Foundation of Super Analysis Lecture 6 – 08-11.14 A. Inoue

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5.1 Functions on the superspace

5.1.1 Grassmann continuation

Let $\phi(q)$ be a $\mathfrak{C}$-valued function on an open set $\Omega \subset \mathbb{R}^m$, that is,

$$\phi(q) = \sum_{I \in \mathcal{I}} \phi_I(q)\sigma^I \quad \text{with} \quad \phi_I : \Omega \ni q \to \phi_I(q) \in \mathbb{C}.$$

By the definition of the topology of $\mathfrak{C}$, we have

$$\lim_{q \to q_0} \phi(q) = \sum_{I \in \mathcal{I}} \big(\lim_{q \to q_0} \phi_I(q) \big) \sigma^I.$$

The differentiation and integration of such $\phi(q)$ are defined by

$$\frac{\partial}{\partial q_j}\phi(q) = \sum_{I \in \mathcal{I}} \frac{\partial}{\partial q_j}\phi_I(q)\sigma^I \quad \text{and} \quad \int_\Omega dq\, \phi(q) = \sum_{I \in \mathcal{I}} \bigg(\int_\Omega dq\, \phi_I(q) \bigg) \sigma^I.$$

We say $\phi \in C^\infty(\Omega : \mathfrak{C})$ if $\phi_I \in C^\infty(\Omega : \mathbb{C})$ for each $I \in \mathcal{I}$.

Remark 5.1 *If we use Banach-Grassmann algebra instead of Fréchet-Grassmann algebra, we need to check whether $\sum_{I \in \mathcal{I}} |\phi_I(q)| < \infty$, etc., which seems cumbersome or rather impossible to check for applying it to concrete problems.*

Lemma 5.1 *Let $\phi(t)$ and $\Phi(t)$ be continuous $\mathfrak{C}$-valued functions on an interval $[a,b]\subset\mathbb{R}$. Then,*

(1) $\int_a^b dt\,\phi(t)$ exists,

(2) if $\Phi'(t)=\phi(t)$ on $[a,b]$, then $\displaystyle\int_a^b dt\,\phi(t)=\Phi(b)-\Phi(a)$,

(3) if $\lambda\in\mathfrak{C}$ is a constant, then

$$\int_a^b dt\,(\phi(t)\cdot\lambda)=\Big(\int_a^b dt\,\phi(t)\Big)\cdot\lambda \quad \text{and} \quad \int_a^b dt\,(\lambda\cdot\phi(t))=\lambda\cdot\int_a^b dt\,\phi(t).$$

Moreover, we may generalize above lemma for a $\mathfrak{C}$-valued function $\phi(q)$ on an open set $\Omega\subset\mathbb{R}^m$.

Definition 5.1 *A set $U_{\rm ev}\subset=\mathfrak{R}^m_{\rm ev}$ is called an* <u>*even superdomain*</u> *if $U_{\rm ev,B}=\pi_{\rm B}(U_{\rm ev})\subset\mathbb{R}^m$ is open and connected and $\pi_{\rm B}^{-1}(\pi_{\rm B}(U_{\rm ev}))=U_{\rm ev}$. When $U\subset\mathfrak{R}^{m|n}$ is represented by $U=U_{\rm ev}\times\mathfrak{R}^n_{\rm od}$ with a even superdomain $U_{\rm ev}\subset$, U is called a* <u>*superdomain*</u> *in $\mathfrak{R}^{m|n}$.*

Porposition 5.1 *Let $U_{\rm ev}\subset$ be a even superdomain. Assume that f is a smooth function from $\mathbb{R}^m\supset U_{\rm ev,B}=\pi_{\rm B}(U_{\rm ev})$ into $\mathfrak{C}$, denoted simply by $f\in C^\infty(U_{\rm ev,B}:\mathfrak{C})$. That is, we have the expression*

$$f(q)=\sum_{J\in\mathcal{I}}f_J(q)\,\sigma^J \quad \text{with} \quad f_J(q)\in C^\infty(U_{\rm ev,B}:\mathbb{C}) \quad \text{for each } J\in\mathcal{I}. \tag{5.1}$$

Then, we may define a mapping $\tilde f$ of $U_{\rm ev}$ into $\mathfrak{C}$, called the Grassmann continuation of f, by

$$\tilde f(x)=\sum_{|\alpha|\ge0}\frac{1}{\alpha!}\partial_q^\alpha f(x_{\rm B})\,x_{\rm S}^\alpha \quad \text{where} \quad \partial_q^\alpha f(x_{\rm B})=\sum_J\partial_q^\alpha f_J(x_{\rm B})\,\sigma^J. \tag{5.2}$$

Here, we put $x=(x_1,\cdots,x_m)$, $x=x_{\rm B}+x_{\rm S}$ with $x_{\rm B}=(x_{1,\rm B},\cdots,x_{m,\rm B})=(q_1,\cdots,q_m)=q\in U_{ev,\rm B}$, $x_{\rm S}=(x_{1,\rm S},\cdots,x_{m,\rm S})$ and $x^\alpha=x_1^{\alpha_1}\cdots x_m^{\alpha_m}$.

Proof. [Main point of this proposition is to see whether this mapping (5.2) is well-defined. Therefore, by using the degree argument, we need to define $\tilde f^{[k]}$, the k-th degree component of $\tilde f$.]

Denoting by $x_{1,\rm S}^{[k_1]}$, the k_1-th degree component of $x_{1,\rm S}$, we get

$$(x_{1,\rm S}^{\alpha_1})^{[k_1]}=\sum(x_{1,\rm S}^{[r_1]})^{p_{1,1}}\cdots(x_{1,\rm S}^{[r_\ell]})^{p_{1,\ell}}.$$

Here, the summation is taken for all partitions of an integer α_1 into $\alpha_1=p_{1,1}+\cdots+p_{1,\ell}$ satisfying $\sum_{i=1}^\ell r_ip_{1,i}=k_1$, $r_i\ge0$. Using these notations, we put

$$\tilde f^{[k]}(x)=\sum_{\substack{|\alpha|\le k,\ k_0+k_1+\cdots+k_m=k\\ k_1,\cdots,k_m\ \text{are even}}}\frac{1}{\alpha!}(\partial_q^\alpha f)^{[k_0]}(x_{\rm B})\,(x_{1,\rm S}^{\alpha_1})^{[k_1]}\cdots(x_{m,\rm S}^{\alpha_m})^{[k_m]} \tag{5.3}$$

where

$$(\partial_q^\alpha f)^{[k_0]}(x_{\rm B})=\sum_{|J|=k_0}\partial_q^\alpha f_J(x_{\rm B})\,\sigma^J.$$

Or more precisely, we have

$$\begin{aligned}
\tilde f^{[0]}(x) &= f^{[0]}(x_{\rm B}),\\
\tilde f^{[1]}(x) &= f^{[1]}(x_{\rm B}),\\
\tilde f^{[2]}(x) &= f^{[2]}(x_{\rm B}) + \sum_{j=1}^m (\partial_{q_j} f)^{[0]}(x_{\rm B})(x_{j,{\rm S}})^{[2]},\\
\tilde f^{[3]}(x) &= f^{[3]}(x_{\rm B}) + \sum_{j=1}^m (\partial_{q_j} f)^{[1]}(x_{\rm B})(x_{j,{\rm S}})^{[2]},\\
\tilde f^{[4]}(x) &= f^{[4]}(x_{\rm B}) + \sum_{j=1}^m (\partial_{q_j} f)^{[2]}(x_{\rm B})(x_{j,{\rm S}})^{[2]}\\
&\qquad + \frac12 \sum_{j=1}^m (\partial_{q_j}^2 f)^{[0]}(x_{\rm B})(x_{j,{\rm S}}^2)^{[4]} + \sum_{j\neq k} (\partial_{q_j\,q_k}^2 f)^{[0]}(x_{\rm B})(x_{j,{\rm S}})^{[2]}(x_{k,{\rm S}})^{[2]}, \quad \textit{etc.}
\end{aligned}$$

Since $\tilde f^{[j]}(x) \neq \tilde f^{[k]}(x)$ $(j\neq k)$ in $\mathfrak{C}$, we may take the sum $\sum_{j=0}^\infty \tilde f^{[j]}(x) \in \mathfrak{C} = \oplus_{k=0}^\infty \mathfrak{C}^{[k]}$, which is denoted by $\tilde f(x)$. Therefore, rearranging the above 'summation', we get rather the 'familiar' expression as in (5.2). □

Corollary 5.1 *If f and $\tilde f$ be given as above, then*

(i) $\tilde f$ is continuous and

(ii) $\tilde f(x) = 0$ in $U_{\rm ev}$ implies $f(x_{\rm B}) = 0$ in $U_{\rm ev,B}$.

Moreover, if we define the partial derivative of $\tilde f$ in the j-direction by

$$\partial_{x_j}\tilde f(x) = \frac{d}{dt}\tilde f(x+te_{(j)})\Big|_{t=0} \quad \textit{where} \quad e_{(j)} = (\overbrace{0,\cdots,0,1}^{j},0,\cdots,0) \in , \tag{5.4}$$

then we get

$$\partial_{x_j}\tilde f(x) = \widetilde{\partial_{q_j} f}(x) \quad \textit{for} \quad j=1,\cdots,m. \tag{5.5}$$

Proof. Let $y_j = y_{j,{\rm B}} + y_{j,{\rm S}} \in \mathfrak{R}_{\rm ev}$. For $y_{(j)} = y_j e_{(j)} = y_{j,{\rm B}} e_{(j)} + y_{j,{\rm S}} e_{(j)} = y_{(j),{\rm B}} + y_{(j),{\rm S}} \in$, as

$$\frac{d}{dt}\tilde f(x+ty_{(j)}) = \frac{d}{dt}\left\{\sum_\alpha \frac{1}{\alpha!}\left(\sum_J \partial_q^\alpha f_J(x_{\rm B}+ty_{(j),{\rm B}})\sigma^J\right)(x_{\rm S}+ty_{(j),{\rm S}})^\alpha\right\},$$

we get easily,

$$\begin{aligned}
\frac{d}{dt}\tilde f(x+ty_{(j)})\Big|_{t=0} &= y_{(j),{\rm B}}\sum_\alpha \frac{1}{\alpha!}\left(\sum_J \partial_q^\alpha f_J(x_{\rm B})\sigma^J\right)x_{\rm S}^\alpha + y_{(j),{\rm S}}\sum_{\check\alpha}\frac{1}{\check\alpha!}\left(\sum_J \partial_q^{\check\alpha}\partial_{q_j} f_J(x_{\rm B})\sigma^J\right)x_{\rm S}^{\check\alpha}\\
&= y_j \sum_\alpha \frac{1}{\alpha!}\partial_q^\alpha \partial_{q_j} f(x_{\rm B})\, x_{\rm S}^\alpha = y_j \widetilde{\partial_{q_j} f}(x).
\end{aligned}$$

Here $\check\alpha = (\alpha_1,\cdots,\alpha_j-1,\cdots,\alpha_m)$. Putting $y_j = y_{j,{\rm B}} + y_{j,{\rm S}} = 1$ in the above, we have (5.5). □

Remark 5.2 *By the same argument as above, we get, for $y=(y_1,\cdots,y_m) \in$,*

$$\frac{d}{dt}\tilde f(x+ty)\Big|_{t=0} = \sum_{j=1}^m y_j \sum_\alpha \frac{1}{\alpha!}\partial_q^\alpha\partial_{q_j} f(x_{\rm B})\, x_{\rm S}^\alpha = \sum_{j=1}^m y_j \partial_{x_j}\tilde f(x). \tag{5.6}$$

Therefore, (5.6) implies that $\tilde{f}$, the Grassmann continuation of f, is super Gâteaux-differentiable at x in the direction y, that is, there exists $\tilde{f}'_F(x;y) \in \mathfrak{C}$ such that for each y

$$\tilde{f}(x+ty) - \tilde{f}(x) - t\tilde{f}'_F(x;y) \to 0 \quad \text{in } \mathfrak{C} \quad \text{when} \quad t \to 0, \text{ or } \frac{d}{dt}\tilde{f}(x+ty)\big|_{t=0} = \tilde{f}'_F(x;y). \tag{5.7}$$

Moreover, the Grassmann continuation $\tilde{f}$ of f has the following property:

Lemma 5.2 [1] *Let f be real analytic on $\mathbb{R}^m$. Then, its Grassmann continuation $\tilde{f}$*

$$\tilde{f}(x) = \tilde{f}(x_{\rm B} + x_{\rm S}) = \sum_{|\alpha|=0}^{\infty} \frac{1}{\alpha!}\partial_q^\alpha f(x_{\rm B}) x_{\rm S}^\alpha \quad \textit{with} \quad \partial_q^\alpha f = \partial_{q_1}^{\alpha_1}\cdots\partial_{q_m}^{\alpha_m} f,\ \alpha = (\alpha_1, \cdots, \alpha_m)$$

is super Fréchet-differentiable at x, i.e. there exist $F_j(x) \in \mathfrak{C}$ and $\epsilon_j(x,y) \in \mathfrak{C}$ such that

$$\tilde{f}(x+y) = \tilde{f}(x) + \sum_{j=1}^m y_j F_j(x) + \sum_{j=1}^m \epsilon_j(x,y) y_j, \quad \textit{with} \quad \epsilon_j(x,y) \to 0 \quad \textit{in } \mathfrak{C} \quad \textit{when} \quad y \to 0 \quad \textit{in } \mathfrak{C}.$$

Proof. For the sake of simplicity, we consider only the case $m = 1$. Below where we use the real analyticity, we require that $|y_{\rm B}| \le \delta(x_{\rm B})$ where $\delta(x_{\rm B})$ is the convergence radius of f at $x_{\rm B}$. Then, we have

$$\begin{aligned}
\tilde{f}(x+y) &= \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x_{\rm B} + y_{\rm B})(x_{\rm S} + y_{\rm S})^n \\
&= \sum_{n=0}^{\infty} \frac{1}{n!}\bigg(\sum_{\ell=0}^{\infty}\frac{1}{\ell!} f^{(\ell+n)}(x_{\rm B}) y_{\rm B}^\ell\bigg)\bigg(\sum_{k=0}^{n}\frac{n!}{k!(n-k)!} x_{\rm S}^{n-k} y_{\rm S}^k\bigg) \quad \text{(by real analyticity of } f(q)), \\
&= \sum_{n=0}^{\infty} \bigg(\sum_{\ell=0}^{\infty}\frac{1}{\ell!} f^{(\ell+j+k)}(x_{\rm B}) y_{\rm B}^\ell\bigg)\bigg(\sum_{j+k=n}\frac{1}{k!j!} x_{\rm S}^{j} y_{\rm S}^k\bigg) \quad \text{(renumbering by } j = n-k), \\
&= \sum_{n=0}^{\infty} \frac{1}{n!}\bigg(\sum_{j=0}^{\infty}\frac{1}{j!} f^{(n+j)}(x_{\rm B}) x_{\rm S}^j\bigg)\bigg(\sum_{\ell+k=n}\frac{n!}{\ell!k!} y_{\rm B}^{\ell} y_{\rm S}^k\bigg) \quad \text{(by reordering for } \ell + k = n), \\
&= \sum_{n=0}^{\infty} \frac{1}{n!}\bigg(\sum_{j=0}^{\infty}\frac{1}{j!} f^{(n+j)}(x_{\rm B}) x_{\rm S}^j\bigg)(y_{\rm B} + y_{\rm S})^n \quad \text{(by putting } n = \ell + k), \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \tilde{f}^{(n)}(x) y^n.
\end{aligned}$$

Therefore,

$$\tilde{f}(x+y) - \tilde{f}(x) = \sum_{n=1}^{\infty} \frac{1}{n!}\tilde{f}^{(n)}(x) y^n = \tilde{f}^{(1)}(x) y + [\sum_{n=2}^{\infty}\frac{1}{n!}\tilde{f}^{(n)}(x) y^{n-1}] y,$$

with

$$\epsilon(x,y) = \sum_{n=2}^{\infty} \frac{1}{n!}\tilde{f}^{(n)}(x) y^{n-1} \to 0 \quad \text{in } \mathfrak{C} \quad \text{when} \quad y \to 0. \qquad \square$$

More generally,

Lemma 5.3 *Let $f(q) \in C^\infty(\mathbb{R}^m)$, we have the Taylor expansion for $\tilde{f}$: For any N, there exists $\tilde{\tau}_N(x,y) \in \mathfrak{C}$ such that*

$$\tilde{f}(x+y) = \sum_{|\alpha|=0}^{N} \frac{1}{\alpha!}\partial_x^\alpha \tilde{f}(x) y^\alpha + \tilde{\tau}_N(x,y), \tag{5.8}$$

[1] S. Matsumoto and K. Kakazu, *A note on topology of supermanifolds*, J.Math.Phys.27(1986), pp. 2690-2692, S. Matsumoto, S Uehara and Y Yasui, *A superparticle on the super Riemann surface*, J.Math.Phys.31(1990), pp. 476-501.

with

$$\tilde{\tau}_N(x,y)=\sum_{|\alpha|=N+1} y^\alpha \int_0^1 dt \frac{1}{N!}(1-t)^N \partial_x^\alpha \tilde{f}(x+ty).$$

Proof. Substituting $q=x_{\rm B}$ and $q'=y_{\rm B}$ in

$$f(q+q')=\sum_{|\alpha|=0}^N \frac{1}{\alpha!}\partial_q^\alpha f(q) q'^\alpha + \sum_{|\alpha|=N+1} q'^\alpha \int_0^1 dt \frac{1}{N!}(1-t)^N \partial_q^\alpha f(q+tq'),$$

and extending both sides, we have the desired result by (5.5). □

Notation: For notational simplicity, we denote $\tilde{f}$ simply by f.

5.1.2 Supersmooth functions and their derivatives

The differentiability of mappings between Banach spaces

Comparison 5.1 *Let $(X,\|\cdot\|_X)$ and $(Y,\|\cdot\|_Y)$ be two Banach spaces.*
(i) A function $\Phi : X \to Y$ is called Gâteaux-differentiable at $x \in X$ in the direction $h \in X$ if there exists an element $\Phi'_G(x;h) \in Y$ such that

$$\|\Phi(x+th)-\Phi(x)-t\Phi'_G(x;h)\|_Y \to 0 \quad \text{when} \quad t\to 0, \text{ i.e. } \frac{d}{dt}\Phi(x+th)\big|_{t=0}=\Phi'_G(x;h).$$

$\Phi'_G(x;h)$ is also denoted by $\Phi'_G(x)(h)$, $d_G\Phi(x;h)$ or $(d_G\Phi(x))(h)$.

Moreover, $\Phi : X\to Y$ is called Fréchet-differentiable at $x\in X$ if there exist a bounded linear operator $\Phi'_F(x) : X \to Y$ and an element $\tau(x,h)\in Y$ such that

$$\Phi(x+h)-\Phi(x)-\Phi'_F(x)h=\tau(x,h) \quad \text{with} \quad \|\tau(x,h)\|_Y = o(\|h\|_X).$$

It is clear that if $\Phi'_F(x)$ (or $d_F\Phi(x)$) exists, then $\Phi'_G(x)$ exists also and $\Phi'_G(x)=\Phi'_F(x)$.

Theorem 5.1 (Theorem 2.1.13 of Berger[2]) *If $\Phi : X \to Y$ be Fréchet-differentiable at x, it is Gâteaux-differentiable at x. Conversely, if the Gâteaux derivative of Φ at x, $d_G\Phi(x,h)$, is linear in h and is continuous in x as a map from $X \to L(X:Y)$, then Φ is Fréchet-differentiable at x. In either case, we have $\Phi'_G(x)y=\Phi'_F(x,y)$.*

Problem 5.1 *How does one extend these notion of differentiability to those on functions on $\mathfrak{R}^{m|n}$?*

Definition 5.2 *A set $U_{\rm ev} \subset= \mathfrak{R}^m_{\rm ev}$ is called an even superdomain if $U_{\rm ev,B}=\pi_{\rm B}(U_{\rm ev})\subset \mathbb{R}^m$ is open and connected and $\pi_{\rm B}^{-1}(\pi_{\rm B}(U_{\rm ev}))=U_{\rm ev}$. When $U\subset \mathfrak{R}^{m|n}$ is represented by $U=U_{\rm ev}\times \mathfrak{R}^n_{\rm od}$ with a even superdomain $U_{\rm ev}\subset$, U is called a superdomain in $\mathfrak{R}^{m|n}$.*

Definition 5.3 *(1) Let $U_{\rm ev}\subset \mathfrak{R}^{m|0}$ be a even super domain. A mapping F from $U_{\rm ev}$ to $\mathfrak{C}$ is called supersmooth if there exists a smooth mapping f from $U_{\rm ev,B}=\pi_{\rm B}(U_{\rm ev})$ to $\mathfrak{C}$ such that $F=\tilde{f}$. We denote the set of supersmooth functions on $U_{\rm ev}$ as $\mathcal{C}_{\rm SS}(U_{\rm ev}:\mathfrak{C})$.*

[2]M.S. Berger, Nonlinearity and Functional Analysis–Lectures on Nonlinear Problems in Mathematical Analysis, Academic Press, NewYork, 1977.

(2) Let U be a superdomain in $\mathfrak{R}^{m|n}$. A mapping f from U to $\mathfrak{C}$ is called supersmooth if it is decomposed as

$$f(x,\theta)=\sum_{|a|\le n} f_a(x)\theta^a. \tag{5.9}$$

Here, $a=(a_1,\cdots a_n)\in\{0,1\}^n$, $\theta^a=\theta_1^{a_1}\cdots\theta_n^{a_n}$ and $f_a(x)\in\mathcal{C}_{\rm SS}(U_{\rm ev}:\mathfrak{C})$. Without mentioning it, we assume always that $f_a(x)\in\mathfrak{C}_{\rm ev}$ ([or $\in\mathfrak{C}_{\rm od}$) for all a, and call them as even (or odd) supersmooth functions denoted by $\mathcal{C}_{\rm SS}(U:\mathfrak{C})$. Moreover,

$$\not{\mathcal{C}}_{\rm SS}=\{f(x,\theta)\in\mathcal{C}_{\rm SS}(U:\mathfrak{C})\,|\,f_a(x)\in\mathbb{C}\}.$$

(3) Let $f\in\mathcal{C}_{\rm SS}(U:\mathfrak{C})$. For $j=1,2,\cdots,m$ and $s=1,2,\cdots,n$, we put

$$\begin{cases} F_j(X)=\displaystyle\sum_{|a|\le n}\partial_{x_j}f_a(x)\theta^a,\\ F_{s+m}(X)=\displaystyle\sum_{|a|\le n}(-1)^{l(a)+p(f_a(x))}f_a(x)\theta_1^{a_1}\cdots\theta_s^{a_s-1}\cdots\theta_n^{a_n}\end{cases} \tag{5.10}$$

with $l(a)=\sum_{j=1}^{s-1}a_j$ and $\theta_s^{-1}=0$. In this case, $F_\kappa(X)$ is the partial derivative of f at $X=(x,\theta)=(X_\mu)$ w.r.t. X_κ

$$\begin{cases} F_j(X)=\dfrac{\partial}{\partial x_j}f(x,\theta)=\partial_{x_j}f(x,\theta)=f_{x_j}(x,\theta)\quad for\quad j=1,2,\cdots,m,\\ F_{m+s}(X)=\dfrac{\partial}{\partial\theta_s}f(x,\theta)=\partial_{\theta_s}f(x,\theta)=f_{\theta_s}(x,\theta)\quad for\quad s=1,2,\cdots,n\end{cases} \tag{5.11}$$

$$F_\kappa(X)=\partial_{X_\kappa}f(X)=f_{X_\kappa}(X)\quad for\quad \kappa=1,\cdots,m+n. \tag{5.12}$$

Remark 5.3 *(1) In this lecture, we use the left odd derivatives. This naming stemms from putting most left the variable w.r.t. which we differentiate. There are some authors[3] who give the name right derivative to this.*

For $f\in\mathcal{C}_{\rm SS}(U:\mathfrak{C})$ with $j=1,2,\cdots,m$ and $s=1,2,\cdots,n$, we note here the right-derivatives:

$$\begin{cases} F_j^{(r)}(X)=\displaystyle\sum_{|a|\le n}\partial_{x_j}f_a(x)\theta^a,\\ F_{s+m}^{(r)}(X)=\displaystyle\sum_{|a|\le n}(-1)^{r(a)}f_a(x)\theta_1^{a_1}\cdots\theta_s^{a_s-1}\cdots\theta_n^{a_n}\end{cases}$$

We put here $r(a)=\sum_{j=s+1}^n a_j$. $F_\kappa^{(r)}(X)$is called the (right) partial κ-derivative w.r.t. X_κ at $X=(x,\theta)$ denoted by

$$F_j^{(r)}(X)=\frac{\partial}{\partial x_j}f(x,\theta)=\partial_{x_j}f(x,\theta),\quad F_{m+s}^{(r)}(X)=f(x,\theta)\frac{\overleftarrow{\partial}}{\partial\theta_s}=f(x,\theta)\overleftarrow{\partial}_{\theta_s}.$$

(2) Since we use the ∞-dimensional Grassmann generators, the decomposition (5.10) *is unique. In fact, if $\sum_a f_a(x)\theta^a\equiv 0$ on U, then $f_a(x)\equiv 0$. (see, p 322 in Vladimirov and Volovich.)*
(3) The higher derivatives are defined analogously. For a multiindex $\alpha=(\alpha_1,\cdots,\alpha_m)\in(\mathbb{N}\cup\{0\})^m$ and $a=(a_1,\cdots,a_n)\in\{0,1\}^n$, we put

$$\partial_x^\alpha=\partial_{x_1}^{\alpha_1}\cdots\partial_{x_m}^{\alpha_m}\quad and\quad {}^{a}_{\theta}=\partial_{\theta_1}^{a_1}\cdots\partial_{\theta_n}^{a_n}.$$

[3] For example, V.S. Vladimirov and I.V. Volovich, *Superanalysis I. Differential calculas*, Theor. Math. Phys. 59(1983), pp. 317-335.

Assume that for $X=(x,\theta), Y=(y,\omega)\in\mathfrak{R}^{m|n}$, we have $X+tY\in U$ (for any $t\in[0,1]$). Repeating the proof used in the proof of corollary 5.1, for $f\in\mathcal{C}_{\mathrm{SS}}(U:\mathfrak{C})$, the following holds:

$$\frac{d}{dt}f(X+tY)\bigg|_{t=0}=\sum_{j=1}^{m}y_j\frac{\partial}{\partial x_j}f(X)+\sum_{s=1}^{m}\omega_s\frac{\partial}{\partial\theta_s}f(X) \tag{5.13}$$

Definition 5.4 *A function f from the super domain $U\subset\mathfrak{R}^{m|n}$ to $\mathfrak{C}$, is called G-differentiable at $X=(x,\theta)$ if*

$$f(x+y,\theta+\omega)-f(x,\theta)=\sum(y_iF_i+\omega_sF_s)+\sum(y_iR_i+\omega_sR_s).$$

Here,

$$d(R_i,0)\to 0,\quad d(R_s,0)\to 0,\quad d_{m|n}((y,\omega),0)\to 0.$$

5.1.3 Taylor's Theorem

For $f\in\mathcal{C}_{\mathrm{SS}}(U:\mathfrak{C})$, we have

$$\frac{d}{dt}f(X+tY)\bigg|_{t=0}=\sum_{j=1}^{m}y_j\frac{\partial}{\partial x_j}f(X)+\sum_{s=1}^{m}\omega_s\frac{\partial}{\partial\theta_s}f(X). \tag{5.14}$$

From this, we define

Definition 5.5 *For a supersmooth function f, we define its differential df as*

$$df(X)=d_Xf(X)=\sum_{\kappa=1}^{m+n}dX_\kappa\frac{\partial f(X)}{\partial X_\kappa},$$

or

$$df(x,\theta)=\sum_{j=1}^{m}dx_j\frac{\partial f(x,\theta)}{\partial x_j}+\sum_{s=1}^{n}d\theta_s\frac{\partial f(x,\theta)}{\partial\theta_s}.$$

From before mentioned definition 5.3, we have

Porposition 5.2 U を $\mathfrak{R}^{m|n}$ 内のスーパー領域とする。任意の $f,g\in\mathcal{C}_{\mathrm{SS}}(U:\mathfrak{C})$ に対し、積 fg も $\mathcal{C}_{\mathrm{SS}}(U:\mathfrak{C})$ に属し、その微分 $d_Xf(X)$ と $d_Xg(X)$ は $\mathfrak{R}^{m|n}$ から $\mathfrak{C}^{m+n}$ への連続線形写像と見なせる。

更に、それらは以下を満たす*:*

(1) 任意の斉次な元 $\lambda,\,\mu\in\mathfrak{C}$ に対し

$$d_X(\lambda f+\mu g)(X)=(-1)^{p(\lambda)p(X)}\lambda\,d_Xf(X)+(-1)^{p(\mu)p(X)}\mu\,d_Xg(X). \tag{5.15}$$

(2) (Leibnitz の公式*)*

$$\partial_{X_\kappa}[f(X)g(X)]=(\partial_{X_\kappa}f(X))g(X)+(-1)^{p(X_\kappa)p(f(X))}f(X)(\partial_{X_\kappa}g(X)). \tag{5.16}$$

証明. (5.15) は明らか。$f,g\in\mathcal{C}_{\mathrm{SS}}(U:\mathfrak{C})$ に対して以下が成立する：

$$\begin{aligned}\frac{d}{dt}f(X+tY)g(X+tY)\bigg|_{t=0}&=\left(\sum_{j=1}^{m}y_j\frac{\partial}{\partial x_j}f(X)+\sum_{s=1}^{m}\omega_s\frac{\partial}{\partial\theta_s}f(X)\right)g(X)\\&\quad+f(X)\left(\sum_{j=1}^{m}y_j\frac{\partial}{\partial x_j}g(X)+\sum_{s=1}^{m}\omega_s\frac{\partial}{\partial\theta_s}g(X)\right).\end{aligned} \tag{5.17}$$

これより、望みの式が従う。□

Porposition 5.3 (Taylor の定理) $U \subset \mathfrak{R}^{m|n}$ を、その中の任意の 2 点 $X=(x,\theta), Y=(y,\omega) \in U$ に対し $Y+t(X-Y) \in U$ $(0 \le \forall t \le 1)$ なるものとする。$f \in \mathcal{C}_{\mathrm{SS}}(U:\mathfrak{C})$ に対し以下の *Taylor* の定理が成立する：任意の正整数 p に対し

$$f(x,\theta) - \sum_{|\alpha|+|a|\le p,\ |a|\le n} \frac{1}{\alpha!}(x-y)^\alpha(\theta-\omega)^a \partial_x^\alpha{}_\theta^a f(y,\omega) = \tau_p(X,Y) \tag{5.18}$$

ここで

$$\tau_p(X,Y) = \sum_{|\alpha|+|a|=p+1,\ |a|\le n} (x-y)^\alpha(\theta-\omega)^a \int_0^1 dt\, \frac{1}{p!}(1-t)^p \partial_x^\alpha{}_\theta^a f(y+t(x-y),\omega+t(\theta-\omega)). \tag{5.19}$$

証明. 以下の等式

$$\begin{aligned}&\int_0^1 dt\, \frac{(1-t)^p}{p!}\left(\frac{d}{dt}\right)^{p+1} f(y+t(x-y),\omega+t(\theta-\omega)) \\ &\qquad = \sum_{|\alpha|+|a|=p+1} (x-y)^\alpha(\theta-\omega)^a \int_0^1 dt\, \frac{1}{p!}(1-t)^p \partial_x^\alpha{}_\theta^a f(y+t(x-y),\omega+t(\theta-\omega)).\end{aligned}$$

を用い、左辺で部分積分をすれば式 (5.19) が従う。 □