Q^2} = \int_{\mathfrak{R}^{2|2}} \frac{dx_1 dx_2}{2\pi} d\theta_1 d\theta_2\, e^{-(x_1^2 + x_2^2 + 2\theta_1\theta_2)} = 1. \tag{4.77}$$

We may diagonalize the matrix Q by using the change of variables

$$\begin{cases} y_1 = x_1 + \dfrac{\theta_1\theta_2}{x_1 - ix_2},\ y_2 = x_2 - \dfrac{i\theta_1\theta_2}{x_1 - ix_2}, \\ \omega_1 = \dfrac{\theta_1}{x_1 - ix_2},\ \omega_2 = -\dfrac{\theta_2}{x_1 - ix_2}, \end{cases} \tag{4.78}$$

or

$$\begin{cases} x_1 = y_1 + \omega_1\omega_2(y_1 - iy_2),\ x_2 = y_2 - i\omega_1\omega_2(y_1 - iy_2), \\ \theta_1 = \omega_1(y_1 - iy_2),\ \theta_2 = -\omega_2(y_1 - iy_2), \end{cases} \tag{4.79}$$

such that

$$GQG^{-1} = \begin{pmatrix} y_1 & 0 \\ 0 & iy_2 \end{pmatrix}, \quad GQ^2G^{-1} = \begin{pmatrix} y_1^2 & 0 \\ 0 & -y_2^2 \end{pmatrix} \tag{4.80}$$

where

$$G = \begin{pmatrix} 1+2^{-1}\omega_1\omega_2 & \omega_1 \\ \omega_2 & 1-2^{-1}\omega_1\omega_2 \end{pmatrix}, \quad G^{-1} = \begin{pmatrix} 1+2^{-1}\omega_1\omega_2 & -\omega_1 \\ -\omega_2 & 1-2^{-1}\omega_1\omega_2 \end{pmatrix}.$$

It is clear that

$$x_1 - ix_2 = y_1 - iy_2, \quad \text{and} \quad \operatorname{str} Q^2 = x_1^2 + x_2^2 + 2\theta_1\theta_2 = y_1^2 + y_2^2.$$

On the other hand, the so-called Berezinian is given by

$$dQ = \frac{dx_1 dx_2}{2\pi} d\theta_1 d\theta_2 = -\frac{dy_1 dy_2}{2\pi} d\omega_1 d\omega_2 \, (y_1 - iy_2)^{-2}.$$

"An ambiguity" related to the formula for the integration under the change of variables, occurs because

$$-\int \frac{dy_1 dy_2}{2\pi} d\omega_1 d\omega_2 \, (y_1 - iy_2)^{-2} e^{-(y_1^2+y_2^2)} = 0.$$

Remark. An explanation given in Fyodorov [82] (pp. 501-502) or Constantinescu and de Groote [52] (p.991) is at least outside of my comprehension.

To resolve these ambiguity, there exist two methods which are apparently no relation.

(I) First of all, we give the approach due to Vladimirov and Volovich [215] in pp.759-760.

Theorem 4.4.10 *Let*

$$x = x(y,\omega), \quad \theta = \theta(y,\omega) \tag{4.81}$$

be a supersmooth diffeomorphism from the neighbourhood $\mathcal{O}_1$ of the foliated singular manifold $S_1(\varphi_1,\Omega)$ in $\mathfrak{R}^{m|n}$ onto the neighbourhood $\mathcal{O}$ of the foliated singular manifold $S(\varphi,\Omega)$ in $\mathfrak{R}^{m|n}$: We denote $L(\theta) = \varphi(\Omega,\theta)$, $L_1(\omega) = \varphi_1(\Omega,\omega)$, where the function φ_1 is related to φ by $\varphi(q,\omega) = x(\varphi_1,\omega)$. Putting

$$M = \begin{bmatrix} A & C \\ D & B \end{bmatrix}, \quad \begin{cases} A = \dfrac{\partial x}{\partial y}, & C = \dfrac{\partial x}{\partial \omega}, \\ D = \dfrac{\partial \theta}{\partial y}, & B = \dfrac{\partial \theta}{\partial \omega}, \end{cases}$$

we assume that

(1) either $\det A|_{\omega=0}$ and $\det(B - DA^{-1}C)|_{\omega=0}$, or $\det B|_{\omega=0}$ and $\det(A - CB^{-1}D)|_{\omega=0}$, are invertible for all $y \in L_1(0)$ and

(2) $L(\theta) = L$ does not depend on θ.

Then, for any function $f \in \mathcal{C}_{\rm SS}(\mathcal{O} : \mathfrak{C})$ which is integrable on S, we have the change of variables formula

$$\int_S dx d\theta \, f(x,\theta) = \int_{S_1} dy d\omega \, f(x(y,\omega),\theta(y,\omega)) (\operatorname{sdet} M)(y,\omega). \tag{4.82}$$

Proof. We fix an arbitrary $x \in L = \varphi(G)$. Then from the first equation of (4.81), using the implicit function theorem, we have

$$y = \bar{y}(x,\omega), \quad \text{i.e.} \quad x = x(\bar{y}(x,\omega),\omega) \tag{4.83}$$

because the matrix A is invertible for all $\theta \in \mathfrak{R}^n_{\rm od}$. Then from the second equation of (4.81), we have

$$\theta = \theta(\bar{y}(x,\omega),\omega) = \bar{\theta}(x,\omega). \tag{4.84}$$

Using this change of variables in the integral $\int d\theta \, f(x,\theta)$ and assuming that the matrix $\partial\bar{\theta}/\partial\omega$ is invertible for all $\omega \in \mathfrak{R}^n_{\rm od}$, we find, from (**??**),

$$\int d\theta \, f(x,\theta) = \int d\omega \, f(x,\bar{\theta}(x,\omega)) \det{}^{-1} \frac{\partial\bar{\theta}}{\partial\omega}.$$

From (4.84), we obtain

$$\frac{\partial\bar{\theta}}{\partial\omega} = \frac{\partial\theta}{\partial\omega} + \frac{\partial\theta}{\partial y}\frac{\partial\bar{y}}{\partial\omega}. \tag{4.85}$$

Moreover, by (4.83),

$$\frac{\partial\bar{y}}{\partial\omega} = -(\frac{\partial x}{\partial y})^{-1}\frac{\partial x}{\partial\omega}\bigg|_{y=\bar{y}(x,\omega)} = -A^{-1}C|_{y=\bar{y}(x,\omega)}.$$

Therefore (4.85) takes the form

$$\frac{\partial\bar{\theta}}{\partial\omega} = (B - DA^{-1}C)|_{y=\bar{y}(x,\omega)}.$$

For fixed x, we have

$$\int d\theta\, f(x,\theta) = \int d\omega\, f(x,\bar{\theta}(x,\omega))\mathrm{det}^{-1}(B - DA^{-1}C)|_{y=\bar{y}(x,\omega)}. \tag{4.86}$$

$$\begin{aligned}\int_S dxd\theta\, f(x,\theta) &= \int_L dx\left[\int d\theta\, f(x,\theta)\right]\\ &= \int_L dx\left[\int d\omega\, f(x,\bar{\theta}(x,\omega))\mathrm{det}^{-1}(B - DA^{-1}C)|_{y=\bar{y}(x,\omega)}\right]\\ &= \int d\omega\left[\int_L dx\, f(x,\bar{\theta}(x,\omega))\mathrm{det}^{-1}(B - DA^{-1}C)|_{y=\bar{y}(x,\omega)}\right]\end{aligned}$$

Making the inner integral the change of variables $x = x(y,\omega)$ for fixed ω, we have

$$\begin{aligned}\int_S dxd\theta\, f(x,\theta) &= \int d\omega\left[\int_{L_1(\omega)} dy\, f(x(y,\omega),\theta(y,\omega))\mathrm{det}^{-1}(B - DA^{-1}C)\det A\right]\\ &= \int_{S_1} dyd\omega\, f(x(y,\omega),\theta(y,\omega))(\mathrm{sdet}\, M)(y,\omega)\end{aligned}$$

where $L_1(\omega) = \varphi_1(G,\omega)$, $\varphi_1(t,\omega) = \bar{y}(\varphi(t),\omega)$. □

Using this theorem, we have

Example 1′. Let $S = L \times \mathfrak{R}^2_{\mathrm{od}}$ be a singular foliated manifold in $\mathfrak{R}^{1|2}$, where $L = \varphi(G)$, $G = (0,1)$, $\varphi(t) = ct$. Under the change of variables

$$x = y + \omega_1\omega_2 = x(y,\omega),\ \theta_1 = \omega_1 = \theta_1(\omega),\ \theta_2 = \omega_2 = \theta_2(\omega),$$

we have

$$\begin{gathered}\mathrm{sdet}\ M = 1,\ L_1(\omega) = \varphi_1(G,\omega),\ \varphi_1(t,\omega) = ct - \omega_1\omega_2, t \in (0,1),\\ S_1 = \{(y,\omega)\ \big|\ y \in L_1(\omega), \omega \in \mathfrak{R}^2_{\mathrm{od}}\}.\end{gathered}$$

In this case, we have

$$\begin{aligned}\int_S dxd\theta\, f(x,\theta) &= \int d\theta\left[\int_G dt\, f(\varphi(t),\theta)\mathrm{det}\frac{\partial\varphi(t)}{\partial t}\right] = \int_0^1 dt\int d\theta\, f(et,\theta)\\ &= \int_0^1 dt\,\frac{\partial}{\partial\omega_2}\frac{\partial}{\partial\omega_1}f(et,\omega)\bigg|_{\omega=0}\\ &= \int d\omega\left[\int_G dt\, f(\varphi_1(t,\omega),\omega)\mathrm{det}\frac{\partial\varphi_1(t,\omega)}{\partial t}\right] = \int_{S_1} dyd\omega\, f(y+\omega_1\omega_2,\omega).\end{aligned}$$ □

Example 3. Let S be the manifold in $\mathfrak{R}^{1|2}$ defined by

$$G = (a,b),\ \varphi(t,\theta) = et + \theta_1\theta_2,\ L(\theta) = \varphi(G,\theta).$$

The change of variables

$$x=y,\ \theta_1=y\omega_1,\ \theta_2=y\omega_2 \quad \text{with} \quad \operatorname{sdet} M=\operatorname{sdet}\begin{pmatrix}1&0&0\\ \omega_1&y&0\\ \omega_2&0&y\end{pmatrix}=y^{-2}$$

carries S to $S_1\subset\mathfrak{R}^{1|2}$ defined by $\varphi_1(t,\omega)$ from $et+\varphi_1^2\omega_1\omega_2=\varphi_1$, i.e. $\varphi_1(t,\omega)=et+t^2\omega_1\omega_2$. In this case, we have

$$\begin{aligned}\int_S dxd\theta\, f(x,\theta)&=\int_a^b dt\,\frac{\partial}{\partial x}f(x,0)|_{x=et}+\int_a^b dt\,[\int d\theta f(et,\theta)]\\ &=\int_{S_1} dyd\omega\, f(y,y\omega)y^{-2}\end{aligned}$$

(II) We use the prescription of Rothstein [193] (see also Martellini and Teofilatto [155], Zirnbauer [?]) to remedy this ambiguity by altering the notion of the volume form on the superspace.

The change of variables (4.79), which is called degree increasing in [193] or non-splitting in [155], may be generated by a vector field $Y(y,\omega)$ with $(x,\theta)=e^{Y(y,\omega)}(y,\omega)$, given by

$$e^{Y(y,\omega)}=1+\omega_1\omega_2(y_1-iy_2)\frac{\partial}{\partial y_1}-i\omega_1\omega_2(y_1-iy_2)\frac{\partial}{\partial y_2}+(y_1-iy_2-1)\omega_1\frac{\partial}{\partial\omega_1}-(y_1-iy_2+1)\omega_2\frac{\partial}{\partial\omega_2}.$$

By the prescription given in [193] and more precisely in [155], we should have

$$\frac{dx_1dx_2}{2\pi}d\theta_1d\theta_2=-\frac{dy_1dy_2}{2\pi}d\omega_1d\omega_2[1-\big(\omega_1\omega_2(y_1-iy_2)\frac{\partial}{\partial y_1}-i\omega_1\omega_2(y_1-iy_2)\frac{\partial}{\partial y_2}\big)](y_1-iy_2)^{-2}.$$

Using this, we have

$$\begin{aligned}\int\frac{dx_1dx_2}{2\pi}d\theta_1d\theta_2\, e^{-(x_1^2+x_2^2+2\theta_1\theta_2)}=&-\int\frac{dy_1dy_2}{2\pi}d\omega_1d\omega_2\,(y_1-iy_2)^{-2}e^{-(y_1^2+y_2^2)}\\ &+\int\frac{dy_1dy_2}{2\pi}d\omega_1d\omega_2\,\omega_1\omega_2(y_1-iy_2)(\frac{\partial}{\partial y_1}-i\frac{\partial}{\partial y_2})(y_1-iy_2)^{-2}e^{-(y_1^2+y_2^2)}.\end{aligned}$$

Though the singularity $(y_1-iy_2)^{-2}$ must be treated by deleting $|y|<\epsilon$ from the integration domain and making $\epsilon\to0$, after calculating w.r.t. odd variables, we have

$$-\int\frac{dy_1dy_2}{2\pi}(y_1-iy_2)(\frac{\partial}{\partial y_1}-i\frac{\partial}{\partial y_2})(y_1-iy_2)^{-2}e^{-(y_1^2+y_2^2)}=1.$$

In fact,

$$\lim_{\epsilon\to0}\int_{U_\epsilon}\frac{dy_1dy_2}{2\pi}d\omega_1d\omega_2\,(y_1-iy_2)^{-2}e^{-(y_1^2+y_2^2)}=0 \quad \text{where} \quad U_\epsilon=\{(y,\omega)\in\mathfrak{R}^{2|2}\bigm| |y_{\rm B}|\ge\epsilon\},$$

and

$$\begin{aligned}&\int_{|y_{\rm B}|\ge\epsilon}\frac{dy_1dy_2}{2\pi}(y_1-iy_2)(\frac{\partial}{\partial y_1}-i\frac{\partial}{\partial y_2})(y_1-iy_2)^{-2}e^{-(y_1^2+y_2^2)}\\ &\quad=\int_{\partial\{|y_{\rm B}|\ge\epsilon\}}\frac{dy_1dy_2}{2\pi}(y_1-iy_2)^{-1}e^{-(y_1^2+y_2^2)}-\int_{|y_{\rm B}|\ge\epsilon}\frac{dy_1dy_2}{2\pi}(y_1-iy_2)^{-2}(\frac{\partial}{\partial y_1}-i\frac{\partial}{\partial y_2})((y_1-iy_2)e^{-(y_1^2+y_2^2)})\end{aligned}$$

with

$$\int_{\partial\{|y_{\rm B}|\ge\epsilon\}}\frac{dy_1dy_2}{2\pi}(y_1-iy_2)^{-1}e^{-(y_1^2+y_2^2)}=\int_0^{2\pi}d\phi\Big[\frac{re^{-r^2}}{\cos\phi-i\sin\phi}\Big]\Big|_{r=\epsilon}^{\infty}\to0 \quad \text{when} \quad \epsilon\to0$$

and

$$\lim_{\epsilon\to0}\int_{|y_{\rm B}|\ge\epsilon}\frac{dy_1dy_2}{2\pi}(y_1-iy_2)^{-2}(\frac{\partial}{\partial y_1}-i\frac{\partial}{\partial y_2})((y_1-iy_2)e^{-(y_1^2+y_2^2)})=1.$$

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