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WHAT IS SUPERANALYSIS AND HOW TO USE IT?

ATSUSHI INOUE

Department of Mathematics, Tokyo Institute of Technology,
Oh-okayama, Meguro-ku, Tokyo, 152-0033, Japan,
e-mail:inoue@math.titech.ac.jp

Claim: As a tiny little old mathematician, I dare to say the following for young people: For analysis in 21-th century, we should find out something-like "PDE on infinite dimensional phase space" which governs all Green functions of quantum field theory, and the star product below with respect to $\hbar$ would be "asymptotic" in some sense relating to these Green functions expanded in $\hbar$. In the following, I enumerate what I think interesting and relating the final goal to **analysis on infinite dimensional configuration manifolds with photons and electrons on equal footing**.

Feynman's problem

Feynman introduced the expression

$$E(t,s:q,q') = \int_{C_{t,s;q,q'}} d_F\gamma\, e^{i\hbar^{-1}\int_s^t L(\tau,\gamma(\tau),\dot\gamma(\tau))d\tau}, \quad L(t,\gamma,\dot\gamma) = \frac{1}{2}|\dot\gamma|^2 - V(t,\gamma)$$

$$\text{where} \quad C_{t,s;q,q'} \sim \{\gamma(\cdot) \in C([s,t]:\mathbb{R}^m) \,|\, \gamma(s)=q',\ \gamma(t)=q\},$$

and rederived the Schrödinger equation, not by substituting $-i\hbar\partial_q$ into $H(t,q,p)=\frac{1}{2}|p|^2+V(t,q)$.

(A) This expression contains the notorious Feynman measure $d_F\gamma$. That is, it is proved unfortunately that there exists no non-trivial 'Feynman measure' on ∞-dimensional spaces $C_{t,q,q'}$. Therefore, one of our main concern is how to 'justify' the results obtained by using such a notorious measure.

Why it is necessary to do so? Because, even if the usage of the Feynman measure is prohibited in mathematics, they get new insights in "quantum area" by 'using' it, for example, works done by E. Witten and other physicists.

Problem 1: Without using measure theory, how should one define path-integrals? A candidate of this direction would be to develop a theory of functional derivative equations, see, Inoue [18, 19, 20], or Gaussian functional integral such as Simon [42, 43], Glim and Jaffe [15]. Another one would be the generalized Riemann integral developped by Henstock [16], Pfeffer [40], Bartle [2], etc, which proceed without measure theoretic preparation.

Remark : It is well-known that the so-called Feynman measure does not exist. It seems better to explain this fact in the simplest case.

[**Non-existence of Feynman measure**] We cite from lecture note of Kuo [36]:

Definition 0.1. *Borel measure is called Lebesgue-like if the following conditions are satisfied:*
(1) The measure of any bounded Borel set is finite, if not empty, then it is positive.
(2) The measure is invariant under translation or certain other transformations.

Theorem 0.1. *Any ∞-dimensional, separable Hilbert space H has no nontrivial Lebesgue-like Borel measure.*

Proof. Let $\{e_1, e_2, \cdots\}$ be an orthonormal basis of H. Let μ be nontrivial Lebesgue-like Borel measure on H, defined on Borel algebra $\mathcal{B}(H)$. Define open sets as

$$B_n = \{u \in H \,\big|\, \|u-e_n\| < \frac{1}{2}\} \quad \text{and} \quad B = \{u\in H \,\big|\, \|u\| < 2\}.$$

Then, by the property (2) of Lebesgue-like measure, we have

$$0 < \mu(B_1) = \mu(B_2) = \cdots < \infty, \quad n \neq m \text{ and } \quad B_n \cap B_m = \emptyset \quad \text{and} \quad B_n \subset B.$$

By the property of measure,

$$\infty = \sum_{n=1}^{\infty} \mu(B_n) \le \mu(B) < \infty. \quad \text{contradiction!} \quad \square$$

(B) On the other hand, Feynman's derivation is efficiently used to construct a fundamental solution of the Schrödinger equation for suitable potentials. That is, a Fourier Integral Operator

$$U(t,s)u(q) = (2\pi\hbar)^{-m/2} \int_{\mathbb{R}^m} dq'\, D^{1/2}(t,s;q,q') e^{i\hbar^{-1}S(t,s;q,q')} u(q')$$

gives a "good parametrix" of the Schrödinger equation (shown by Fujiwara [13]). Here, $S(t,s;q,q')$ satisfies **the Hamilton-Jacobi equation** and $D(t,s;q,q')$, **the van Vleck determinant** of $S(t,s;q,q')$, satisfies the continuity equation. (**Good parametrix** means that not only it gives a parametrix but also its dependence of $\hbar$ is explicit, in other word, **Bohr's correspondence principle** is easily recovered via **stationary phase method** making $\hbar \to 0$. Feynman's main idea is to apply the stationary phase method to the path-integral formulation of the quantum mechanics when $\hbar \to 0$, by that we may recover the classical mechanics rather transparently.)

This formula is reformulated (by Inoue [24]) in the Hamiltonian form as

$$\begin{aligned} U_H(t,s)\, u(q) &= (2\pi\hbar)^{-m/2} \int_{\mathbb{R}^m} dp\, D_H^{1/2}(t,s\,;q,p) e^{i\hbar^{-1}S_H(t,s\,;q,p)} \hat{u}(p) \\ &= (2\pi\hbar)^{-m} \int_{\mathbb{R}^m \times \mathbb{R}^m} dp\, dq'\, D_H^{1/2}(t,s\,;q,p) e^{i\hbar^{-1}(S_H(t,s\,;q,p)-q'p)} u(q'). \end{aligned}$$

On the other hand, Feynman posed the following problem:

> ...... **path integrals suffer grievously from a serious defect**. They do not permit a discussion of spin operators or other such operators in a simple and lucid way. They find their greatest use in systems for which coordinates and their conjugate momenta are adequate. Nevertheless, spin is a simple and vital part of real quantum-mechanical systems. It is a serious limitation that the half-integral spin of the electron does not find a simple and ready representation. It can be handled if the amplitudes and quantities are considered as quaternions instead of ordinary complex numbers, but the lack of commutativity of such numbers is a serious complication.

Now, we give a partial answer of this problem by taking the Weyl equation as the simplest model with spin. That is, we rederive the Weyl equation from the Hamiltonian mechanics on superspace (called pseudo classical mechanics). More precisely speaking, introducing fermion variables to decompose the matrix structure, we define a Hamiltonian function on the superspace from which we construct solutions of the superspace version of the Hamilton-Jacobi and the continuity equations, respectively. (The fermion variables are assumed to have the inner structure represented by a countable number of Grassmann generators with the Fréchet topology.) Defining a Fourier Integral Operator with phase and amplitude given by these solutions, we may define the good parametrix for the (super) Weyl equation. This means, back to the ordinary matrix-valued representation, that we rederive the Weyl equation and therefore we give a partial solution of Feynman's problem ("partial" because we have not yet constructed an explicit integral representation of the fundametal solution itself using superanalysis. The desired explicit expression contains, for example, Maslov index and Berry's phase etc.).

We reformulate the above problem in mathematical language as follows:

Problem 2: Find a "**good representation**" of $\psi(t,q) : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}^2$ satisfying

$$\text{(W)} \qquad \begin{cases} i\hbar \dfrac{\partial}{\partial t}\psi(t,q) = \mathbb{H}(t)\psi(t,q), \\ \psi(\underline{t},q) = \underline{\psi}(q). \end{cases}$$

Here, $\underline{t}$ is arbitrarily fixed and

$$\mathbb{H}(t) = \mathbb{H}\Big(t,q,\frac{\hbar}{i}\frac{\partial}{\partial q}\Big) = \sum_{k=1}^{3} c\boldsymbol{\sigma}_k \Big(\frac{\hbar}{i}\frac{\partial}{\partial q_k} - \frac{\varepsilon}{c}A_k(t,q)\Big) + \varepsilon A_0(t,q) \tag{0.1}$$

with the Pauli matrices $\{\boldsymbol{\sigma}_j\}$.

In order to get a good parametrix, we transform the Weyl equation (W) on the Euclidian space $\mathbb{R}^3$ with value $\mathbb{C}^2$ to the super Weyl equation (SW) on the superspace $\mathfrak{R}^{3|2}$ with value $\mathfrak{C}$:

$$\text{(SW)}\qquad \begin{cases} i\hbar\dfrac{\partial}{\partial t}u(t,x,\theta) = \mathcal{H}\Big(t,x,\dfrac{\hbar}{i}\dfrac{\partial}{\partial x},\theta,\dfrac{\partial}{\partial\theta}\Big)u(t,x,\theta),\\ u(\underline{t},x,\theta) = \underline{u}(x,\theta). \end{cases}$$

Remark. For example, the operators

$$\sigma_1\Big(\theta,\frac{\partial}{\partial\theta}\Big) = \theta_1\theta_2 - \frac{\partial^2}{\partial\theta_1\partial\theta_2},\quad \sigma_2\Big(\theta,\frac{\partial}{\partial\theta}\Big) = i\Big(\theta_1\theta_2 + \frac{\partial^2}{\partial\theta_1\partial\theta_2}\Big),\quad \sigma_3\Big(\theta,\frac{\partial}{\partial\theta}\Big) = 1-\theta_1\frac{\partial}{\partial\theta_1}-\theta_2\frac{\partial}{\partial\theta_2},$$

act on $u(\theta_1,\theta_2)=u_0+u_1\theta_1\theta_2$ as $\boldsymbol{\sigma}_1 = \begin{pmatrix}0&1\\1&0\end{pmatrix}$, $\boldsymbol{\sigma}_2 = \begin{pmatrix}0&i\\-i&0\end{pmatrix}$, $\boldsymbol{\sigma}_3 = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$, respectively.

Theorem 0.2. *Let $\{A_j(t,q)\}_{j=0}^3 \in C^\infty(\mathbb{R}\times\mathbb{R}^3:\mathbb{R})$ satisfy, for any $k=0,1,2,\cdots$,*

$$\|\!|A_j|\!\|_{k,\infty} = \sup_{t,q,|\gamma|=k} |(1+|q|)^{|\gamma|-1}\partial_q^\gamma A_j(t,q)| < \infty \quad \textit{for}\quad j=0,\cdots,3. \tag{0.2}$$

We have a "good parametrix" for (SW) *represented by*

$$\mathcal{U}(t,\underline{t})\,u(x,\theta) = (2\pi\hbar)^{-3/2}\hbar\int_{\mathfrak{R}^{3|2}} d\xi\,d\pi\,\mathcal{D}^{1/2}(t,\underline{t}\,;x,\theta,\xi,\pi)e^{i\hbar^{-1}\mathcal{S}(t,\underline{t}\,;x,\theta,\xi,\pi)}\mathcal{F}\underline{u}(\xi,\pi).$$

Here, $\mathcal{S}(t,\underline{t}\,;x,\theta,\xi,\pi)$ and $\mathcal{D}(t,\underline{t}\,;x,\theta,\xi,\pi)$ satisfy the Hamilton-Jacobi equation and the continuity equation, respectively:

$$\text{(H}-\text{J)}\ \begin{cases} \dfrac{\partial}{\partial t}\mathcal{S}+\mathcal{H}\Big(t,x,\dfrac{\partial\mathcal{S}}{\partial x},\theta,\dfrac{\partial\mathcal{S}}{\partial\theta}\Big)=0,\\ \mathcal{S}(\underline{t},\underline{t}\,;x,\theta,\xi,\pi)=\langle x|\xi\rangle+\langle\theta|\pi\rangle, \end{cases} \quad \textit{and}\quad \text{(C)}\ \begin{cases} \dfrac{\partial}{\partial t}\mathcal{D}+\dfrac{\partial}{\partial x}\Big(\mathcal{D}\dfrac{\partial\mathcal{H}}{\partial\xi}\Big)+\dfrac{\partial}{\partial\theta}\Big(\mathcal{D}\dfrac{\partial\mathcal{H}}{\partial\pi}\Big)=0,\\ \mathcal{D}(\underline{t},\underline{t}\,;x,\theta,\xi,\pi)=1. \end{cases}$$

Here, for $u(x,\theta)=u_0(x)+u_1(x)\theta_1\theta_2$, Fourier transformation $\mathcal{F}$ is defined by

$$\mathcal{F}u(\xi,\pi) = (2\pi\hbar)^{-3/2}\hbar\int_{\mathfrak{R}^{3|2}} dx d\theta\, e^{i\hbar^{-1}(\langle x|\xi\rangle+\langle\theta|\pi\rangle)}u(x,\theta) = \hat{u}_1(\xi)+\hat{u}_0(\xi)\pi_1\pi_2.$$

Using the identification maps

$$\sharp: L^2(\mathbb{R}^3:\mathbb{C}^2)\to \mathcal{L}^2_{\mathrm{SS,ev}}(\mathfrak{R}^{3|2}) \quad\text{and}\quad \flat: \mathcal{L}^2_{\mathrm{SS,ev}}(\mathfrak{R}^{3|2})\to L^2(\mathbb{R}^3:\mathbb{C}^2),$$

$$\sharp\begin{pmatrix}\psi_1\\\psi_2\end{pmatrix}(x,\theta) = u_0(x)+u_1(x)\theta_1\theta_2 \quad\text{with}\quad u_j(x) = \sum_{|\alpha|=0}^\infty \frac{1}{\alpha!}\partial_q^\alpha\psi_{j+1}(x_{\mathrm{B}})x_{\mathrm{S}}^\alpha \quad\text{for}\quad x=x_{\mathrm{B}}+x_{\mathrm{S}},\ j=0,1,$$

$$(\flat u)(q) = \begin{pmatrix}\psi_1(q)\\\psi_2(q)\end{pmatrix} \quad\text{with}\quad \psi_1(q)=u(x,\theta)\big|_{x=q,\theta=0},\ \psi_2(q) = \frac{\partial^2}{\partial\theta_2\partial\theta_1}u(x,\theta)\bigg|_{x=q,\theta=0},$$

we get

Corollary 0.3. *Let $\{A_j(t,q)\}_{j=0}^3 \in C^\infty(\mathbb{R}\times\mathbb{R}^3:\mathbb{R})$ satisfy* (0.2). *We have a good parametrix for* (W) *represented by*

$$\mathbb{U}(t,\underline{t})\underline{\psi}(q) = \flat(2\pi\hbar)^{-3/2}\hbar\int_{\mathfrak{R}^{3|2}} d\xi\,d\pi\,\mathcal{D}^{1/2}(t,\underline{t}\,;x,\theta,\xi,\pi)e^{i\hbar^{-1}\mathcal{S}(t,\underline{t}\,;x,\theta,\xi,\pi)}\mathcal{F}(\sharp\underline{\psi})(\xi,\pi)\Big|_{x_{\mathrm{B}}=q}.$$

Remark: Feynman's problem should be considered also as the question of obtaining explicit solution formula for a certain PDE. For example, Strichartz estimate for a linear Schrödinger equation is derived by using the explicit representation of the solution. See, Taylor [45], Burq et al [7].

For example in [45], the fundamental solution $S(t,x)=e^{-it\Delta}\delta(x)$ of

$$i\frac{\partial u}{\partial t} = \Delta u \quad\text{with}\quad u(0,x)=\delta(x)\quad\text{on } S^d$$

has the special form at time $t=2\pi m/n$. That is, when $d=1$, we have

$$S(2\pi m/n,x) = \frac{1}{n}\sum_{j=0}^{n-1} G(m,n,j)\delta_{2\pi j/n} \quad\text{with}\quad G(m,n,j) = \sum_{\ell=0}^{n-1} e^{2\pi i(\ell^2 m+\ell j)/n}.$$

Problem 3: Find the deep and subtle relation between $H(t,x)=e^{t\Delta}\delta(x)$ and $S(t,x)$ above.

Problem for system of PDE. We regard Feynman's problem as calling a **new methodology of solving systems of PDE**. By the way, a system of PDE has **two non-commutativities**:
(i) One from $[\partial_q, q] = 1$ (Heisenberg relation),
(ii) the other from $[A, B] \neq 0$ (A, B are coefficient matrices of a system of PDE).

Non-commutativity from Heisenberg relation is nicely treated by using Fourier transformations (the theory of Ψ.D.Op).

Here, we want to give a **new method of treating non-commutativity** $[A, B] \neq 0$, using Fourier transformations on functions on superspace $\mathfrak{R}^{m|n}$.

Dogmatic opinion. For a given system of PDE, if we may reduce that system to scalar PDEs by diagonalization, then we must doubt whether it is truely necessary to use matrix representation. Therefore, if we need to represent some equations using matrices, we should try to treat system of PDE as it is, **without diagonalization**.

Remark. We may consider the method employed here, as a trial to **extend the "method of characteristics" to PDE with matrix-valued coefficients**.

Efetov's representation formula

Let $\mathfrak{U}_N$ be a set of Hermitian $N \times N$ matrices, which is identified with $\mathbb{R}^{N^2}$ as a topological space. In this set, we introduce a probability measure $d\mu_N(H)$ on $\mathfrak{U}_N$ by

$$
\begin{gathered}
d\mu_N(H) = \prod_{k=1}^N d(\Re H_{kk}) \prod_{j<k}^N d(\Re H_{jk}) d(\Im H_{jk}) P_{N,J}(H), \\
P_{N,J}(H) = Z_{N,J}^{-1} \exp\big[-\frac{N}{2J^2} \operatorname{tr} H^* H\big]
\end{gathered}
\tag{0.3}
$$

where $H = (H_{jk})$, $H^* = (H^*_{jk}) = (\overline{H}_{kj}) = {}^t\overline{H}$, $\prod_{k=1}^N d(\Re H_{kk}) \prod_{j<k}^N d(\Re H_{jk}) d(\Im H_{jk})$ being the Lebesgue measure on $\mathbb{R}^{N^2}$, and $Z_{N,J}^{-1}$ is the normalizing constant given by $Z_{N,J} = 2^{N/2}(J^2\pi/N)^{3N/2}$.

Let $E_\alpha = E_\alpha(H)$ ($\alpha = 1, \cdots, N$) be real eigenvalues of $H \in \mathfrak{U}_N$.

We put

$$
\rho_N(\lambda) = \rho_N(\lambda; H) = N^{-1} \sum_{\alpha=1}^N \delta(\lambda - E_\alpha(H)), \tag{0.4}
$$

where δ is the Dirac's delta. Denoting

$$
\langle f \rangle_N = \langle f(\cdot) \rangle_N = \int_{\mathfrak{U}_N} d\mu_N(H)\, f(H),
$$

for a function f on $\mathfrak{U}_N$, we get

Theorem 0.4 (Wigner's semi-circle law)**.**

$$
\lim_{N\to\infty} \langle \rho_N(\lambda) \rangle_N = w_{sc}(\lambda) = \begin{cases} (2\pi J^2)^{-1}\sqrt{4J^2 - \lambda^2} & \text{for} \quad |\lambda| < 2J, \\ 0 & \text{for} \quad |\lambda| > 2J. \end{cases} \tag{0.5}
$$

Remark. By definition, the limit in (0.5) is interpreted as

$$
\lim_{N\to\infty} \langle \phi, \int_{\mathfrak{U}_N} d\mu_N(H)\, N^{-1} \sum_{\alpha=1}^N \delta(\cdot - E_\alpha(H)) \rangle = \langle \phi, w_{sc} \rangle = \int_{\mathbb{R}} d\lambda\, \phi(\lambda) w_{sc}(\lambda)
$$

for any $\phi \in C_0^\infty(\mathbb{R}) = \mathcal{D}(\mathbb{R})$. $\langle \cdot, \cdot \rangle$ stands for the duality between $\mathcal{D}(\mathbb{R})$ and $\mathcal{D}'(\mathbb{R})$. We need more interpretation to give the meaning to $\int_{\mathfrak{U}_N} d\mu_N(H)\, N^{-1} \sum_{\alpha=1}^N \delta(\cdot - E_\alpha(H))$.

Seemingly, there exist several methods to prove this fact. Here, we want to explain a new derivation of this fact using odd variables obtained by Efetov. That is, the key expression obtained by introducing new auxiliary variables, is

$$
\langle \rho_N(\lambda) \rangle_N = \pi^{-1} \Im \int_{\mathfrak{Q}} dQ\, \big(\{(\lambda - i0)I_2 - Q\}^{-1} \big)_{bb} \exp\left[-N\mathcal{L}(Q)\right] \tag{0.6}
$$

where I_n stands for $n\times n$-identity matrix and

$$\mathcal{L}(Q)=\mathrm{str}\,[(2J^2)^{-1}Q^2+\log((\lambda-i0)I_2-Q)],$$

$$\mathfrak{Q}=\big\{Q=\begin{pmatrix}x_1 & \rho_1\\ \rho_2 & ix_2\end{pmatrix}\,|\,x_1,x_2\in\mathfrak{R}_{\mathrm{ev}},\ \rho_1,\rho_2\in\mathfrak{R}_{\mathrm{od}}\big\}\cong\mathfrak{R}^{2|2},\ \ dQ=\frac{dx_1dx_2}{2\pi}d\rho_1d\rho_2, \tag{0.7}$$

$$\big(((\lambda-i0)I_2-Q)^{-1}\big)_{bb}=\frac{(\lambda-i0-x_1)(\lambda-i0-ix_2)+\rho_1\rho_2}{(\lambda-i0-x_1)^2(\lambda-i0-ix_2)}.$$

Here in (0.6), **the parameter N appears only in one place**. This formula is formidably charming but **not yet directly justified**, like Feynman's expression of certain quantum objects using his measure.

Moreover, they claim without proof that they may apply **the method of steepest descent** to (0.6) when $N\to\infty$. That is, as

$$\delta\mathcal{L}(Q)\tilde{Q}=\frac{d}{d\epsilon}\mathcal{L}(Q+\epsilon\tilde{Q})\Big|_{\epsilon=0},$$

they seek solutions of

$$\delta\mathcal{L}(Q)=\mathrm{str}\,\big(\frac{Q}{J^2}-\frac{1}{\lambda-Q}\big)=0.$$

As a candidate of effective saddle points, they take

$$Q_c=\big(\frac{1}{2}\lambda+\frac{1}{2}\sqrt{\lambda^2-4J^2}\big)I_2,$$

and they have

$$\lim_{N\to\infty}\big\langle\rho_N(\lambda)\big\rangle_N=\pi^{-1}\Im(\lambda-Q_c)^{-1}_{bb}=w_{sc}(\lambda).\quad\square$$

Remark. Not only the expression (0.6) nor the applicability of the saddle point method to it are not so clear. To get the mathematical rigour, we **dare to loose such a beautiful expression** like (0.6), but we have the two formulae, rather ugly compared with (0.6), which lead to our results.

Proceeding mathematically, we get, as a byproduct of this new treatise,

Theorem 0.5 (A refined version of Wigner's semi-circle law)**.** *For each λ with $|\lambda|<2J$, when $N\to\infty$, we have*

$$\big\langle\rho_N(\lambda)\big\rangle_N=\frac{\sqrt{4J^2-\lambda^2}}{2\pi J^2}-\frac{(-1)^NJ}{\pi(4J^2-\lambda^2)}\cos\left(N\left[\frac{\lambda\sqrt{4J^2-\lambda^2}}{2J^2}+2\arcsin\left(\frac{\lambda}{2J}\right)\right]\right)N^{-1}+O(N^{-2}). \tag{0.8}$$

When λ satisfies $|\lambda|>2J$, there exist constants $C_\pm(\lambda)>0$ and $k_\pm(\lambda)>0$ such that

$$\Big|\big\langle\rho_N(\lambda)\big\rangle_N\Big|\le C_\pm(\lambda)\exp\left[-k_\pm(\lambda)N\right] \tag{0.9}$$

with $k_\pm(\lambda)\to0$ and $C_\pm(\lambda)\to\infty$ for $\lambda\searrow2J$ or $\lambda\nearrow-2J$, respectively.

Theorem 0.6 (The spectrum edge problem)**.** *Let $z\in[-1,1]$. We have*

$$\begin{aligned}\big\langle\rho_N(2J-zN^{-2/3})\big\rangle_N&=N^{-1/3}f(z/J)+O(N^{-2/3})\qquad\text{as } N\to\infty,\\ \big\langle\rho_N(-2J+zN^{-2/3})\big\rangle_N&=-N^{-1/3}f(z/J)+O(N^{-2/3})\qquad\text{as } N\to\infty,\end{aligned} \tag{0.10}$$

where

$$f(w)=\frac{1}{4\pi^2J}(\mathrm{Ai}'(w)^2-\mathrm{Ai}''(w)\,\mathrm{Ai}(w)),\quad \mathrm{Ai}(w)=\int_{\mathbb{R}}dx\,\exp\left[-\frac{i}{3}x^3+iwx\right].$$

Problem 4: In our paper [32], we didn't use the full power of superanalysis, because at that time there exists no steepest descent or stationary phase method in Super Analysis. Develop these methods to recover physicists claims mathmatically!

Atiyah-Singer's index theorem

After extending C^∞-manifold M as a supermanifold $\widetilde{\mathcal{M}}$, we may describe a Hamiltonian $\mathcal{H}$ on $\mathcal{T}^*\widetilde{\mathcal{M}}$ which extends supersymmetrically a Hamiltonian H corresponding to a Riemannian metric g on M. The classical mechanics on T^*M defined by H is extended to that on $\mathcal{T}^*\widetilde{\mathcal{M}}$ defined by $\mathcal{H}$. Using L^2-scheme on $\widetilde{\mathcal{M}}$, we may construct a 'heat operator'(=super heat operator) with its fundamental solution. Calculating the trace of that fundamental solution of that super heat operator, we get Gauss-Bonnet-Chern Theorem which is considered in some sense the extension of Weyl Theorem for the asymptotic behavior of the heat kernel on a manifold M when t tends to 0.

Theorem 0.7 (Weyl Theorem)**.** *Let (M,g) be a closed orientable Riemannian manifold with $\dim M=m$. Let $H(t,q,q')$ be a fundamental solution of the heat equation $\dfrac{\partial}{\partial t}-\dfrac{1}{2}\Delta$, then we have*

$$\text{tr } e^{\frac{1}{2}t\Delta}=\int_M dqH(t,q,q)\sim(2\pi t)^{-m/2}volM+O(t^{(-m/2)+1})\quad as\ t\to 0. \tag{0.11}$$

Here, the Riemann metric is given by

$$g=g_{ij}(q)dq^idq^j,\quad g(q)=\det(g_{ij}(q)),\quad \Delta=\frac{1}{\sqrt{g(q)}}\frac{\partial}{\partial q^i}(\sqrt{g(q)}g^{ij}(q)\frac{\partial}{\partial q^j}).$$

Remark 1. $A_t\sim B_t+O(t^\alpha)$ as $t\to 0$ stands for $|A_t-B_t|\le Ct^\alpha$ for suitable constant C when $0<t\ll 1$.

Remark 2. It is well-known that (0.11) stemms readily from the asymptotic behavior of the heat kernel near $t=0$ given by

$$H(t,q,q')\sim(2\pi t)^{-m/2}(1+a_1t+a_2t^2+\cdots)e^{-\frac{d(q,q')^2}{2t}} \tag{0.12}$$

where $d(q,q')$=the distance between q and q' w.r.t. the Riemann metric g and a_j are represented by the geometrical terms.

More precisely,

Theorem 0.8. *Let (M,g) be a closed, orientable smooth Riemannian manifold of $\dim M=m$. For $H(q,p)=\frac{1}{2}g^{jk}(q)p_jp_k\in C^\infty(T^*M)$, there exists a 1-parameter C_0-semigroup $\{T_t\}_{t\ge 0}$ on $\mathbf{B}(L^2(M))$ such that for any $u\in C_0^\infty(M)$,*

$$\begin{cases}\dfrac{\partial}{\partial t}(T_tu)(q)=\hat{H}(q,\partial_q)(T_tu)(q) \quad with\quad \hat{H}=\hat{H}(q,\partial_q)=\dfrac{1}{2}\Delta-\dfrac{1}{12}R,\\ \text{s-lim}_{t\to 0}T_tu=u,\end{cases} \tag{0.13}$$

where R stands for the scalar curvature of M. Moreover, for $t>0$, $T_t=e^{t\hat{H}}$ is of trace class in $\mathbf{B}(L^2(M))$ satisfying

$$\lim_{t\to 0}(2\pi t)^{m/2}\text{ tr } T_t=vol\ M. \tag{0.14}$$

Remark. This theorem is obtained by combining Inoue-Maeda [27] and Maeda [38] under more general conditions.

Theorem 0.9 (Gauss-Bonnet-Chern Theorem)**.** *Let (M,g) be a closed orientable Riemannian manifold of even dimension $m=2\ell$. Then*

$$\chi(M)=\int_M dqK(q) \tag{0.15}$$

where $\chi(M)$ is the Euler number of M,

$$K(q)=\frac{(-1)^\ell}{(4\pi)^\ell\ell!2^\ell}\sum_{\vartheta,\varpi\in\wp_d}(-1)^{|\varpi|}(-1)^{|\vartheta|}R^{\varpi_1\varpi_2}{}_{\vartheta_1\vartheta_2}(q)\cdots R^{\varpi_{m-1}\varpi_m}{}_{\vartheta_{m-1}\vartheta_m}(q), \tag{0.16}$$

$\wp_m$ is the permutation group of order m, $|\varpi|$ is the parity of $\varpi=(\varpi_1,\cdots,\varpi_m)\in\{1,\cdots,m\}^m$, and $R^{ij}{}_{kl}(q)$ is the curvature tensor of (M,g) at $q\in M$.

Claim 0.1. *There exists mathematically a link behind these two theorems.*

In other words, we give a new proof of Gauss-Bonnet-Chern Theorem, which is considered as the super version of Weyl Theorem. This proof is suggested by physicists; Witten [47, 48], Alvarez-Gaumé [1], Mañes & Zumino [39], Friedan & Windey [11]. See also Getzler [14], Cycon et al [9].

Following is not yet published in this form:

Theorem 0.10. *Let (M,g) be a closed, orientable smooth Riemannian manifold of $\dim M=m$. Then, we have the following:*

1. *There exists canonically a supermanifold $\widetilde{\mathcal{M}}$, called a super extension of M.*
2. *The Hamiltonian $H(q,p)$ has a superextension $\mathcal{H}=\mathcal{H}(x,\xi,\theta,\pi)\in\mathcal{C}_{SS}(\mathcal{T}^*\tilde{M})$ given by*

$$\mathcal{H}=\frac{1}{2}g^{jk}(\xi_j+i\omega_{abj}\theta^a\pi^b)(\xi_k+i\omega_{cdk}\theta^c\pi^d)-\frac{1}{4}R_{abcd}\theta^a\pi^b\theta^c\pi^d. \tag{0.17}$$

3. *There exists a 1-parameter C_0-semigroup $\{\mathcal{T}_t\}_{t\geq 0}$ on $\mathbf{B}(\mathfrak{L}^2_{SS}(\widetilde{\mathcal{M}}))$ satisfying*

$$\begin{cases} \dfrac{\partial}{\partial t}(\mathcal{T}_t u)(x,\theta) = \hat{\mathcal{H}}(x,\partial_x,\theta,\partial_\theta)(\mathcal{T}_t u)(x,\theta) \quad \text{for} \quad u \in \mathfrak{C}_{SS,0}(\tilde{M}), \\ \text{s-lim}_{t\to 0}\ \mathcal{T}_t u = u. \end{cases} \tag{0.18}$$

Here, $\hat{\mathcal{H}}(x,\partial_x,\theta,\partial_\theta)$ is the Weyl quantization of $\mathcal{H}(x,\xi,\theta,\pi)$.

4. *Ther exists an operator $\hat{\mathcal{P}} \in \mathbf{B}(\mathfrak{L}^2_{SS}(\tilde{M}))$ such that for $t>0$, $\hat{\mathcal{P}}\mathcal{T}_t$ is of trace class in $\mathbf{B}(\mathfrak{L}^2_{SS}(\tilde{M}))$ and*

$$\chi(M) = \lim_{t\to 0} \mathrm{tr}\ \hat{\mathcal{P}}\mathcal{T}_t = \begin{cases} \int_M dq K(q) & \text{when } m = 2\ell, \\ 0 & \text{otherwise.} \end{cases} \tag{0.19}$$

Problem 5: In case of ordinary manifold, there is the inverse statement of Theorem 0.8 called " the most probable path", see Fujita-Kotani [12], Takahashi-Watanabe [44]. How about the super case?

Weyl quantization for bosonic and fermionic cases

Weyl quantization for bosonic case. For $B(q,p)$, we define the so-called Weyl quantization

$$\hat{B}u(q) = \frac{1}{(2\pi\hbar)^m}\int_{\mathbb{R}^m}\int_{\mathbb{R}^m} e^{i\hbar^{-1}p(q-q')} B\left(\frac{q+q'}{2},p\right)u(q')dqdp.$$

Then, it is well-known that

(0): (Linearity): $\hat{B}\longmapsto B(q,p)$ is linear map.

(1): (Reality): $B^*\longmapsto \overline{B(q,p)}$.

(2): If $\hat{B}$ is of trace class, then

$$\mathrm{tr}\ \hat{B} = \frac{1}{(2\pi\hbar)^m}\int_{T^*(\mathbb{R}^m)} B(q,p)dqdp.$$

(3): (Covariance) If $G \in Sp(2m:\mathbb{R})$ is a linear canonical transformation with metaplctic representation $\pi_M(G)$, then $\hat{B}\longmapsto B(q,p)$ implies

$$\pi_M(G)^*\hat{B}\pi_M(G)\longmapsto B(G(q,p)).$$

Stratonovich-Weyl correspondence for j-spin. The correspondence which gives a function W_A from S^2 to $\mathbb{C}$, i.e. $W_A \in F(S^2:\mathbb{C})$, for a given operator A on the Hilbert space $\mathbb{C}^{2j+1}$:

$$L(\mathbb{C}^{2j+1}:\mathbb{C}^{2j+1}) \ni A\longmapsto W_A \in F(S^2:\mathbb{C})$$

is called Stratonovich-Weyl symbol W_A associated to A. This satisfies

(0): (Linearity): $A \to W_A$ is one-to-one, linear map.

(1): (Reality): $W_{A^*}(\mathbf{n}) = \overline{W_A(\mathbf{n})}$.

(2): (Standardization): $\frac{2j+1}{4\pi}\int_{S^2} W_A(\mathbf{n})d\mathbf{n} = \mathrm{tr}\ A$

(3): (Traciality): $\frac{2j+1}{4\pi}\int_{S^2} W_A(\mathbf{n})W_B(\mathbf{n})d\mathbf{n} = \mathrm{tr}\,(AB)$

(4): (Covariance):$W_{g\cdot A}(\mathbf{n}) = W_A(g^{-1}\cdot\mathbf{n})$ where for $g \in SU(2)$, we define $g\cdot A = \pi_j(g)A\pi_j(g)^{-1}$.

For $\mathbf{n} \in S^2$, we may write

$$W_A(\mathbf{n}) = \mathrm{tr}\,(A\Delta^j(\mathbf{n}))$$

where Δ^j is an operator-valued function on S^2, called Stratonovich-Weyl operator kernel. Moreover, we get

$$A = \frac{2j+1}{4\pi}\int_{S^2} W_A(\mathbf{n})\Delta^j(\mathbf{n})d\mathbf{n}.$$

This satisfies

(1): $\Delta^j(\mathbf{n})^* = \Delta^j(\mathbf{n})\mathbf{n} \in S^2$.

(2): $\frac{2j+1}{4\pi}\int_{S^2}\Delta^j(\mathbf{n})d\mathbf{n} = \mathbb{I}_{2j+1}$.

(3): $\frac{2j+1}{4\pi}\int_{S^2}\mathrm{tr}\,(\Delta^j(\mathbf{m})\Delta^j(\mathbf{n}))\Delta^j(\mathbf{n})d\mathbf{n} = \Delta^j(\mathbf{m})$.

(4): $\Delta^j(g\cdot\mathbf{n}) = \pi_j(g)\Delta^j(\mathbf{n})\pi_j(g)^{-1}$.

Problem 6: This approach to treat j-spin is proposed, for example, in Bolte, Glaser and Keppeler [4], Várilly and Gracia-Bondia [46]. Make this approach to the one appreciable even for innocent mathematicians, and use that for systems of PDE efficiently, if possible. For example, Brummelhuis and Nourrigat [6], Keppeler [34] and Bolte and Keppeler [5]. Of course, it is desirable to describe the supersymmetry using this quantization.

Deformation quantization

We follow the presentation in Hirshfeld and Henselder [17].

Deformation quantization achieves the passage from classical mechanics to quantum mechanics by the replacement of the pointwise multiplication of functions on phase space by the star product.

Let M be an open subset of $\mathbb{R}^d$ with a Poisson structure

$$\{f,g\} = \alpha^{ij}(x)\partial_i f(x)\partial_j g(x) \quad \text{with} \quad \alpha^{il}\partial_l\alpha^{jk} + \alpha^{jl}\partial_l\alpha^{ki} + \alpha^{kl}\partial_l\alpha^{ij} = 0$$

Find an associative product $\bigstar$ on $C^\infty(M)[[\hbar]]$ such that for $f,g\in C^\infty(M)$

$$f\bigstar g(x) = f(x)g(x) + \frac{i\hbar}{2}\{f,g\}(x) + O(\hbar^2).$$

Kontsevich [35] gave a solution of this problem, therefore the Field medal was awarded.

A prototype is the construction of quantum electrodynamics (QED) by quantization of Maxwell's classical electrodynamics. Quantization proceeds according to Dirac's prescription of replacing the classical Poisson brackets of phase space variables by the commutators of quantum mechanical operators.

On the other hand, because of the Pauli principle, ferminionic systems do not have a classical limit in the sense that bosonic systems do.

Problem 7: How do we extend Kontsevich's result containing fermions, not only quantum mechanics but also quantum field theory?

See, for example, Cattaneo and Felder [8]. Here, they write down symbolically

$$f\bigstar g(x) = \int_{\gamma(\pm\infty)=x} f(\gamma(1))g(\gamma(0))e^{i\hbar^{-1}\int_\gamma d^{-1}\omega}d\gamma, \tag{0.20}$$

where (M,ω) is the symplectic manifold and the integral over trajectories $\gamma:\mathbb{R}\to M$ is to be understood as an expansion around the classical solution $\gamma(t) = x$, which is a constant function of time since the Hamiltonian vanishes. In that paper, they extend also the expression (0.20), to the bosonic quantum field theory.

By the way, photon and electron should be treated on the same footing, so claimed in Berezin and Marinov [3] and they introduced a prototype of superanalysis.

Problem 8: Some years before, the constructive quantum field theory is in fashion, for example Glim and Jaffe [15]. Can we use this theory to give a transparent and simple way of getting Kontsevich's result.

Other Problems

Problem 9: Recently, I found an article entitled *The Cauchy-Riemann equations in infinite dimensions* [37]. Relate this to functional derivative equations (FDE)!

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