Foundation of Super Analysis Lecture 7 – 08-11.21 A. Inoue

1 Necessity of the non-commutative analysis and its merit

2 Dirac and Weyl equations

3 Super number and Superspace

4 Linear algebra on the superspace

5 Elementary Analysis on the superspace

5.1 Functions on the superspace and their derivatives

5.1.1 Grassmann continuation

5.1.2 Supersmooth functions and their derivatives

5.1.3 Characterization of supersmooth functions

Comparison 5.1 *Let a function $f(z)$ from $\mathbb{C}$ to $\mathbb{C}$ be decomposed as*

$$f(z) = u(x,y) + iv(x,y), \quad u(x,y) = \Re f(z) \in \mathbb{R},\ v(x,y) = \Im f(z) \in \mathbb{R},$$

where $z = x + iy$, $|z| = \sqrt{x^2+y^2}$ with $z_0 = x_0 + iy_0$. Since

$$\begin{aligned}|f(z) - f(z_0)| &= |u(x,y) + iv(x,y) - (u(x_0,y_0) + iv(x_0,y_0))| \\ &= \sqrt{(u(x,y) - u(x_0,y_0))^2 + (v(x,y) - v(x_0,y_0))^2},\end{aligned}$$

we have that if $f(z)$ is continuous at $z = z_0$, then as real-valued 2 real variables function, $u(x,y), v(x,y)$ are continuous at (x_0, y_0).

Let a function $f(z)$ from $\mathbb{C}$ to $\mathbb{C}$ be differentiable at $z = z_0$, roughly expressed as

$$\frac{f(z_0+h) - f(z_0)}{w} \to \gamma \in \mathbb{C} \quad (|w| \to 0,\ w \in \mathbb{C}). \tag{5.1}$$

Here, $\gamma = \alpha + i\beta$ $(\alpha, \beta \in \mathbb{R})$ and denoted by $\gamma = f'(z_0)$. In other word, we denote it as

$$|f(z_0+w) - f(z_0) - \gamma w| = o(|w|) \qquad (|w| \to 0). \tag{5.2}$$

Using the expression $w = h + ik$ $(h, k \in \mathbb{R})$ and $\gamma w = (\alpha + i\beta)(h + ik) = (h\alpha - k\beta) + i(k\alpha + h\beta)$, we have

$$
\begin{aligned}
|f(z_0 + w) - f(z_0) - \gamma w| \\
&= |u(x_0 + h, y_0 + k) - u(x_0, y_0) - (h\alpha - k\beta) \\
&\qquad + i(v(x_0 + h, y_0 + k) - v(x_0, y_0) - (k\alpha + h\beta))| \\
&= \Big([u(x_0 + h, y_0 + k) - u(x_0, y_0) - (h\alpha - k\beta)]^2 \\
&\qquad + [v(x_0 + h, y_0 + k) - v(x_0, y_0) - (k\alpha + h\beta)]^2 \Big)^{1/2}.
\end{aligned}
$$

Therefore, when $(h, k) \to 0$ (i.e. $\sqrt{h^2 + k^2} \to 0$), we get

$$
\begin{aligned}
|u(x_0 + h, y_0 + k) - u(x_0, y_0) - (h\alpha - k\beta)| &= o(\sqrt{h^2 + k^2}), \\
|v(x_0 + h, y_0 + k) - v(x_0, y_0) - (k\alpha + h\beta)| &= o(\sqrt{h^2 + k^2}).
\end{aligned}
$$

From the first equation above, putting $h = 0$, we have $\beta = -u_y(x_0, y_0)$ and putting $k = 0$, $\alpha = u_x(x_0, y_0)$. Using the 2nd equation above, we have $\alpha = v_y(x_0, y_0)$ and $\beta = v_x(x_0, y_0)$. In other word, we have a system of PDEs

$$
u_x = v_y, \quad u_y = -v_x. \tag{5.3}
$$

This is called the Cauchy-Riemann equation and the solution $u(x, y) + iv(x, y)$ of this equation defines the analytic function in $z = x + iy$. Moreover, we may denote, without notational confusion,

$$
\frac{d}{dz} f(z) = f'(z) = \alpha + i\beta = u_x - iu_y = v_y + iv_x.
$$

Problem: *Does there exist a Cauchy-Riemann type equation corresponding supersmooth functions?*

Bosonic case *(Differentiablity for functions on $\mathfrak{R}^{m|0}$ or $\mathfrak{C}^{m|0}$ with values in $\mathfrak{C}$): Let U_{ev} (or V_{ev}) be an even superdomain in $\mathfrak{R}^{m|0}$ (or $\mathfrak{C}^{m|0}$).*

Definition 5.1 (super Gâteaux- or Fréchet-differentiability and super analyticity) *We assume that $g : \mathfrak{R}^{m|0} \supset U_{\mathrm{ev}} \to \mathfrak{C}$ (or $g : \mathfrak{C}^{m|0} \supset V_{\mathrm{ev}} \to \mathfrak{C}$).*

(i) A function g is said to be 1-time <u>super Gâteaux-differentiable at x in the direction y</u>, there exists an element $g'_G(x; \cdot) : \mathfrak{R}^{m|0} \to \mathfrak{C}$ (or $g'_G(x; \cdot) : \mathfrak{C}^{m|0} \to \mathfrak{C}$) such that when $t \to 0$,

$$
g(x + ty) - g(x) - tg'_G(x; y) \to 0 \quad \text{in } \mathfrak{C}, \quad \text{or} \quad \frac{d}{dt} g(x + ty)\Big|_{t=0} = g'_G(x; y).
$$

Analogously, g is said to be N-times super Gâteaux-differentiable at x in the direction $(y^{(1)}, \cdots, y^{(N)}) \in (\mathfrak{R}^{m|0})^N$, if there exists

$$
\frac{\partial^N}{\partial t_1 \cdots \partial t_N} g\big(x + \sum_{j=1}^{N} t_j y^{(j)}\big)\Big|_{t_1 = \cdots = t_N = 0} = g^{(N)}_G(x; y^{(1)}, \cdots, y^{(N)}).
$$

(ii) A function f is said to be 1-time <u>super Fréchet-differentiable at x</u>, there exists a linear operator $f'_F(x) \in \mathcal{L}(\mathfrak{R}^{m|0} : \mathfrak{C}^m)$ (or $f'_F(x) \in \mathcal{L}(\mathfrak{C}^{m|0} : \mathfrak{C}^m)$) such that

$$
\begin{aligned}
&(i)\ f(x + h) - f(x) = \langle h | f'_F(x) \rangle + \langle h | \epsilon(x, h) \rangle \quad \text{for} \quad h \in \mathfrak{R}^{m|0} (\text{or } \mathfrak{C}^{m|0}), \\
&(ii)\ \epsilon(x, h) \to 0 \ \text{in} \ \mathfrak{C}^m \quad \text{when} \quad h \to 0 \ \text{in} \ \mathfrak{R}^{m|0} (\text{or } \mathfrak{C}^{m|0}).
\end{aligned}
$$

f is said to be 2-times super Fréchet-differentiable at x when f'_F from $x \in \mathfrak{R}^{m|0}$ to $\mathfrak{C}^m$ is also super Fréchet-differentiable at x. Moreover, we may define N-times super Fréchet-differentiablity at x and we say it supersmooth if it is ∞-times super Fréchet-differentiable.

(iii) When $f : \mathfrak{C}^{m|0} \to \mathfrak{C}$, f is called to be superanalytic at $z \in V_{\rm ev} \subset \mathfrak{C}^{m|0}$ iff there exists a function $\epsilon(z,w) \in \mathfrak{C}^m$ and an element $\alpha = f'_F(z) \in \mathfrak{C}^m$ such that

$$(i)\ f(z+w) - f(z) = \langle w|\alpha\rangle + \langle w|\epsilon(z,w)\rangle \quad for \quad w \in \mathfrak{C}^{m|0}$$
$$(ii)\ \epsilon(z,w) \to 0 \ \ in \ \ \mathfrak{C}^m \quad when \quad w \to 0 \ \ in \ \ \mathfrak{C}^{m|0}.$$

Remarks. (i) If f is super Fréchet-differentiable at x, we have

$$f'_G(x;h) = f'_F(x)h = \sum_{j=1}^m h_j f_j(x) = \langle h|(f_j(x))\rangle$$
$$with \quad f_j(x) = \frac{\partial f}{\partial x_j}(x) = f'_G(x;e_j),\ e_j = (\overbrace{0,\cdots,0,1}^{j},\overbrace{0,\cdots,0}^{m-j}).$$

(ii) It is clear that if $f : \mathfrak{C}^{m|0} \to \mathfrak{C}$ is superanalytic, then $f_{\rm B}(z)$ for $z \in \mathbb{C}^m$ is analytic in ordinary sense.
(iii) The superanalyticity is another name of super Fréchet-differentiability for functions from $\mathfrak{C}^{m|0}$ to $\mathfrak{C}$.

Bosonic "analytic" case :

***Lemma 5.1 ((1.1.17) of dW, Theorem 1 of MK for** $m=1$**)** [1] *Let f be a analytic function on an open set $V \subset \mathbb{C}$ to $\mathbb{C}$. Then, we may extend f uniquely to a function $\tilde{f}$*

$$\tilde{f}(z) = \sum_{n=0}^\infty \frac{1}{n!} f^{(n)}(z_{\rm B}) z_{\rm S}^n \quad for \quad z = z_{\rm B} + z_{\rm S} \quad with \quad z_{\rm B} \in V, \tag{5.4}$$

which is superanalytic.

Proof. See the proof of Lemma 5.3 below, more precise than that in Theorem 1 of MK, and it gives also that $\tilde{f}$ is superanalytic on $U = \pi_{\rm B}^{-1}(V)$. Moreover, with slight modification of the proof of Theorem 1 of MK, we claim that

"for a superanalytic function g on U, if $g(z_{\rm B}) = 0$ on $z_{\rm B} \in V$ implies $g(z) = 0$ on $z \in U$".

Since g has value in $\mathfrak{C}$, we have the expansion

$$g(z) = \sum_{I \in \mathcal{I}} g_I(z)\sigma^I \quad with \quad g_I : U \ni z \to \mathbb{C}.$$

To relate the conventional analyticity, we introduce two sets $\mathfrak{C}_L^{(m)}$ and $\mathfrak{C}^{(m)}$:

We define a set $\Lambda_L^{\mathbb{C}}$ as

$$\begin{aligned} \Lambda_L^{\mathbb{C}} &= \Big\{ X = \sum_{I \in \mathcal{I}_L} X_I \sigma^I \,|\, X_I \in \mathbb{C} \Big\} \quad with \quad \mathcal{I}_L = \{I = (i_1, i_2, \cdots, i_L, 0, \cdots)\} \subset \mathcal{I} \\ &\cong \Lambda(\mathbb{R}^L : \mathbb{C}) (= the\ exterior\ algebra\ of\ forms\ on\ \mathbb{R}^L\ with\ coefficients\ in\ \mathbb{C}) \cong \mathbb{C}^{2^L}, \end{aligned} \tag{5.5}$$

[1] deWitt:Supermanifolds, London, Cambridge Univ. Press, 1984.
Matsumoto and Kakazu:*A note on topology of supermanifolds*, J.Math.Phys.27(1986), pp. 2690-2692.

and we take its projective limit $\Lambda^{\mathbb{C}}$. In fact, for $M > L$, defining maps $\psi_{LM} : \Lambda_L^{\mathbb{C}} \to \Lambda_L^{\mathbb{C}}$ by $\psi_{LM}(\sum_{I\in\mathcal{I}_M} X_I\sigma^I) = \sum_{I\in\mathcal{I}_L} X_I\sigma^I$, we have the set $(\Lambda_L^{\mathbb{C}}, \psi_{LM})$ which forms a projective system and yields a projective limit $\Lambda^{\mathbb{C}}$. More precisely, the topology of $\Lambda^{\mathbb{C}}$ is defined as follows: Elements $X^{(n)}$ converges to X in $\Lambda^{\mathbb{C}}$ if and only if for any $\epsilon > 0$ and I, there exists an integer $n_0 = n_0(\epsilon, I)$ such that $|X_I^{(n)} - X_I| < \epsilon$ when $n > n_0$.

For any L, take $m \le L$, we define

$$\mathfrak{C}_L^{(m)} = \Big\{X = \sum_{I\in\mathcal{I}_L, |I|\le m} X_I\sigma^I \mid X_I \in \mathbb{C}\Big\}.$$

It is clear that

$$\begin{gathered}\mathfrak{C}^{(m)} = \lim_{L\to\infty} \mathfrak{C}_L^{(m)}, \quad \mathfrak{C}_L^{(m)} = \{X = \sum_{I\in\mathcal{I}_{m,L}} X_I\sigma^I\}, \\ \text{with} \quad \mathcal{I} = \cup_{m=0}^{\infty}\mathcal{I}_m,\ \mathcal{I}_m = \{I = (i_1, i_2, \cdots) \in \mathcal{I} \mid |I| \le m\}, \\ \mathcal{I}_{m,L} = \{I = (i_1, i_2, \cdots, i_L, 0, \cdots \mid |I| \le m\} \to \mathcal{I}_m\ (L\to\infty).\end{gathered} \tag{5.6}$$

We have also

$$\begin{gathered}\mathfrak{R}_L^{(m)} = \Big\{X = \sum_{I\in\mathcal{I}_L, |I|\le m} X_I\sigma^I \mid X_{\tilde{0}} \in \mathbb{R},\ X_I \in \mathbb{C}\,(|I| \ge 1)\Big\} \to \mathfrak{R}^{(m)}\ (L\to\infty), \\ \mathfrak{R}_L^{[m]} = \Big\{X = \sum_{I\in\mathcal{I}_{m,L}, |I|=m} X_I\sigma^I \mid X_I \in \mathbb{C}\Big\} \to \mathfrak{R}^{[m]}\ (L\to\infty).\end{gathered} \tag{5.7}$$

For any fixed I, take L and m such that $I \in \mathcal{I}_{m,L}$, we restrict the domain of definition of $g_I(z)$ to $U \cap \mathfrak{C}_L^{(m)}$. Then, this gives an ordinary analytic function of variables $\{z_J\}_{J\in\mathcal{I}_{m,L}}$. From $g(z_{\mathrm{B}}) = 0$ on $z_{\mathrm{B}} \in V$, we get $\dfrac{\partial}{\partial z_J} g_I(z) = 0$ for all $J \in \mathcal{I}_{m,L}$. Therefore,

$$g_I(z) = 0 \quad \text{for} \quad \forall z \in U \cap \mathfrak{C}^{(m)},$$

which implies

$$g(z) = 0 \quad \text{for} \quad \forall z \in U \cap (\cup_{m=1}^{\infty}\mathfrak{C}^{(m)}) = U \cap \mathfrak{C}.$$

That is, $g(z) = 0$ for $\forall z \in U$. □

Lemma 5.2 ((1.1.18) of dW, Theorem 2 of MK for** $m = 1$**) *Let $U \subset \mathfrak{C}_{\mathrm{ev}}$ be a connected open set. If f is superanalytic on U to $\mathfrak{C}$, then*

$$f(z) = \sum_{I\in\mathcal{I}} f_I(z)\sigma^I, \quad f_I(z) = \sum_{n=0}^{\infty} \frac{1}{n!} f_I^{(n)}(z_{\mathrm{B}}) z_{\mathrm{S}}^n.$$

Proof. Since, for each I, $f_I(z_{\mathrm{B}})$ is analytic from $V = \pi_{\mathrm{B}}(U) \subset \mathbb{C}$ to $\mathbb{C}$, it has the superanalytic extension which equals to $f_I(z)$. □

Corollary 5.1 (Theorem 3 of MK for** $m = 1$**) *Let f be superanalytic on a connected open set $U \subset \mathfrak{C}_{\mathrm{ev}}$ to $\mathfrak{C}$. Then, f has the unique extension to a superanalytic function on $\tilde{U}$. Here, we put*

$$U_{\mathrm{B}} = \{z \in \mathbb{C} \mid z = w_{\mathrm{B}} \text{ for some } w \in U\}, \quad \tilde{U} = \{w \in \mathfrak{C}_{\mathrm{ev}} \mid w_{\mathrm{B}} \in U_{\mathrm{B}}\}.$$

Proof. *From above two lemmas, we get this claim.* □

As is mentioned before, we have

Lemma 5.3 *Let f be real analytic on $\mathbb{R}^m$. Then, its Grassmann continuation $\tilde f$*

$$\tilde f(x) = \tilde f(x_{\rm B} + x_{\rm S}) = \sum_{|\alpha|=0}^{\infty} \frac{1}{\alpha!}\partial_q^\alpha f(x_{\rm B}) x_{\rm S}^\alpha \quad \text{with} \quad \partial_q^\alpha f = \partial_{q_1}^{\alpha_1}\cdots\partial_{q_m}^{\alpha_m} f,\ \alpha = (\alpha_1,\cdots,\alpha_m)$$

is super Fréchet-differentiable at x, i.e. there exist $F_j(x) \in \mathfrak{C}$ and $\epsilon_j(x,y) \in \mathfrak{C}$ such that

$$\tilde f(x+y) = \tilde f(x) + \sum_{j=1}^m y_j F_j(x) + \sum_{j=1}^m \epsilon_j(x,y) y_j, \quad \text{with} \quad \epsilon_j(x,y) \to 0 \quad \text{in } \mathfrak{C} \quad \text{when} \quad y \to 0 \quad \text{in } \mathfrak{C}.$$

Bosonic "smooth" case *(Without analyticity, we may proceed analogously for functions on $\mathfrak{R}^{m|0}$): As is noted in p.6 of dW, we may define the Grassmann continuation of a smooth function as follows.*

Porposition 5.1 *Let $U_{\rm ev} \subset$ be a even superdomain. Assume that f is a smooth function from $\mathbb{R}^m \supset U_{\rm ev,B} = \pi_{\rm B}(U_{\rm ev})$ into $\mathfrak{C}$, denoted simply by $f \in C^\infty(U_{\rm ev,B} : \mathfrak{C})$. That is, we have the expression*

$$f(q) = \sum_{J\in\mathcal{I}} f_J(q)\,\sigma^J \quad \text{with} \quad f_J(q) \in C^\infty(U_{\rm ev,B} : \mathbb{C}) \quad \text{for each } J \in \mathcal{I}. \tag{5.8}$$

Then, we may define a mapping $\tilde f$ of $U_{\rm ev}$ into $\mathfrak{C}$, called the Grassmann continuation of f, by

$$\tilde f(x) = \sum_{|\alpha|\ge 0} \frac{1}{\alpha!}\partial_q^\alpha f(x_{\rm B})\, x_{\rm S}^\alpha \quad \text{where} \quad \partial_q^\alpha f(x_{\rm B}) = \sum_J \partial_q^\alpha f_J(x_{\rm B})\,\sigma^J. \tag{5.9}$$

Here, we put $x = (x_1,\cdots,x_m)$, $x = x_{\rm B} + x_{\rm S}$ with $x_{\rm B} = (x_{1,\rm B},\cdots,x_{m,\rm B}) = (q_1,\cdots,q_m) = q \in U_{ev,\rm B}$, $x_{\rm S} = (x_{1,\rm S},\cdots,x_{m,\rm S})$ and $x^\alpha = x_1^{\alpha_1}\cdots x_m^{\alpha_m}$.

Corollary 5.2 *If f and $\tilde f$ be given as above, then*

(i) $\tilde f$ is continuous and

(ii) $\tilde f(x) = 0$ in $U_{\rm ev}$ implies $f(x_{\rm B}) = 0$ in $U_{\rm ev,B}$.

Moreover, if we define the partial derivatives of $\tilde f$ by

$$\partial_{x_j}\tilde f(x) = \frac{d}{dt}\tilde f(x + te_{(j)})\Big|_{t=0} \quad \text{where} \quad e_{(j)} = (\overbrace{0,\cdots,0,1}^{j},0,\cdots,0) \in , \tag{5.10}$$

then we get

$$\partial_{x_j}\tilde f(x) = \widetilde{\partial_{q_j} f}(x) \quad \text{for} \quad j = 1,\cdots,m. \tag{5.11}$$

Remark 5.1 *By the same argument as above, we get, for $y = (y_1,\cdots,y_m) \in$,*

$$\frac{d}{dt}\tilde f(x+ty)\Big|_{t=0} = \sum_{j=1}^m y_j \sum_\alpha \frac{1}{\alpha!}\partial_q^\alpha \partial_{q_j} f(x_{\rm B})\, x_{\rm S}^\alpha = \sum_{j=1}^m y_j \partial_{x_j}\tilde f(x). \tag{5.12}$$

Therefore, (5.12) *implies that $\tilde f$, the Grassmann continuation of f, is super Gâteaux-differentiable at x in the direction y, that is, there exists $F_j(x) \in \mathfrak{C}$ such that for each y*

$$\tilde f(x+ty) - \tilde f(x) - t\sum_{j=1}^m y_j F_j(x) \to 0 \quad \text{in } \mathfrak{C} \quad \text{when} \quad t \to 0. \tag{5.13}$$

More generally,

Lemma 5.4 *Let $f(q) \in C^\infty(\mathbb{R}^m)$, we have the Taylor expansion for $\tilde{f}$: For any N, there exists $\tilde{\tau}_N(x,y) \in \mathfrak{C}$ such that*

$$\tilde{f}(x+y) = \sum_{|\alpha|=0}^{N} \frac{1}{\alpha!}\partial_x^\alpha \tilde{f}(x) y^\alpha + \tilde{\tau}_N(x,y), \tag{5.14}$$

with

$$\tilde{\tau}_N(x,y) = \sum_{|\alpha|=N+1} y^\alpha \int_0^1 dt \frac{1}{N!}(1-t)^N \partial_x^\alpha \tilde{f}(x+ty).$$

Corollary 5.3 *Let u be ∞-times super-Fréchet differentiable, then for any N, we have*

$$u(y) = \sum_{|\alpha|\le N} \frac{1}{\alpha!}\partial_x^\alpha u(x)(y-x)^\alpha + \tau_N(u;x,y)$$

with

$$\tau_N(u;x,y) = \sum_{|\alpha|=N+1} y^\alpha \int_0^1 dt \frac{1}{N!}(1-t)^N \partial_x^\alpha u(x+t(y-x)).$$

5.1.4 Grassmann continuation of composite functions

Lemma 5.5 *Let $U \subset \mathbb{R}^m$, $U' \subset \mathbb{R}^{m'}$ be open sets. For given $g \in C^\infty(U:\mathbb{R}^{m'})$, $f \in (U':\mathbb{R}^{m''})$ such that $g(U) \subset U'$, we have*

$$\widetilde{(f\circ g)}(x) = (\tilde{f}\circ\tilde{g})(x) \quad \text{for} \quad x \in \tilde{U}\times\mathfrak{R}_{\rm od}^m \quad \text{where} \quad \tilde{U} = \pi_{\rm B}^{-1}(U) \subset \mathfrak{R}_{\rm ev}^m.$$

Proof. Without loss of generality, we may assume $m''=1$.
We begin with the simplest case $m=m'=1$. By Faà di Bruno's formula and the definition of Grassmann continuation, we have

$$\begin{aligned}\widetilde{(f\circ g)}(x) &= \sum_{n=0}^\infty \frac{(f\circ g)^{(n)}(x_{\rm B})}{n!} x_{\rm S}^n \\ &= \sum_{n=0}^\infty \Bigg(\sum_{\substack{k_1+\cdots+k_n=n,\\ k_1+2k_2+\cdots+nk_n=n}} \frac{1}{k_1!\cdots k_n!} \prod_{j=1}^n \left(\frac{g^{(j)}(x_{\rm B})}{j!}\right)^{k_j} f^{(k_1+\cdots+k_n)}(g(x_{\rm B}))\Bigg) x_{\rm S}^n.\end{aligned} \tag{5.15}$$

Putting $\tilde{g}(x) = g(x_{\rm B}) + \sum_{l=1}^\infty \frac{g^{(l)}(x_{\rm B})}{l!} x_{\rm S}^l = y_{\rm B} + y_{\rm S}$, we have

$$(\tilde{f}\circ\tilde{g})(x) = \sum_{m=0}^\infty \frac{f^{(m)}(y_{\rm B})}{m!} y_{\rm S}^m = f(y_{\rm B}) + \frac{f^{(1)}(y_{\rm B})}{1!} y_{\rm S} + \frac{f^{(2)}(y_{\rm B})}{2!} y_{\rm S}^2 + \cdots \tag{5.16}$$

with

$$y_{\rm S}^m = \left(\sum_{l=1}^\infty \frac{g^{(l)}(x_{\rm B})}{l!} x_{\rm S}^l\right)^m = \sum_{n=0}^\infty x_{\rm S}^n \Bigg(\sum_{\substack{k_1+\cdots+k_n=n,\\ k_1+2k_2+\cdots+nk_n=n}} \frac{1}{k_1!\cdots k_n!} \prod_{j=1}^n \left(\frac{g^{(j)}(x_{\rm B})}{j!}\right)^{k_j}\Bigg) \tag{5.17}$$

which leads to (5.15) after reordering summation.

*For general m, m', we use Faà di Bruno's formula for general dimensions as in [**?**] and we get*

$$\widetilde{\partial_q^\alpha (f\circ g)}(x) = \partial_x^\alpha \widetilde{(f\circ g)}(x). \qquad \square$$

Notation: Hereafter, for the sake of notational simplicity, $\tilde{f}$ is denoted simply by f unless there occurs confusion.

Remark. Concerning the meaning of the Grassmann continuation $\tilde{f}$ of f, in p.7 of his book, de Witt claimed as follows:

> *"The presence of a soul in the independent variable evidently has little practical effect on the variety of functions with which one may work in applications of the theory. In this respect $\mathfrak{R}_{\mathrm{ev}}$ is a harmless generalization of its own subspace $\mathbb{R}$, the real line."*

In fact, we have

Porposition 5.2 *Let a function G from $\mathfrak{R}^{m|0}$ to $\mathfrak{C}$ be given and super Fréchet differentiable. Putting $g(q) = G(q)$ for $q \in \mathbb{R}^m$, we have $\tilde{g} = G$.*

Proof. If G is superanalytic from $\mathfrak{C}^{m|0}$ to $\mathfrak{C}$, then g is at least real analytic from $\mathbb{R}^m$ to $\mathbb{C}$ and $\tilde{g} = G$ by Lemma 5.1.
If G is supersmooth from $\mathfrak{R}^{m|0}$ to $\mathfrak{C}$ which is 0 on $\mathbb{R}^m$, then G is 0 on $\mathfrak{R}^{m|0}$. In fact, since $\partial_q^\alpha g(x_{\mathrm{B}}) = 0$ for any α in (5.14), we have

$$
\begin{aligned}
G(x_{\mathrm{B}} + x_{\mathrm{S}}) = \tau_N(G; x_{\mathrm{B}}, x_{\mathrm{S}}) &= \sum_{|\alpha|=N+1} x_{\mathrm{S}}^\alpha \int_0^1 dt \frac{1}{N!}(1-t)^N \partial_x^\alpha G(x_{\mathrm{B}} + t x_{\mathrm{S}}) \\
&= \sum_{|\beta|=N+2} x_{\mathrm{S}}^\beta \int_0^1 dt \frac{1}{(N+1)!}(1-t)^{N+1} \partial_x^\beta G(x_{\mathrm{B}} + t x_{\mathrm{S}}).
\end{aligned}
$$

As we noted before that $f_I(x)\sigma^I \to 0$ in $\mathfrak{C}$ when $|I|$ tends to ∞ even if $f_I(x) \to \infty$, we have $\tau_N(G; x_{\mathrm{B}}, x_{\mathrm{S}}) \to 0$ in $\mathfrak{C}$ when $N \to \infty$. In fact, for any $\epsilon > 0$, we take N such that $N^{N+1}e^{-N}2^{-2^N} < \epsilon$, then $(x_{\mathrm{S}})^\alpha$ contains Grassmann generators with degree more than $r(I) \sim 2^{2^N}$ whose number of components $N_d = \#\{\alpha \mid |\alpha| = N+1\} \le N! \sim \sqrt{2\pi} N^{N+1/2} e^{-N}$, therefore $p_I(\tau_N(g; x_{\mathrm{B}}, x_{\mathrm{S}})) \le N_d 2^{-2^N} \le \epsilon$ for N sufficiently big. □