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5.2 Integration on the superspace

Integration (even case) Now, we define the integration of a supersmooth function $u(x)$ on an even superdomain $U_{ev} \subset$, which is similar to the integral of holomorphic functions on a complex domain. (See, Rogers [?, ?, ?, ?].)

Definition 5.1 *Let $u(x)$ be a supersmooth function defined on a even super domain $U_{ev} \subset \mathfrak{R}^{1|0}$. Let $\lambda = \lambda_{\mathrm{B}} + \lambda_{\mathrm{S}}$, $\mu = \mu_{\mathrm{B}} + \mu_{\mathrm{S}} \in U_{ev}$ and let a continuous and piecewise C^1-curve $c : [\lambda_{\mathrm{B}}, \mu_{\mathrm{B}}] \to U_{ev}$ be given such that $c(\lambda_{\mathrm{B}}) = \lambda$, $c(\mu_{\mathrm{B}}) = \mu$. We define*

$$\int_c dx\, u(x) = \int_{\lambda_{\mathrm{B}}}^{\mu_{\mathrm{B}}} dt\, u(c(t))\dot{c}(t) \in \tag{5.1}$$

and call it the integral of u *along the curve c.*

Using the integration by parts, we get the following fundamental result (see de Witt [?]).

Porposition 5.1 *Let $u(t) \in C^\infty([\lambda_{\mathrm{B}}, \mu_{\mathrm{B}}] : \mathfrak{C})$ and let $u(x)$ be the Grassmann continuation of $u(t)$. Suppose that there exists a function $U(t) \in C^\infty([\lambda_{\mathrm{B}}, \mu_{\mathrm{B}}] : \mathfrak{C})$ satisfying $U'(t) = u(t)$ on $[\lambda_{\mathrm{B}}, \mu_{\mathrm{B}}]$. Then, for any continuous and piecewise C^1-curve $c : [\lambda_{\mathrm{B}}, \mu_{\mathrm{B}}] \to U_{ev} \subset \mathfrak{R}^{1|0}$ such that $c(\lambda_{\mathrm{B}}) = \lambda$, $c(\mu_{\mathrm{B}}) = \mu$, we have*

$$\int_c dx\, u(x) = U(\lambda) - U(\mu). \tag{5.2}$$

Proof. By definition, we get

$$\begin{aligned}
\int_{\lambda_{\mathrm{B}}}^{\mu_{\mathrm{B}}} dt\, u(c(t))\dot{c}(t) &= \int_{\lambda_{\mathrm{B}}}^{\mu_{\mathrm{B}}} dt \sum_{\ell\geq 0} \frac{1}{\ell!} u^{(\ell)}(c_{\mathrm{B}}(t)) c_{\mathrm{S}}(t)^\ell (\dot{c}_{\mathrm{B}}(t) + \dot{c}_{\mathrm{S}}(t)) \\
&= \int_{\lambda_{\mathrm{B}}}^{\mu_{\mathrm{B}}} dt\, u(c_{\mathrm{B}}(t))\dot{c}_{\mathrm{B}}(t) + \int_{\lambda_{\mathrm{B}}}^{\mu_{\mathrm{B}}} dt \sum_{k\geq 1} \frac{1}{k!} u^{(k)}(c_{\mathrm{B}}(t)) \dot{c}_{\mathrm{B}}(t) c_{\mathrm{S}}(t)^k \\
&\qquad\qquad + \int_{\lambda_{\mathrm{B}}}^{\mu_{\mathrm{B}}} dt \sum_{\ell\geq 0} \frac{1}{\ell!} u^{(\ell)}(c_{\mathrm{B}}(t)) c_{\mathrm{S}}(t)^\ell \dot{c}_{\mathrm{S}}(t) \\
&= U(\mu_{\mathrm{B}}) - U(\lambda_{\mathrm{B}}) + \sum_{\ell\geq 0} \frac{1}{(\ell+1)!} \left\{ U^{(\ell+1)}(\mu_{\mathrm{B}})(\mu_{\mathrm{S}})^{\ell+1} - U^{(\ell+1)}(\lambda_{\mathrm{B}})(\lambda_{\mathrm{S}})^{\ell+1} \right\} \\
&= U(\mu) - U(\lambda). \square
\end{aligned}$$

Corollary 5.1 *Let $u(x)$ be a supersmooth function defined on a even superdomain $U_{ev} \subset \mathfrak{R}^{1|0}$ into . Let c_1, c_2 be continuous and piecewise C^1-curves from $[\lambda_{\mathrm{B}}, \mu_{\mathrm{B}}] \to U_{ev}$ such that $\lambda = c_1(\lambda_{\mathrm{B}}) = c_2(\lambda_{\mathrm{B}})$ and $\mu = c_1(\mu_{\mathrm{B}}) = c_2(\mu_{\mathrm{B}})$. If c_1 is homotopic to c_2, then*

$$\int_{c_1} dx\, u(x) = \int_{c_2} dx\, u(x). \tag{5.3}$$

Thus, if $[\lambda_{\mathrm{B}}, \mu_{\mathrm{B}}] \subset \pi_{\mathrm{B}}(U_{ev})$, we have

$$\int_\lambda^\mu dx\, u(x) = \int_{\lambda_{\mathrm{B}}}^{\mu_{\mathrm{B}}} dt\, u(t). \tag{5.4}$$

Because of (5.4), we have

Definition 5.2 *(1) Let I_{ev} be a even superdomain in such that $\pi_{\mathrm{B}}(I_{ev}) = \prod_{j=1}^m (a_j, b_j) \subset \mathbb{R}^m$ with $-\infty < a_j < b_j < \infty$, which is called a even supercube. For $u \in \mathcal{C}_{\mathrm{SS}}(I_{ev} : \mathfrak{C})$, we define*

$$\int_{I_{ev}} dx\, u(x) = \int_{a_1}^{b_1} dq_1 \cdots \int_{a_m}^{b_m} dq_m\, u(q_1, \cdots, q_m) = \int_{\pi_{\mathrm{B}}(I_{ev})} dx_{\mathrm{B}}\, u(x_{\mathrm{B}}). \tag{5.5}$$

(2) For any even superdomain $U_{\mathrm{ev}} \subset$ such that $\pi_{\mathrm{B}}(U_{\mathrm{ev}})$ is of definite area, we may put

$$\int_{U_{\mathrm{ev}}} dx\, u(x) = \int_{\pi_{\mathrm{B}}(U_{\mathrm{ev}})} dx_{\mathrm{B}}\, u(x_{\mathrm{B}}) \tag{5.6}$$

for $u \in \mathcal{C}_{\mathrm{SS}}(U_{\mathrm{ev}} : \mathfrak{C})$.

Remarks. (1) The formula (5.6) stemms easily from the well-known procedures to define multiple integrals in Riemannian integration.
(2) The reason why we should use 'contour integration' is explained precisely in Rogers [?]. As we treat

only even superdomains here, her arguments there are simplified considerably. But we should change the role of the 'body' in our treatment, if we need to catch up all arguments of Rogers, which is noted in the remark after Propositon ??.

Super p-forms ω: In order to define "super"-form ω of degree p over a p-dimensional singular manifold L, Volvich-Vladimirov [?] introduced the following:

Let M be a p-dimensional oriented manifold of the class C^1 and the mapping $\varphi : M \to U \subset \mathbb{R}^m$ be of the class C^1. The set $L = \varphi(M)$, or, more precisely, the pair (M, φ), is called a p-dimensional singular manifold. That is, every p-dimensional manifold $M \subset \mathbb{R}^m$ is a p-dimensional singular manifold with respect to the identity mapping.

The pairs (M, φ) and (M_1, φ_1) are said to be equivalent if $L = \varphi(M) = \varphi_1(M_1)$ and there exists a diffeomorphism $f : M \to M_1$ such that $\varphi = \varphi_1 \circ f$.

$$\int_M \varphi^* \omega = \int_{M_1} \varphi_1^* \omega. \tag{5.7}$$

We may interprete the equation (5.7) as a change of variables as follows:

????

Integration (odd case) It seems natural to put formally

$$d\theta_j = \sum_{I \in \mathcal{I}, |I| = odd} d\theta_{j,I} \sigma^I \quad \text{for} \quad \theta_j = \sum_{I \in \mathcal{I}, |I| = odd} \theta_{j,I} \sigma^I.$$

Then, we have

$$d\theta_j \wedge d\theta_k = d\theta_k \wedge d\theta_j.$$

This make us imagine that even if there exists the notion of integration, it differs much from the ordinary one.

Let v be a polynomial of odd variables $\theta = (\theta_1, \cdots, \theta_n) \in \mathfrak{R}^n_{\rm od}$ such that

$$v(\theta_1, \cdots, \theta_n) = \sum_{|b| \le n} v_b \theta^b \quad \text{with homogeneous } v_b \theta^b \in \mathfrak{C} \text{ for each } b.$$

Denote by $P_n(\mathfrak{C})$ the set of all v as above.

Definition 5.3 *For $v \in P_n(\mathfrak{C})$, we put*

$$\int d\theta \, v(\theta) = \int d\theta_n \cdots d\theta_1 \, v(\theta_1, \cdots, \theta_n) = ({}_{\theta_n} \cdots {}_{\theta_1} v)(0)$$

and we call it the integral of v on .

Above definition yields readily that

$$\int d\theta_n \cdots d\theta_1 \, \theta_1 \cdots \theta_n = 1.$$

Moreover, we have

Porposition 5.2 *Given $v, w \in P_n(\mathfrak{C})$, we have the following:*

(1) ($\mathfrak{C}$-linearity) For any homogeneous $\lambda, \mu \in \mathfrak{C}$,

$$\int d\theta(\lambda v + \mu w)(\theta) = (-1)^{np(\lambda)}\lambda \int d\theta\, v(\theta) + (-1)^{np(\mu)}\mu \int d\theta\, w(\theta). \tag{5.8}$$

(2) (Translational invariance) For any $\rho \in$, we have

$$\int d\theta\, v(\theta+\rho) = \int d\theta\, v(\theta). \tag{5.9}$$

(3) (Integration by parts) For $v \in P_n(\mathfrak{C})$ such that $p(v) = 1$ or 0, we have

$$\int d\theta\, v(\theta)_{\theta_s} w(\theta) = -(-1)^{p(v)} \int d\theta\, ({}_{\theta_s} v(\theta))w(\theta). \tag{5.10}$$

(4) (Linear change of variables) Let $A = (A_{jk})$ with $A_{jk} \in$ be invertible. Then,

$$\int d\theta\, v(\theta) = (\det A)^{-1} \int d\omega\, v(A\cdot\omega). \tag{5.11}$$

(5) (Iteration of integrals)

$$\int d\theta\, v(\theta) = \int_{\mathfrak{R}^{0|n-k}} d\theta_n \cdots d\theta_{k+1} \left(\int_{\mathfrak{R}^{0|k}} d\theta_k \cdots d\theta_1\, v(\theta_1, \cdots, \theta_k, \theta_{k+1}, \cdots, \theta_n) \right). \tag{5.12}$$

(6) (Odd change of variables) Let $\theta = \theta(\omega)$ be an odd change of variables such that $\theta(0) = 0$ and $\det \left.\dfrac{\partial\theta(\omega)}{\partial\omega}\right|_{\omega=0} \neq 0$*. Then, for any $v \in P_n(\mathfrak{C})$,*

$$\int d\theta\, v(\theta) = \int d\omega\, v(\theta(\omega)) \det{}^{-1} \frac{\partial\theta(\omega)}{\partial\omega}. \tag{5.13}$$

(7) For $v \in P_n(\mathfrak{C})$ and $\omega \in$,

$$\int d\theta\, (\theta_1 - \omega_1) \cdots (\theta_n - \omega_n) v(\theta) = v(\omega). \tag{5.14}$$

Remarks. (1) All above assertions are easily obtained by following the arguments in pp.755-757 of Vladimirov and Volovivh [?], so proofs are omitted here.
(2) (5.14) allows us to put $\delta(\theta-\omega)$= $(\theta_1 - \omega_1) \cdots (\theta_n - \omega_n)$, though $\delta(-\theta) = (-1)^n \delta(\theta)$.

Integration (mixed case) Finally, we define

Definition 5.4 *Let $U = U_{ev} \times \mathfrak{R}^n_{\mathrm{od}} \subset \mathfrak{R}^{m|n}$ be a superdomain and let $u \in \mathcal{C}_{\mathrm{SS}}(U : \mathfrak{C})$, that is, $u(x,\theta) = \sum u_a(x)\theta^a$ with $u_a(x) \in \mathcal{C}_{\mathrm{SS}}(U_{ev} : \mathfrak{C})$. Then, we define*

$$\begin{aligned} \int_U dx d\theta\, u(x,\theta) &= \int_{U_{ev}} dx \left\{ \int d\theta\, u(x,\theta) \right\} \\ &= \int_{\pi_{\mathrm{B}}(U_{ev})} dx_{\mathrm{B}}\, u_{\tilde{1}}(x_{\mathrm{B}}) \quad \textit{with} \quad \tilde{1} = (1, \cdots, 1) \\ &= \int d\theta \left\{ \int_{U_{ev}} dx\, u(x,\theta) \right\}. \end{aligned} \tag{5.15}$$

More generally, we consider the following situation.

Definition 5.5 *Let Ω be a domain in $\mathbb{R}^m$.*

(1) Let $\varphi \in C^\infty(\Omega : \mathfrak{R}^m_{\rm ev})$ with image $L = \varphi(\Omega)$, $\varphi(q) = \sum_{I\in\mathcal{I},|I|=even} \varphi_I(q)\sigma^I$, $\varphi_I(q) \in C^\infty(\Omega : \mathbb{R}^m)$. For $f(x,\theta) = \sum_{|a|\le n} f_a(x)\theta^a$ with $f_a(x)$ being the Grassmann extension of $f_a(x_{\rm B})$ such that $f(x,\theta)$ is integrable in L. In this case, we call the following expression "the integral of the function $f(x,\theta)$ over the manifold $S = L \times \mathfrak{R}^n_{\rm od}$" where

$$\int_S dxd\theta\, f(x,\theta) = \int d\theta \bigg[\int_L dx\, f(x,\theta)\bigg].$$

Since we define, for each θ,

$$\int_L dx\, f(x,\theta) = \int_\Omega dq\, f(\varphi(q)) \det\left(\frac{\partial\varphi(q)}{\partial q}\right),$$

we have

$$\int_S dxd\theta\, f(x,\theta) = \int_\Omega dq\, \frac{\partial}{\partial\theta_n}\cdots\frac{\partial}{\partial\theta_1} f(\varphi(q),0) \det\left(\frac{\partial\varphi(q)}{\partial q}\right).$$

In this case, it is clear that

$$\int_S dxd\theta\, f(x,\theta) = \int_L dx \bigg[\int d\theta\, f(x,\theta)\bigg].$$

(2) For $\theta \in^n$, we put a set $\varphi(\Omega,\theta)$ (called, a singular manifold) in $\mathfrak{R}^m_{\rm ev}$ by

$$\varphi(\Omega,\theta) = \{x \in \mathfrak{R}^m_{\rm ev} \mid x = \varphi(q,\theta),\, q \in \Omega \subset \mathbb{R}^m\} \quad \text{for} \quad \theta \in \mathfrak{R}^n_{\rm od}.$$

Here,

$$\varphi(q,\theta) = \varphi(q) + \sum_{2\le|a|=even\le n} \varphi_a(q)\theta^a \quad \text{with} \quad \varphi(q), \varphi_a(q) \in C^\infty(\Omega : \mathbb{R}).$$

We call a set in $\mathfrak{R}^{m|n}$ of the form

$$S = S(\varphi,\Omega) = \{(x,\theta) \in \mathfrak{R}^{m|n} \mid \theta \in \mathfrak{R}^n_{\rm od},\, x \in \varphi(\Omega,\theta)\}$$

a foliated singular manifold. In this case, we define

$$\int_S dxd\theta\, f(x,\theta) = \int d\theta \bigg[\int_{L(\theta)} dx\, f(x,\theta)\bigg].$$

Let $f(x,\theta)$ be a function with values in $\mathfrak{C}$ defined on S. We define the integral of f on S as

$$\int_S dxd\theta\, f(x,\theta) = \int d\theta \bigg[\int_{\varphi(\Omega,\theta)} dx\, f(x,\theta)\bigg]$$

Here, the inner integral is understood in the following sense:

$$\int_{\varphi(\Omega,\theta)} dx\, f(x,\theta) = \int_\Omega dq\, f(\varphi(q,\theta),\theta) \det\left(\frac{\partial\varphi(q,\theta)}{\partial q}\right).$$

Change of variables under integral sign (Berezin case) If an integrand has compact support, we have the following:

Theorem 5.1 *Let*

$$x = x(y, \omega), \quad \theta = \theta(y, \omega)$$

be a supersmooth diffeomorphism from $\mathfrak{R}_Y^{m|n}$ *to* $\mathfrak{R}_X^{m|n}$*. Put*

$$M = \begin{pmatrix} A & C \\ D & B \end{pmatrix}, \quad \begin{cases} A = \dfrac{\partial x}{\partial y}, & C = \dfrac{\partial x}{\partial \omega}, \\ D = \dfrac{\partial \theta}{\partial y}, & B = \dfrac{\partial \theta}{\partial \omega}. \end{cases}$$

Then, for any function $f \in \mathcal{C}_{\mathrm{SS}}(\mathfrak{R}_X^{m|n} : \mathfrak{C})$ *with compact support, we have the change of variables formula*

$$\int_{\mathfrak{R}_X^{m|n}} dx d\theta\, f(x, \theta) = \int_{\mathfrak{R}_Y^{m|n}} dy d\omega\, f(x(y, \omega), \theta(y, \omega))(\mathrm{sdet}\, M)(y, \omega).$$

Proof. [The proof borrowed from Berezin [?]]. First of all, we consider two simple cases:
(i) Let a linear coordinate change be given by

$$x_i = \sum_{k=1}^m A_{ik} y_k, \ \theta_j = \sum_{\ell=1}^n B_{j\ell} \omega_\ell$$

with $A_{ik}, B_{j\ell} \in \mathfrak{C}_{\mathrm{ev}}$. Then, $C = D = 0$ and

$$\mathrm{sdet}\, \left(\frac{\partial(x, \theta)}{\partial(y, \omega)} \right) = \det A \det{}^{-1} B.$$

In this case, we get our result easily.
(ii) For more general linear transformation,

$$x_i = \sum_{k=1}^m A_{ik} y_k + \sum_{\ell=1}^n C_{i\ell} \omega_\ell = x_i(y, \omega), \quad \theta_j = \sum_{k=1}^m D_{jk} y_k + \sum_{\ell=1}^n B_{j\ell} \omega_\ell = \theta_j(y, \omega)$$

with $A_{ik}, B_{j\ell} \in \mathfrak{C}_{\mathrm{ev}}$ and $C_{i\ell}, D_{jk} \in \mathfrak{C}_{\mathrm{od}}$, we have

$$\mathrm{sdet}\, \left(\frac{\partial(x, \theta)}{\partial(y, \omega)} \right) = \det A \det{}^{-1}(B - DA^{-1}C).$$

$$\begin{aligned}
&\int dy d\omega\, f(Ay + C\omega, Dy + B\omega) \ (y \to Ay, \ \omega \to B\omega) \\
&\quad = \det{}^{-1} A \ \det B \int dy d\omega\, f(y + CB^{-1}\omega, DA^{-1}y + \omega) \ ((y, \omega) \to (y, \omega + DA^{-1}y)) \\
&\quad = \det{}^{-1} A \ \det B \int dy d\omega\, f(y + CB^{-1}(\omega - DA^{-1}y), \omega) \ ((y, \omega) \to (y - CB^{-1}DA^{-1}y, \omega)) \\
&\quad = \det{}^{-1} A \ \det B \det{}^{-1}(1 - CB^{-1}DA^{-1}) \int dy d\omega\, f(y + CB^{-1}\omega, \omega) \ ((y, \omega) \to (y + CB^{-1}\omega, \omega)) \\
&\quad = \det B \det{}^{-1}(A - CB^{-1}D) \int dy d\omega\, f(y, \omega).
\end{aligned}$$

(iii) We consider the change of variables of the forms

$$x = x(y), \ \theta = \omega, \tag{5.16}$$

or

$$x = y, \ \theta = \theta(\omega). \tag{5.17}$$

For (5.16), we may use the ordinary change of variable formula which yields our result.

In case (5.17), we consider a transformation T which can be included in a 1-parameter group T_t of transformations of form (5.17). Set

$$\theta(t) = T_t\omega,$$
$$g(t) = \int d\omega\, f(x,\theta(t))\mathrm{sdet}\left(\frac{\partial\theta(t)}{\partial\omega}\right).$$

Since

$$\mathrm{sdet}\left(\frac{\partial\theta(t+s)}{\partial\omega}\right) = \mathrm{sdet}\left(\frac{\partial\theta(t+s)}{\partial\theta(t)}\right)\mathrm{sdet}\left(\frac{\partial\theta(t)}{\partial\omega}\right),$$

we have

$$g(t+s) = \int d\omega\, f(x,\theta(t+s))\mathrm{sdet}\left(\frac{\partial\theta(t+s)}{\partial\omega}\right) = \int d\omega\, f(x,\theta(t+s))\mathrm{sdet}\left(\frac{\partial\theta(t+s)}{\partial\theta(t)}\right)\mathrm{sdet}\left(\frac{\partial\theta(t)}{\partial\omega}\right).$$

Putting

$$\Delta(s) = \mathrm{sdet}\left(\frac{\partial\theta(t+s)}{\partial\theta(t)}\right),$$

we get

$$g'(t) = \frac{d}{ds}g(t+s)\bigg|_{s=0} = \int d\omega\left[\sum \frac{d\theta_j(t+s)}{ds}\left(\frac{\partial f}{\partial\theta_j}\right)\Delta(s) + f\frac{d}{ds}\Delta(s)\right]_{s=0}\mathrm{sdet}\,(\frac{\partial\theta(t)}{\partial\omega}). \tag{5.18}$$

Noting that

$$\Delta(s) = \det J(s)^{-1} = \exp(-\,\mathrm{tr}\,\log\,J(s)) \quad \text{where} \quad J(s) = \left(\frac{\partial\theta_j(t+s)}{\partial\theta_k(t)}\right),$$

we get

$$\frac{d}{ds}\Delta(s)\bigg|_{s=0} = -\,\mathrm{tr}\,(J'(s)J^{-1}(s))\exp(-\,\mathrm{tr}\,\log\,J(s))\big|_{s=0} = -\,\mathrm{tr}\,J'(0) = -\sum\frac{d}{ds}\frac{\partial\theta_j(t+s)}{\partial\theta_j(t)}\bigg|_{s=0}.$$

The expression in the square brackets in the right-hand side of (5.20) transforms into

$$\sum_j\left(\theta_j'\frac{\partial f}{\partial\theta_j} - f\frac{\partial\theta_j'}{\partial\theta_j}\right) = -\sum_j\frac{\partial}{\partial\theta_j}(\theta_j' f).$$

Since $T_t\omega_j = \theta_j(t)$, we have

$$\theta_j'(t) = -\Phi_j(\theta).$$

Therefore,

$$g'(t) = \int d\omega\sum_j\frac{\partial}{\partial\theta_j}(\Phi_j f)\mathrm{sdet}\left(\frac{\partial\theta(t)}{\partial\omega}\right).$$

We should remark that (i) $g'(t)$ has the same form as $g(t)$ by replacing f in $g(t)$ with $\sum_j\frac{\partial}{\partial\theta_j}(\Phi_j f)$, and (ii) $g'(0) = 0$ because for $t = 0$, $\theta_j(0) = \omega_j$ and $\mathrm{sdet}\,(\frac{\partial\theta(t)}{\partial\omega})|_{t=0} = 1$, therefore

$$g'(0) = \int d\omega\sum_j\frac{\partial}{\partial\omega_j}(\omega_j f(\omega)) = 0.$$

Repeating this process, we get $g^{(n)}(0) = 0$ for any $n > 0$.

(iv) The change of variables of the form

$$x = x(y,\omega),\ \theta = \omega, \tag{5.19}$$

or

$$x = y,\ \theta = \theta(y, \omega). \tag{5.20}$$

The case (5.20) can easily be reduced to (5.17).

For the case (5.19), we first consider the special type:

$$\begin{aligned} &x_i = y_i + t f_I \omega^I, \quad I = \{i_1, \cdots, i_{2k}\},\ = \omega_{i_1} \cdots \omega_{i_{2k}}, \\ &\theta_i = \omega_i \end{aligned} \tag{5.21}$$

$$\begin{aligned} &x_i = y_i + \sum_k \sum f_I(y) \omega^I, \quad I = \{i_1, \cdots, i_k\},\ = \omega_{i_1} \cdots \omega_{i_k}, \\ &\theta_i = \omega_i \end{aligned} \tag{5.22}$$

(v) The change of variables of the forms

$$x = x(y, \omega),\ \theta = \theta(y, \omega) \tag{5.23}$$

Porposition 5.3 (積分記号下での変数変換則) 写像

$$Y = (y, \omega) \to X = (x, \theta), \quad x = x(y, \omega), \quad \theta = \theta(y, \omega)$$

を $\mathfrak{R}_Y^{m|n}$ から $\mathfrak{R}_X^{m|n}$ へのスーパー微分同型とし、

$$M = \begin{pmatrix} A & C \\ D & B \end{pmatrix}, \quad \begin{cases} A = \dfrac{\partial x}{\partial y}, \quad C = \dfrac{\partial x}{\partial \omega}, \\ D = \dfrac{\partial \theta}{\partial y}, \quad B = \dfrac{\partial \theta}{\partial \omega} \end{cases}$$

とする。コンパクト台を持つ任意の関数 $f \in \mathcal{C}_{\mathrm{SS}}(\mathfrak{R}_X^{m|n} : \mathfrak{C})$ に対し、次の変数変換則が成立する：

$$\int_{\mathfrak{R}_X^{m|n}} dx d\theta\, f(x, \theta) = \int_{\mathfrak{R}_Y^{m|n}} dy d\omega\, f(x(y, \omega), \theta(y, \omega))(\mathrm{sdet}\, M)(y, \omega).$$

超行列法と称して「統計力学的事象の解析」に使われる事実は以下のものである。

Porposition 5.4 (Gaussian 型積分) A を $N \times N$-正定値行列とする。$z = {}^t(z_1, \cdots, z_N) \in \mathbb{C}^N$ とし、共役複素数を $\bar{z} = (\bar{z}_1, \cdots, \bar{z}_N) \in \mathbb{C}^N$ と書くとき、

$$\int_{\mathbb{C}^{2N}} d[\bar{z}, z]\, e^{-\bar{z} \cdot A z} = \frac{1}{\det A}, \quad d[\bar{z}, z] = (-2\pi i)^{-N} d\bar{z}_1 dz_1 \cdots d\bar{z}_N dz_N.$$

$\theta = (\theta_1, \cdots, \theta_N) \in \mathfrak{R}_\theta^{0|N}$ のみならず別の奇変数 $\bar{\theta} = (\bar{\theta}_1, \cdots, \bar{\theta}_N) \in \mathfrak{R}_{\bar{\theta}}^{0|N}$ を用意すると

$$\int_{\mathfrak{R}_\theta^{0|N} \times \mathfrak{R}_{\bar{\theta}}^{0|N}} d[\bar{\theta}, \theta] e^{-\bar{\theta} \cdot A \theta} = \det A, \quad d[\bar{\theta}, \theta] = d\bar{\theta}_N d\theta_N \cdots d\bar{\theta}_1 d\theta_1.$$

5.3 Fourier 変換

今や Fourier 変換が存在しない偏微分方程式の研究は考えにくい。「系」を考えるときの基礎も奇変数に関する Fourier 変換になる！

$$(F_e v)(\xi) = (2\pi\hbar)^{-m/2}\int dx\, e^{-i\hbar^{-1}\langle x|\xi\rangle}v(x), \quad (\bar{F}_e w)(x) = (2\pi\hbar)^{-m/2}\int d\xi\, e^{i\hbar^{-1}\langle x|\xi\rangle}w(\xi),$$
$$(F_o v)(\pi) = \hbar^{n/2}\iota_n\int d\theta\, e^{-i\hbar^{-1}\langle\theta|\pi\rangle}v(\theta), \quad (\bar{F}_o w)(\theta) = \hbar^{n/2}\iota_n\int d\pi\, e^{i\hbar^{-1}\langle\theta|\pi\rangle}w(\pi).$$

$$\langle x|\xi\rangle = \sum_{j=1}^{m} x_j\xi_j,\ \langle\theta|\pi\rangle = \sum_{k=1}^{n}\theta_k\pi_k,\ \iota_n = e^{-\frac{\pi i}{4}n(n-2)}.$$

上の公式は

$$\langle X|\Xi\rangle = \langle x|\xi\rangle + \langle\theta|\pi\rangle \in \mathfrak{R}_{\mathrm{ev}},\ c_{m,n} = (2\pi\hbar)^{-m/2}\hbar^{n/2}\iota_n,$$

なる記号を用いて

$$(\mathcal{F}u)(\xi,\pi) = c_{m,n}\int_{\mathfrak{R}^{m|n}} dX\, e^{-i\hbar^{-1}\langle X|\Xi\rangle}u(X) = \sum_a [(F_e u_a)(\xi)][(F_o\theta^a)(\pi)],$$
$$(\bar{\mathcal{F}}v)(x,\theta) = c_{m,n}\int_{\mathfrak{R}^{m|n}} d\Xi\, e^{i\hbar^{-1}\langle X|\Xi\rangle}v(\Xi) = \sum_a [(\bar{F}_e v_a)(x)][(\bar{F}_o\pi^a)(\theta)].$$

Integration: We define

$$\begin{aligned}\int_{\mathfrak{R}^{m|n}} dxd\theta\, u(x,\theta) &= \int\!\!\!\!\!\!\diagup dx\left\{\int\!\!\!\!\!\!\diagup d\theta\, u(x,\theta)\right\}\\ &= \int_{\mathbb{R}^m} dX_{\mathrm{B}}(\partial_{\theta_n}\cdots\partial_{\theta_1}u)(X_{\mathrm{B}}) \quad (\pi_{\mathrm{B}}() = \mathbb{R}^m)\\ &= \int\!\!\!\!\!\!\diagup d\theta\left\{\int\!\!\!\!\!\!\diagup dx\, u(x,\theta)\right\} = \int_{\mathfrak{R}^{m|n}} d\theta dx\, u(x,\theta).\end{aligned}$$

Especially for odd integration, we have the following curious looking but well-known relations

$$\int\!\!\!\!\!\!\diagup d\theta_n\cdots d\theta_1\,\theta_1\cdots\theta_n = 1 \quad \text{and} \quad \int\!\!\!\!\!\!\diagup d\theta_n\cdots d\theta_1\, 1 = 0 \quad \text{(Berezin integral)}.$$

Remarks for the need of ∞ number of Grassmann generators.

(i) Though $\mathfrak{C}$ does not form a field because $X^2 = 0$ for any $X \in \mathfrak{C}_{\mathrm{od}}$, but if $X, Y \in \mathfrak{C}$ satisfy $XY = 0$ for any $Y \in \mathfrak{C}_{\mathrm{od}}$, then $X = 0$. This property holds only when the number of generators is infinite. By this, we may determine the derivative $\partial_X u(X)$ uniquely.

(ii) In general, we need at least countable number of operations in doing analysis. If the number of Grassmann generators is finite, then the effect of odd variables may vanish after finitely many operations.

Remark. Though the differential calculus on Fréchet spaces has some difficulties in general, such calculus on Fréchet-Grassmann algebra holds safely in our case. For example, the implicit and inverse function theorems, and the chain rule for differentiation. See, Inoue and Maeda [**?**], Inoue [**?**, **?**].

============ メモ ============

さて何名程の諸君が聴講してくれるか？