Foundation of Super Analysis Lecture 5 – 08-11.07 A. Inoue

1 Necessity of the non-commutative analysis and its merit

2 Dirac and Weyl equations

3 Super number and Superspace

4 Linear algebra on the superspace

4.1 Matrix algebras on the superspace

4.2 Supertrace, superdeterminant

Lemma 4.1 *Let V, W be two rectangular matrices with odd elements, $m \times n$, $n \times m$, respectively. We have*
(1) $\mathrm{tr}\,(VW)^k = -\,\mathrm{tr}\,(WV)^k$ *for any* $k = 1, 2, \cdots$.
(2) $\det(\mathbb{I}_m + VW) = \det(\mathbb{I}_n + WV)^{-1}$.

Comparison 4.1 *If* $A = (a_{ij}) \in \mathrm{Mat}\,(m \times n : \mathfrak{C}_{\mathrm{ev}})$, $B = (b_{jk}) \in \mathrm{Mat}\,(n \times m : \mathfrak{C}_{\mathrm{ev}})$, *then we have*
(1) $\mathrm{tr}\,(AB)^k = \mathrm{tr}\,(BA)^k$,
(2) $\det(\mathbb{I}_m + AB) = \det(\mathbb{I}_n + BA)$.

Definition 4.1 *Let* $M = \begin{bmatrix} A & C \\ D & B \end{bmatrix} \in \mathrm{Mat}\,[m|n : \mathfrak{C}]$. *We define the supertrace of M by*

$$\mathrm{str}\ M = \mathrm{tr}\ A - (-1)^{p(M)}\,\mathrm{tr}\ B.$$

Using Lemma 4.1, we get readily

Porposition 4.1 *(a) Let* $M, N \in \mathrm{Mat}\,[m|n : \mathfrak{C}]$ *such that* $p(M) + p(N) \equiv 0 \bmod 2$. *Then, we have*

$$\mathrm{str}\,(M + N) = \mathrm{str}\ M + \mathrm{str}\ N.$$

(b) M is a matrix of size $(m+n) \times (r+s)$ and N is a matrix of size $(r+s) \times (m+n)$. Then,

$$\mathrm{str}\,(MN) = (-1)^{p(M)p(N)}\,\mathrm{str}\,(NM).$$

Definition 4.2 *Let* $B = (B_{jk})$ *be $(\ell \times \ell)$-matrix with elements in* $\mathfrak{C}_{\mathrm{ev}}$, *denoted by,* $B \in \mathrm{Mat}\,[\ell : \mathfrak{C}_{\mathrm{ev}}]$. *As* $\mathfrak{C}_{\mathrm{ev}}$ *is a commutative ring, we may define* $\det B$ *as usual:*

$$\det B = \sum_{\rho \in \wp_\ell} \mathrm{sgn}\,(\rho) B_{1\,\rho(1)} \cdots B_{\ell\,\rho(\ell)}.$$

Then, we have, as ordinary case,

$$\det(AB) = \det A \det B, \quad \det(\exp A) = \exp(\mathrm{tr}\ A) \quad \text{for} \quad A, B \in \mathrm{Mat}\,[\ell : \mathfrak{C}_{\mathrm{ev}}]. \tag{4.1}$$

Moreover, we have

Comparison 4.2 *Let*

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad M = \begin{pmatrix} \mathbb{I}_m & 0 \\ -A_{22}^{-1}A_{21} & \mathbb{I}_n \end{pmatrix},$$

be block matrices of even elements. Then, we have

$$\det A = \det AM = \det \begin{pmatrix} A_{11} - A_{12}A_{22}^{-1}A_{21} & A_{12} \\ 0 & A_{22} \end{pmatrix} = \det(A_{11} - A_{12}A_{22}^{-1}A_{21}) \det A_{22}.$$

Now, we define

Definition 4.3 *Let M be a supermatrix. When $\det B_{\mathrm{B}} \neq 0$, we put*

$$\mathrm{sdet}\ M = (\det(A - CB^{-1}D))(\det B)^{-1}$$

and call it superdeterminant or Berezinian of M.

Corollary 4.1 *When $\det B_{\mathrm{B}} \neq 0$ and $\mathrm{sdet}\ M \neq 0$, then $\det A_{\mathrm{B}} \neq 0$.*

Exercise 4.1 *Prove the above corollary.*

Now, we study the properties of sdet M.

Lemma 4.2 *(1) Let $L \in \mathrm{Mat}_{\mathrm{ev}}[\ell : \mathfrak{C}_{\mathrm{ev}}]$ such that the product of any two entries of it is zero. Then*

$$(\mathbb{I}_\ell + L)^{-1} = \mathbb{I}_\ell - L, \quad \det(\mathbb{I}_\ell + L) = 1 + \mathrm{tr}\ L.$$

(2) Let $M \in \mathrm{Mat}_{\mathrm{ev}}[m|n : \mathfrak{C}]$ such that the product of any two entries of it is zero. Then

$$\mathrm{sdet}\,(\mathbb{I}_{m+n} + M) = 1 + \mathrm{str}\ M.$$

Proof. (1) Remarking

$$(\mathbb{I}_\ell + L)^{-1} = \mathbb{I}_\ell - L + L^2 - L^3 + \cdots \quad \text{かつ} \quad \det(e^L) = e^{\mathrm{tr}\ L},$$

we get the result readily.
(2) For $M = \begin{bmatrix} A & C \\ D & B \end{bmatrix}$, satisfying $C(\mathbb{I}_n + B)^{-1}D = 0$ and $\mathrm{tr}\ A\,\mathrm{tr}\ B = 0$ garanteed by the product of any two entries of M being zero,

$$\begin{aligned} \mathrm{sdet}\,(\mathbb{I}_{m+n} + M) &= \det(\mathbb{I}_m + A - C(\mathbb{I}_n + B)^{-1}D) \det(\mathbb{I}_n + B)^{-1} \\ &= \det(\mathbb{I}_m + A) \det(\mathbb{I}_n - B) = 1 + \mathrm{tr}\ A - \mathrm{tr}\ B = 1 + \mathrm{str}\ M. \qquad \square \end{aligned}$$

Theorem 4.1 *Let* $M, N \in \mathrm{Mat}\,[m|n : \mathfrak{C}]$.

(1) If M is invertible, then we have sdet $M \neq 0$. *Moreover, if A is nonsingular, then*

$$(\mathrm{sdet}\ M)^{-1} = (\det A)^{-1}(\det(B - DA^{-1}C)). \tag{4.2}$$

(2) Multiplicativity of sdet *:*

$$\mathrm{sdet}\,(MN) = \mathrm{sdet}\ M \cdot \mathrm{sdet}\ N. \tag{4.3}$$

(3) str *and* sdet *are matrix invariants. That is, if N is invertible, then*

$$\mathrm{str}\ M = (-1)^{p(M)+p(N)}\,\mathrm{str}\,(NMN^{-1}), \quad \mathrm{sdet}\ M = \mathrm{sdet}\,(NMN^{-1}). \tag{4.4}$$

Proof (due to Leites [**?**]). (1) By

$$\begin{bmatrix} A & C \\ D & B \end{bmatrix} = \begin{bmatrix} \mathbb{I}_m & 0 \\ DA^{-1} & \mathbb{I}_n \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & B - DA^{-1}C \end{bmatrix} \begin{bmatrix} \mathbb{I}_m & A^{-1}C \\ 0 & \mathbb{I}_n \end{bmatrix} \quad \text{if } \det A_{\mathrm{B}} \neq 0, \tag{4.5}$$

we have readily by definition, sdet $M = \det A(\det(B - DA^{-1}C))^{-1}$, which yields (4.2).

(2) [Step 1]: Let $\mathcal{G}_+$, $\mathcal{G}_0$ and $\mathcal{G}_-$ be subgroups of $\mathrm{GL}[m|n : \mathfrak{C}]$, given by

$$\mathcal{G}_+ = \left\{ \begin{bmatrix} \mathbb{I}_m & C \\ 0 & \mathbb{I}_n \end{bmatrix} \right\}, \quad \mathcal{G}_0 = \left\{ \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \right\}, \quad \mathcal{G}_- = \left\{ \begin{bmatrix} \mathbb{I}_m & 0 \\ D & \mathbb{I}_n \end{bmatrix} \right\}.$$

Then, we have, $M = M_+ M_0 M_-$ with $M_+ \in \mathcal{G}_+$, $M_0 \in \mathcal{G}_0$ and $M_- \in \mathcal{G}_-$. i.e., for any $M \in \mathrm{GL}[m|n : \mathfrak{C}]$,

$$M = \begin{bmatrix} A & C \\ D & B \end{bmatrix} = \begin{bmatrix} \mathbb{I}_m & CB^{-1} \\ 0 & \mathbb{I}_n \end{bmatrix} \begin{bmatrix} A - CB^{-1}D & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} \mathbb{I}_m & 0 \\ B^{-1}D & \mathbb{I}_n \end{bmatrix} \quad \text{if } \det B_{\mathrm{B}} \neq 0. \tag{4.6}$$

Remarking that

$$\begin{bmatrix} \mathbb{I}_m & C \\ 0 & \mathbb{I}_n \end{bmatrix} \times \begin{bmatrix} \mathbb{I}_m & C' \\ 0 & \mathbb{I}_n \end{bmatrix} = \begin{bmatrix} \mathbb{I}_m & C + C' \\ 0 & \mathbb{I}_n \end{bmatrix},$$

we introduce the notion of elemantary matrices having the form

$$\begin{bmatrix} \mathbb{I}_m & E \\ 0 & \mathbb{I}_n \end{bmatrix}$$

where E has only one non-zero entry.

[Step 2]: We claim sdet (MN) = sdet $M\cdot$ sdet N whenever $M \in \mathcal{G}_+$ or $M \in \mathcal{G}_0$, and similarly, whenever $N \in \mathcal{G}_0$ or $N \in \mathcal{G}_-$. For example, when

$$M = \begin{bmatrix} \mathbb{I}_m & C' \\ 0 & \mathbb{I}_n \end{bmatrix} \in \mathcal{G}_+ \quad N = \begin{bmatrix} A & C \\ D & B \end{bmatrix},$$

we have

$$\begin{aligned} \mathrm{sdet}\,(MN) &= \mathrm{sdet} \begin{bmatrix} \mathbb{I}_m & C' \\ 0 & \mathbb{I}_n \end{bmatrix} \begin{bmatrix} A & C \\ D & B \end{bmatrix} = \mathrm{sdet} \begin{bmatrix} A + C'D & C + C'B \\ D & B \end{bmatrix} \\ &= \det(A + C'D - (C + C'B)B^{-1}D)(\det D)^{-1} = \det(A - CB^{-1}D)(\det D)^{-1} \\ &= \mathrm{sdet}\ M \cdot \mathrm{sdet}\ N. \end{aligned}$$

Check other cases analogously.

[Step 3]: We claim that $\mathrm{sdet}\,(MN) = \mathrm{sdet}\ M\cdot \mathrm{sdet}\ N$ for any elementary matrix N

$$N = \begin{bmatrix} \mathbb{I}_m & E \\ 0 & \mathbb{I}_n \end{bmatrix} \in \mathcal{G}_+.$$

Since we have

$$\begin{aligned}
&\mathrm{sdet}\,(MN) = \mathrm{sdet}\,(M_+(M_0M_-N)) = \mathrm{sdet}\ M_+\cdot \mathrm{sdet}\,(M_0(M_-N))) = \mathrm{sdet}\ M_0\cdot \mathrm{sdet}\,(M_-N),\\
&\mathrm{sdet}\ M\cdot \mathrm{sdet}\ N = \mathrm{sdet}\ M_0\cdot \mathrm{sdet}\ M_-\cdot \mathrm{sdet}\ N,
\end{aligned}$$

by Step 1 and Step 2, we need to prove

$$\mathrm{sdet}\,(M_-N) = \mathrm{sdet}\ M_-\cdot \mathrm{sdet}\ N = 1$$

when N is an elementary matrix. By definition,

$$\mathrm{sdet}\begin{bmatrix} \mathbb{I}_m & 0 \\ D & \mathbb{I}_n \end{bmatrix}\begin{bmatrix} \mathbb{I}_m & E \\ 0 & \mathbb{I}_n \end{bmatrix} = \mathrm{sdet}\begin{bmatrix} \mathbb{I}_m & E \\ D & \mathbb{I}_n + DE \end{bmatrix} = \det(1 - E(1+DE)^{-1}D)\det(1+DE)^{-1}.$$

As E has only one non-zero entry, the product of any two of the matrices E, DE, $E(1+DE)^{-1}D$ is zero. Applying Lemma, we get, by $(1+DE)^{-1} = 1 - DE$ and $E\cdot DE = 0$,

$$\mathrm{sdet}\,(M_-N) = \det(1-DE)(\det(1+DE))^{-1} = (1 - \mathrm{tr}\ DE)(1+\mathrm{tr}\ DE)^{-1}.$$

As $\mathrm{tr}\ DE = -\,\mathrm{tr}\ ED$, we have

$$\mathrm{sdet}\,(M_-N) = 1 = \mathrm{sdet}\ M_-\ \mathrm{sdet}\ N.$$

[Step 4]: Put

$$\mathcal{G} = \Big\{N \in \mathrm{GL}[m|n:\mathfrak{R}] \mid \mathrm{sdet}\,(MN) = \mathrm{sdet}\ M\cdot \mathrm{sdet}\ N \quad \text{for any} \quad M \in \mathrm{GL}[m|n:\mathfrak{R}]\Big\}.$$

For $N_1, N_2 \in \mathcal{G}$, we have

$$\begin{aligned}
\mathrm{sdet}\,(M\cdot N_1N_2) &= \mathrm{sdet}\,((MN_1)N_2) = \mathrm{sdet}\,(MN_1)\cdot \mathrm{sdet}\ N_2\\
&= \mathrm{sdet}\ M\cdot \mathrm{sdet}\ N_1\cdot \mathrm{sdet}\ N_2 = \mathrm{sdet}\ M\cdot \mathrm{sdet}\,(N_1N_2),
\end{aligned} \tag{4.7}$$

which implies $\mathcal{G}$ froms a group. By Steps 2 and 3, $\mathcal{G}$ contains $\mathcal{G}_-$ and $\mathcal{G}_0$ and all elementary matrices $N \in \mathcal{G}_+$. By Step1, $\mathrm{GL}[m|n:\mathfrak{C}]$ is generated by these matrices, we have $\mathcal{G} = \mathrm{GL}[m|n:\mathfrak{C}]$, that is, $\mathrm{sdet}\,(MN) = \mathrm{sdet}\ M\cdot \mathrm{sdet}\ N$.

(3) Let N, M be given. Then, using (4.7), we get

$$\mathrm{str}\ NMN^{-1} = (-1)^{p(N)p(MN^{-1})}\,\mathrm{str}\ MN^{-1}N = (-1)^{p(N)+p(M)}\,\mathrm{str}\ M,$$

since $p(MN^{-1}) = p(M) + p(N^{-1}) \mod 2$ and $0 = p(NN^{-1}) = p(N) + p(N^{-1}) \mod 2$, we have $p(N)p(MN^{-1}) = p(N) + p(M) \mod 2$.

Using (4.7), we have $\mathrm{sdet}\,(MN) = \mathrm{sdet}\,(NM)$ which implies $\mathrm{sdet}\,(NMN^{-1}) = \mathrm{sdet}\,(N^{-1}NM) = \mathrm{sdet}\ M$. $\square$

Theorem 4.2 (Liouville's theorem) *Let $M(t) \in \mathrm{Mat}\,[m|n:\mathfrak{C}]$ with a real parameter t. Let $X(t) \in \mathrm{Mat}\,[m|n:\mathfrak{C}]$ satisfy*

$$\frac{d}{dt}X(t) = M(t)X(t), \quad X(0) = \mathbb{I}_{m+n}. \tag{4.8}$$

Then $X(t) \in \mathrm{GL}[m|n:\mathfrak{C}]$, and

$$\mathrm{sdet}\ X(t) = \exp\{\int_0^t ds\, \mathrm{str}\ M(s)\}. \tag{4.9}$$

Proof (due to Berezin [?]). Let $\tilde{X}(t)$ be a solution of

$$\frac{d}{dt}\tilde{X}(t) = -\tilde{X}(t)M(t), \quad \tilde{X}(0) = \mathbb{I}_{m+n}.$$

Then, since

$$\frac{d}{dt}(\tilde{X}(t)X(t)) = 0 \quad \text{with} \quad \tilde{X}(0)X(0) = \mathbb{I}_{m+n},$$

we have $\tilde{X}(t)X(t) = \mathbb{I}_{m+n}$ which implies $X(t) \in \mathrm{GL}[m|n:\mathfrak{C}]$.

Let

$$M(t) = \begin{bmatrix} A(t) & C(t) \\ D(t) & B(t) \end{bmatrix}, \quad X(t) = \begin{bmatrix} X_{11}(t) & X_{12}(t) \\ X_{21}(t) & X_{22}(t) \end{bmatrix}.$$

Then, putting $Y(t) = X_{11}(t) - X_{12}(t)X_{22}^{-1}(t)X_{21}(t)$ and $Z = X_{22}^{-1}(t)$, we have, by simple calculation[1],

$$\frac{d}{dt}Y = (A - X_{12}X_{22}^{-1}D)Y, \quad \frac{d}{dt}Z = -Z(DX_{12}X_{22}^{-1} + B).$$

As all elements appeared in the above equations are even, we may apply the classical Liouville theorem to have

$$\frac{d}{dt}\det Y = \mathrm{tr}\,(A - X_{12}X_{22}^{-1}D)\det Y, \quad \frac{d}{dt}\det Z = -\,\mathrm{tr}\,(DX_{12}X_{22}^{-1} + B)\det Z.$$

As $\mathrm{tr}\,(A - X_{12}X_{22}^{-1}D) = \mathrm{tr}\,(A + DX_{12}X_{22}^{-1})$, we have

$$\frac{d}{dt}\,\mathrm{sdet}\ X = \frac{d}{dt}(\det Y\cdot \det Z) = \mathrm{tr}\,(A-B)\cdot \det Y\cdot \det Z = \mathrm{str}\ M\cdot \mathrm{sdet}\ X \quad \text{with} \quad \mathrm{sdet}\ X(0) = 1.$$

This yields the desired result. □

Corollary 4.2 *For $M, N \in \mathrm{Mat}_{\,\mathrm{ev}}[m|n:\mathfrak{C}]$ we have*

$$\begin{aligned} &\mathrm{sdet}\,(MN) = \mathrm{sdet}\ M\cdot \mathrm{sdet}\ N, \\ &\exp(\mathrm{str}\ M) = \mathrm{sdet}\,(\exp M). \end{aligned} \tag{4.10}$$

Proof. (1) Put $X(t) = (1-t)\mathbb{I}_{m+n} + tM$ and $Y(t) = (1-t)\mathbb{I}_{m+n} + tN$. As $X(t)$ and $Y(t)$ are differentiable in t and invertible except at most one t, we my define

$$A(t) = \frac{dX(t)}{dt}X(t)^{-1}, \quad B(t) = \frac{dX(t)}{dt}Y(t)^{-1}.$$

Then

$$\frac{d}{dt}(X(t)Y(t)) = (A(t) + B_1(t))X(t)Y(t) \quad \text{where} \quad B_1(t) = X(t)B(t)X(t)^{-1}.$$

[1]Differentiating $X_{12}^{-1}X_{12} = \mathbb{I}_n$w.r.t.$t$, we get the second equality of (4.8). Using once more the equality (4.8), we get the first oneof (4.8)

Applying above theorem, we have

$$\mathrm{sdet}\,(MN) = \mathrm{sdet}\,(X(1)Y(1)) = \exp\{\int_0^1 ds\,\mathrm{str}\,(A(t)+B_1(t))\} = \exp\{\int_0^1 ds(\mathrm{str}\ A(t) + \mathrm{str}\ B(t))\}$$
$$= \mathrm{sdet}\ X(1)\cdot \mathrm{sdet}\ Y(1) = \mathrm{sdet}\ M\cdot \mathrm{sdet}\ N.$$

(2) Putting $M(t) = M$, $X(t) = e^{tM}$ and $t = 1$ in theorem above, we get the desired result. □

Comparison 4.3 (cited from "Encyclopaedia of Mathematics" ed. M. Hazewinkel) *Liouville-Ostrogradski formula (or Liouville formula) : A relation that connects the Wronskian of a system of solutions and the coefficients of an ordinary linear differential equation.*

Let $x_1(t), \cdots, x_n(t)$ be an arbitrary system of solutions of a homogeneous system of linear first-order equations

$$x' = A(t)x, \qquad x \in \mathbb{R}^n \tag{4.11}$$

with an operator $A(t)$ that is continuous on an interval I, and let

$$W(x_1(t), \cdots, x_n(t)) = W(t)$$

be the Wronskian of this system of solutions. The Liouville-Ostrogradski formula has the form

$$\frac{d}{dt}W(t) = A(t)\cdot \mathrm{tr}\ A(t), \qquad t \in I \tag{4.12}$$

or, equivalently,

$$W(x_1(t), \cdots, x_n(t)) = W(x_1(\underline{t}), \cdots, x_n(\underline{t}))\cdot \exp\{\int_{\underline{t}}^t ds\ \mathrm{tr}\ A(s)\}, \quad t, \underline{t} \in I. \tag{4.13}$$

Here tr $A(t)$ *is the trace of the operator $A(t)$. The Liouville-Ostrogradski formula can be written by means of the Cauchy operator $X(t,\underline{t})$ of the system* (4.11) *as follows:*

$$\det X(t,\underline{t}) = \exp\{\int_{\underline{t}}^t ds\ \mathrm{tr}\ A(s)\}, \quad t, \underline{t} \in I. \tag{4.14}$$

The geometrical meaning of (4.14) *(or* (4.13) *) is that as a result of the transformation $X(t,\underline{t}) : \mathbb{R}^n \to \mathbb{R}^n$ the oriented volume of any body is increased by a factor* $\exp\{\int_{\underline{t}}^t ds\ \mathrm{tr}\ A(s)\}$.

4.3 Diagonalization

Definition 4.4 *A supermatrix $M = \begin{pmatrix} A & C \\ D & B \end{pmatrix} \in \mathrm{Mat}\,[m|n : \mathfrak{C}]$ is called generic if all eigenvalues of M_{B} as* $\mathrm{Mat}\,[m+n : \mathbb{C}]$ *are different each others.*

Theorem 4.3 (Berezin) *Let $M \in \mathrm{Mat}\,[m|n : \mathfrak{C}]$ be generic. Then, there exists a matrix $X \in \mathrm{GL}[m|n : \mathfrak{C}]$ such that $E = XMX^{-1}$ is diagonal.*

Proof. Decomposing the equality $EX = XM$ with respect to the degree, we have

$$(EX)^{[k]} = \sum_{j=0}^k E^{[j]}X^{[k-j]} = \sum_{j=0}^k X^{[j]}M^{[k-j]} = (XM)^{[k]}. \tag{4.15}$$

From this, we want to construct $X^{[k]}$ and $E^{[k]}$: For $k=0$, we have

$$E^{[0]}X^{[0]} = X^{[0]}M^{[0]}. \tag{4.16}$$

By the assumption, there exist $X_{11}^{[0]}$, $E_{11}^{[0]} = (\lambda_1^{[0]}, \cdots, \lambda_m^{[0]})$ and $X_{22}^{[0]}$,$E_{22}^{[0]} = (\lambda_{m+1}^{[0]}, \cdots, \lambda_{m+n}^{[0]})$ such that

$$X_{11}^{[0]}A_{\mathrm{B}} = E_{11}^{[0]}X_{11}^{[0]} \quad \text{かつ} \quad X_{22}^{[0]}B_{\mathrm{B}} = E_{22}^{[0]}X_{22}^{[0]}.$$

Defining

$$X^{[0]} = \begin{pmatrix} X_{11}^{[0]} & 0 \\ 0 & X_{22}^{[0]} \end{pmatrix},\ E^{[0]} = \begin{pmatrix} E_{11}^{[0]} & 0 \\ 0 & E_{22}^{[0]} \end{pmatrix},$$

we have the desired one satisfying (4.16).

Assume that there exist $X^{[j]}$ and $E^{[j]}$ for $0 \le j \le k-1$ satisfying (4.15). Multiplying $(X^{[0]})^{-1}$ from the right to (4.15) for k, we have

$$E^{[0]}X^{[k]}(X^{[0]})^{-1} - X^{[k]}(X^{[0]})^{-1}E^{[0]} + E^{[k]} = K^{[k]} \tag{4.17}$$

where

$$K^{[k]} = (\sum_{j=0}^{k-1} X^{[j]}M^{[k-j]})(X^{[0]})^{-1} - (\sum_{j=1}^{k-1} E^{[j]}X^{[k-j]})(X^{[0]})^{-1}.$$

By inductive assumption, the matrix $K^{[k]}$ is known and belongs to $\mathrm{Mat}\,[m|n:\mathfrak{C}]$. From (4.17), we have

$$(\lambda_i^{[0]} - \lambda_j^{[0]})(X^{[k]}(X^{[0]})^{-1})_{ij} + \lambda_i^{[k]}\delta_{ij} = (K^{[k]})_{ij}. \tag{4.18}$$

This equation is uniquely solvable since $\lambda_i^{[0]} \neq \lambda_j^{[0]}$ and

$$\begin{cases} \lambda_i^{[k]} = (K^{[k]})_{ii}, \\ (X^{[k]}(X^{[0]})^{-1})_{ij} = \frac{(K^{[k]})_{ij}}{\lambda_i^{[0]} - \lambda_j^{[0]}}, \qquad \text{for} \quad i \neq j. \end{cases}$$

Therefore, we define $X^{[j]}$ and $E_{[j]}$ for any $j \ge 0$. Since $X^{[0]}$ is invertible, $X \in \mathrm{GL}[m|n:\mathfrak{C}]$. This implies X and E are defined as desired. $\square$

Comparison 4.4 *$n \times n$-matrix M on any commutative ring is diagonalizable iff the minimal polynomial $\varphi_M(x)$ w.r.t. M has no multiple root.*

Problem 4.1 *Find a condition for a supermatrix M being diagonalizable?*

4.3.1 A simple example

Let

$$Q = \begin{pmatrix} x_1 & \theta_1 \\ \theta_2 & ix_2 \end{pmatrix} \quad \text{with} \quad x_1,\, x_2 \in \mathfrak{R}_{\mathrm{ev}},\ \theta_1,\, \theta_2 \in \mathfrak{R}_{\mathrm{od}},$$

which maps $\mathfrak{R}^{1|1}$ to $\mathfrak{R}^{1|1}$ or $\mathfrak{R}_{\mathrm{od}} \times i\mathfrak{R}_{\mathrm{ev}}$ to $\mathfrak{R}_{\mathrm{od}} \times i\mathfrak{R}_{\mathrm{ev}}$. This supermatrix appears in Efetov's calculation in Random Matrix Theory.

K.B. Efetov, *Supersymmetry and theory of disordered metals*, Advances in Physics 32(1983) pp. 53-127.

A. Inoue and Y. Nomura, *Some refinements of Wigner's semi-circle law for Gaussian random matrices using superanalysis*, Asymptotic Analysis 23(2000), pp. 329-375.

Invertibility of Q : Find Y for a given V such that

$$QY = V \quad \text{with} \quad Y = \begin{pmatrix} y_1 \\ \omega_2 \end{pmatrix},\ V = \begin{pmatrix} v_1 \\ \rho_2 \end{pmatrix} \in \mathfrak{R}^{1|1},$$

$$x_1 y_1 + \theta_1 \omega_2 = v_1,\ \theta_2 y_1 + i x_2 \omega_2 = \rho_2.$$

If $(x_1 x_2)_{\mathrm{B}} \neq 0$, we have readily

$$y_1 = \frac{i x_2 v_1 - \theta_1 \rho_2}{D_-},\ \omega_2 = \frac{x_1 \rho_2 - \theta_2 v_1}{D_+} \quad \text{with} \quad D_\pm = i x_1 x_2 \pm \theta_1 \theta_2.$$

Analogously, for

$$\tilde{Y} = \begin{pmatrix} \omega_1 \\ i y_2 \end{pmatrix} \in \mathfrak{R}_{\mathrm{od}} \times i\mathfrak{R}_{\mathrm{ev}},\ \tilde{V} = \begin{pmatrix} \rho_1 \\ v_2 \end{pmatrix} \in \mathfrak{R}_{\mathrm{od}} \times \mathfrak{R}_{\mathrm{ev}},$$

satisfying $Q\tilde{Y} = \tilde{V}$, we have

$$\omega_1 = \frac{i x_2 \rho_1 - \theta_1 v_2}{D_-},\ i y_2 = \frac{x_1 v_2 - \theta_2 \rho_1}{D_+}.$$

To relate the above quantity with the sdet Q, we proceed as follows: Let

$$Y = \begin{pmatrix} y_1 & \omega_1 \\ \omega_2 & i y_2 \end{pmatrix} \quad \text{with} \quad QY = YQ = I_2.$$

Then, from $QY = I_2$, we have

$$\begin{gathered} x_1 y_1 + \theta_1 \omega_2 = 1, \quad x_1 \omega_1 + i y_2 \theta_1 = 0, \\ \theta_2 y_1 + i x_2 \omega_2 = 0, \quad \theta_2 \omega_1 - x_2 y_2 = 1. \end{gathered}$$

Therefore, we have

$$Y = \begin{pmatrix} \frac{i x_2}{D_-} & -\frac{\theta_1}{D_-} \\ -\frac{\theta_2}{D_+} & \frac{x_1}{D_+} \end{pmatrix} = (\mathrm{sdet}\ Q)^{-1} \begin{pmatrix} \frac{1}{i x_2} & \frac{\theta_1}{x_2^2} \\ \frac{\theta_2}{x_2^2} & -\frac{x_1 x_2 + 2 i \theta_1 \theta_2}{x_2^3} \end{pmatrix},$$

which yields $YQ = I_2$ also. Here, we used

$$\mathrm{sdet}\ Q = \det(x_1 - \theta_1 (i x_2)^{-1} \theta_2)(\det(i x_2))^{-1} = \frac{i x_1 x_2 - \theta_1 \theta_2}{(i x_2)^2},\ (\mathrm{sdet}\ Q)^{-1} = \frac{i x_1 x_2 + \theta_1 \theta_2}{x_1^2}.$$

Therefore,

$$\begin{pmatrix} y_1 \\ \omega_2 \end{pmatrix} = \frac{D_+}{x_1^2} \begin{pmatrix} \frac{1}{i x_2} & \frac{-\theta_1}{(i x_2)^2} \\ \frac{-\theta_2}{(i x_2)^2} & \frac{i x_1 x_2 - 2 \theta_1 \theta_2}{(i x_2)^3} \end{pmatrix} \begin{pmatrix} v_1 \\ \rho_2 \end{pmatrix} = \frac{D_+}{x_1^2} \begin{pmatrix} \frac{i x_2 v_1 - \theta_1 \rho_2}{(i x_2)^2} \\ \frac{-i x_2 \theta_2 v_1 + (i x_1 x_2 - 2 \theta_1 \theta_2) \rho_2}{(i x_2)^3} \end{pmatrix},$$

$$\begin{pmatrix} \omega_1 \\ i y_2 \end{pmatrix} = \frac{D_+}{x_1^2} \begin{pmatrix} \frac{1}{i x_2} & \frac{-\theta_1}{(i x_2)^2} \\ \frac{-\theta_2}{(i x_2)^2} & \frac{i x_1 x_2 - 2 \theta_1 \theta_2}{(i x_2)^3} \end{pmatrix} \begin{pmatrix} \rho_1 \\ v_2 \end{pmatrix} = \frac{D_+}{x_1^2} \begin{pmatrix} \frac{i x_2 \rho_1 - i \theta_1 v_2}{(i x_2)^2} \\ \frac{-i x_2 \theta_2 \rho_1 + (i x_1 x_2 - 2 \theta_1 \theta_2) v_2}{(i x_2)^3} \end{pmatrix}.$$

Eigenvalues : Let

$$QU = \lambda U \quad \text{with} \quad U = \begin{pmatrix} u \\ \omega \end{pmatrix}, \quad u \in \mathfrak{R}_{\mathrm{ev}},\ \omega \in \mathfrak{R}_{\mathrm{od}},\ \lambda \in \mathfrak{R}_{\mathrm{ev}}.$$

Then,

$$(x_1 - \lambda) u + \theta_1 \omega = 0,\ \theta_2 u + (i x_2 - \lambda) \omega = 0.$$

Putting

$$D_+(\lambda) = (x_1 - \lambda)(ix_2 - \lambda) + \theta_1\theta_2,\ D_-(\lambda) = (x_1 - \lambda)(ix_2 - \lambda) - \theta_1\theta_2,$$

we have

$$D_-(\lambda)u = 0,\ D_+(\lambda)\omega = 0.$$

To garantee the existence of $u_{\rm B} \neq 0$ satisfying above, we take λ satisfying

$$D_-(\lambda) = \lambda^2 - (x_1 + ix_2)\lambda + ix_1x_2 - \theta_1\theta_2 = 0.$$

This yields

$$\lambda = x_1 + \frac{\theta_1\theta_2}{x_1 - ix_2} \quad (\text{or } \lambda = ix_2 - \tfrac{\theta_1\theta_2}{x_1-ix_2}, \text{ but this is not fitted because } \notin \mathfrak{R}_{\rm ev})$$

and

$$U = \begin{pmatrix} 1 \\ \frac{\theta_2}{x_1-ix_2} \end{pmatrix}, \quad QU = (x_1 + \frac{\theta_1\theta_2}{x_1 - ix_2})U.$$

Analogously, we seek $\tilde{\lambda} \in i\mathfrak{R}_{\rm ev}$, $\tilde{U} \in \mathfrak{R}_{\rm od} \times \mathfrak{R}_{\rm ev}$ satisfying $Q\tilde{U} = \tilde{\lambda}\tilde{U}$ which is given

$$\tilde{U} = \begin{pmatrix} \frac{-\theta_1}{x_1-ix_2} \\ 1 \end{pmatrix}, \quad Q\tilde{U} = (ix_2 + \frac{\theta_1\theta_2}{x_1 - ix_2})\tilde{U}.$$

Therefore,

$$Q\begin{pmatrix} 1 & -\frac{\theta_1}{x_1-ix_2} \\ \frac{\theta_2}{x_1-ix_2} & 1 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{\theta_1}{x_1-ix_2} \\ \frac{\theta_2}{x_1-ix_2} & 1 \end{pmatrix}\begin{pmatrix} x_1 + \frac{\theta_1\theta_2}{x_1-ix_2} & 0 \\ 0 & ix_2 + \frac{\theta_1\theta_2}{x_1-ix_2} \end{pmatrix}.$$

Diagonalization : We may diagonalize the matrix Q by using the change of variables

$$\begin{cases} y_1 = x_1 + \dfrac{\theta_1\theta_2}{x_1 - ix_2},\ y_2 = x_2 - \dfrac{i\theta_1\theta_2}{x_1 - ix_2}, \\ \rho_1 = \dfrac{\theta_1}{x_1 - ix_2},\ \rho_2 = -\dfrac{\theta_2}{x_1 - ix_2}, \end{cases} \tag{4.19}$$

or

$$\begin{cases} x_1 = y_1 + \rho_1\rho_2(y_1 - iy_2),\ x_2 = y_2 - i\rho_1\rho_2(y_1 - iy_2), \\ \theta_1 = \rho_1(y_1 - iy_2),\ \theta_2 = -\rho_2(y_1 - iy_2), \end{cases} \tag{4.20}$$

such that

$$GQG^{-1} = \begin{pmatrix} y_1 & 0 \\ 0 & iy_2 \end{pmatrix}, \quad GQ^2G^{-1} = \begin{pmatrix} y_1^2 & 0 \\ 0 & -y_2^2 \end{pmatrix} \tag{4.21}$$

where

$$G = \begin{pmatrix} 1 + 2^{-1}\rho_1\rho_2 & \rho_1 \\ \rho_2 & 1 - 2^{-1}\rho_1\rho_2 \end{pmatrix}, \quad G^{-1} = \begin{pmatrix} 1 + 2^{-1}\rho_1\rho_2 & -\rho_1 \\ -\rho_2 & 1 - 2^{-1}\rho_1\rho_2 \end{pmatrix}.$$

It is clear that

$$\text{str}\, Q = x_1 - ix_2 = y_1 - iy_2 = \text{str}\, GQG^{-1}, \quad \text{かつ}$$

$$\text{str}\, Q^2 = x_1^2 + x_2^2 + 2\theta_1\theta_2 = y_1^2 + y_2^2 = \text{str}\,(GQG^{-1})^2.$$

4.4 Integration of Gaussian type and Pfaffian

Though we haven't defined the integration w.r.t. odd variables, for the future indication, we mention the relation of Gaussian type integral and determinant and Pfaffian.

Definition 4.5 *For $n \times n$-anti symmeric matrix $\tilde{B} = (\tilde{B}_{jk})$ with even elements, we define the Pfaffian* $\mathrm{Pf}\,(\tilde{B})$ *of B as*

$$\mathrm{Pf}\,(\tilde{B}) = \frac{1}{(n/2)!} \sum_{\rho \in \wp_n} \mathrm{sgn}\,(\rho) \tilde{B}_{\rho(1)\,\rho(2)} \cdots \tilde{B}_{\rho(n-1)\,\rho(n)}. \tag{4.22}$$

Here, $\wp_n$ is the permutation group of degree n, $\mathrm{sgn}\,(\rho)$ is the signature of $\rho \in \wp_n$.

Remark : Let $n$be even, and let $A = (A_{ij})$ be anti symmeric matrix. Then, we have

$$\int d\theta \, \exp(-\frac{1}{2} A_{ij}\theta_i\theta_j) = \mathrm{Pf}\,(A).$$

Moreover, $\mathrm{Pf}\,(A)^2 = \det(A)$ holds.

The integration w.r.t. odd variables is introduced by Berezin such that

$$\int d\theta \, \theta = 1, \qquad \int d\theta \, 1 = 0 \quad \text{(Berezin integral)}.$$

This integration is defined without measure but using inner-multiplication in exterior algebra such that

$$\frac{\partial}{\partial z_j} \rfloor dz_k = \delta_{jk}.$$

Definition 4.6 *A even super matrix $M = \begin{pmatrix} A & C \\ D & B \end{pmatrix}$ is called positive-definite if the following conditions are satisfied:*

(gs.1) *A has the body part A_{B} which is regular positive definite symmetric matrix.*

(gs.2) *B is a regular anti-symmetric matrix.*

(gs.3) *C and D satisfies ${}^t C + D = 0$.*

For above super matrix M, we define the corresponding bilinear form as

$$\begin{aligned} \langle X, MX \rangle &= {}^t X M X \\ &= \sum_{j,k=1}^{m} x_j A_{jk} x_k + \sum_{j=1}^{m} \sum_{s=1}^{n} x_j C_{j\,m+s} \theta_s + \sum_{k=1}^{m} \sum_{t=1}^{n} \theta_t D_{m+t\,k} x_k + \sum_{s,t=1}^{n} \theta_s B_{m+s\,m+t} \theta_t. \end{aligned}$$

Lemma 4.3 *Let M be a even, positive definite matrix. Then,*

$$\begin{aligned} G(\lambda, M) &= \int_{\mathfrak{R}^{m|n}} dX e^{-\lambda^{-1} 2^{-1} \langle X, MX \rangle} \quad \textit{for} \quad \lambda > 0 \\ &= \begin{cases} 0 & \textit{if } n \textit{ is odd}, \\ (2\pi\lambda)^{m/2} (2\lambda)^{-n/2} (\det\ A)^{-1/2} \, \mathrm{Pf}\,(B - DA^{-1}C) & \textit{if } n \textit{ is even}. \end{cases} \end{aligned} \tag{4.23}$$

Comparison 4.5 *For a positive definite symmetric real matrix H, we have*

$$\int_{\mathbb{R}^m} e^{-\lambda x \cdot Hx/2} dx = \left(\frac{2\pi}{\lambda} \right)^{m/2} (\det H)^{-1/2}. \tag{4.24}$$