Foundation of Super Analysis Lecture 4 – 08-10.31 A. Inoue

1 Necessity of the non-commutative analysis and its merit

2 Dirac and Weyl equations

3 Super number and Superspace

3.1 Super number

3.2 The superspace

3.3 A construction of countable Grassmann generators

4 Linear algebra on the superspace

In this section, we refer results from

F.A. Berezin(ed. A.A. Kirillov), Introduction to Superanalysis, D.Reidel Publ. Company, 1987.

B.S. deWitt, Supermanifolds, London, Cambridge Univ. Press, 1984.

D.A. Leites, *Introduction to the theory of supermanifolds*, Russian Math. Surveys 35(1980), pp.1-64.

Remark 4.1 *Almost all papers prefixed "super", treated the case of finite number of odd variables with the same number of Grassmann generators, or rather, they didn't distinguish odd variables and Grassmann generators. On the other hand, as is claimed in Rogers, if we use countable Grassmann generators with* ℓ^1*-norm, we need to check whether the condition* $\sum_I |X_I| < \infty$ *is satisfied or not. It seems very hard to check it. But in any way, after slight modification if necessary, algebraic operations not affected with the topology is borrowed from these papers.*

4.1 Matrix algebras on the superspace

Definition 4.1 *A rectangular array* M*, whose cells are indexed by pairs consisting of a row number and a column number, is called a supermatrix and denoted by* $M \in \mathrm{Mat}\,((m|n) \times (r|s) : \mathfrak{C})$*, if it satisfies the following:*

1. *A* $(m+n) \times (r+s)$ *matrix* M *is decomposed blockwisely as* $M = \begin{pmatrix} A & C \\ D & B \end{pmatrix}$ *where* A*,* B*,* C *and* D *are* $m \times r$*,* $n \times s$*,* $m \times s$ *and* $n \times r$ *matrices with elements in* $\mathfrak{C}$*, respectively.*

2. One of the following conditions is satisfied: Either

- $p(M)=0$, *that is,* $p(A_{jk})=0=p(B_{uv})$ *and* $p(C_{jv})=1=p(D_{uk})$ *or*
- $p(M)=1$, *that is,* $p(A_{jk})=1=p(B_{uv})$ *and* $p(C_{jv})=0=p(D_{uk})$.

We call M *is even denoted by* $\mathrm{Mat}_{\mathrm{ev}}((m|n)\times(r|s):\mathfrak{C})$ *(resp. odd denoted by* $\mathrm{Mat}_{\mathrm{od}}((m|n)\times(r|s):\mathfrak{C})$*) if* $p(M)=0$ *(resp.* $p(M)=1$*). Therefore, we have*

$$\mathrm{Mat}\,((m|n)\times(r|s):\mathfrak{C})=\mathrm{Mat}_{\mathrm{ev}}((m|n)\times(r|s):\mathfrak{C})\oplus\mathrm{Mat}_{\mathrm{od}}((m|n)\times(r|s):\mathfrak{C}).$$

Moreover, we may decompose M *as* $M=M_{\mathrm{B}}+M_{\mathrm{S}}$ *where*

$$M_{\mathrm{B}}=\begin{cases}\begin{pmatrix}A_{\mathrm{B}} & 0\\ 0 & B_{\mathrm{B}}\end{pmatrix} & \textit{when}\quad p(M)=0,\\[2em] \begin{pmatrix}0 & C_{\mathrm{B}}\\ D_{\mathrm{B}} & 0\end{pmatrix} & \textit{when}\quad p(M)=1.\end{cases}$$

The summation of two matrices in $\mathrm{Mat}_{\mathrm{ev}}((m|n)\times(r|s):\mathfrak{C})$ or in $\mathrm{Mat}_{\mathrm{od}}((m|n)\times(r|s):\mathfrak{C})$ is defined as usual, but the sum of $\mathrm{Mat}_{\mathrm{ev}}((m|n)\times(r|s):\mathfrak{C})$ and $\mathrm{Mat}_{\mathrm{od}}((m|n)\times(r|s):\mathfrak{C})$ is not defined except at least one of them being zero matrix.

It is clear that if M is the $(m+n)\times(r+s)$ matrix and N is the $(r+s)\times(p+q)$ matrix, then we may define the product MN and its parity $p(MN)$ as

$$(MN)_{ij}=\sum_k M_{ik}N_{kj},\quad p(MN)=p(M)+p(N)\mod 2.$$

Moreover, we define $\mathrm{Mat}\,[m|n:\mathfrak{C}]$ as the algebra of $(m+n)\times(m+n)$ supermatrices.

Matrices as Linear Transformations: By definition of matrix operation to vector, we have

$$\mathrm{Mat}_{\mathrm{ev}}((m|n)\times(r|s):\mathfrak{C})\ni M=\begin{pmatrix}A & C\\ D & B\end{pmatrix}:\mathfrak{R}^{r|s}\to\mathfrak{R}^{m|n},$$

$$\mathrm{Mat}_{\mathrm{od}}((m|n)\times(r|s):\mathfrak{C})\ni M=\begin{pmatrix}A & C\\ D & B\end{pmatrix}:\mathfrak{R}^{r|s}\to\mathfrak{R}^{m}_{\mathrm{od}}\times\mathfrak{R}^{n}_{\mathrm{ev}},$$

$$\mathrm{Mat}_{\mathrm{od}}((n|m)\times(m|n):\mathfrak{C})\ni\Lambda_{n,m}=\begin{pmatrix}0 & \mathbb{I}_n\\ \mathbb{I}_m & 0\end{pmatrix}:\mathfrak{R}^{m}_{\mathrm{od}}\times\mathfrak{R}^{n}_{\mathrm{ev}}\to\mathfrak{R}^{n}_{\mathrm{ev}}\times\mathfrak{R}^{m}_{\mathrm{od}}=\mathfrak{R}^{n|m}.$$

We define X^* for $X=\sum X_I\sigma^I$ as

$$X^*=\sum_{I\in\mathcal{I}}\overline{X_I}\,\overline{\sigma^I}=\sum_{I\in\mathcal{I}}(-1)^{\frac{|I|(|I|-1)}{2}}\overline{X_I}\sigma^I.$$

Here, the complex conjugate of $X_I\in\mathbb{C}$ is denoted by $\overline{X_I}$ and for $I=(i_1,\cdots,i_n)$ we put $\overline{\sigma^I}=\sigma_n^{i_n}\cdots\sigma_1^{i_1}$.

For elements $X=(x,\theta)=(x_1,\cdots,x_m,\theta_1,\cdots,\theta_n)$ and $\Xi=(\xi,\pi)=(\xi_1,\cdots,\xi_m,\pi_1,\cdots,\pi_n)$ in $\mathfrak{R}^{m|n}$, we define the scalar product as

$$\langle X|\Xi\rangle_{m|n}=\sum_{j=1}^m\langle x_j|\xi_j\rangle+\sum_{k=1}^n\langle\theta_k|\pi_k\rangle\in\mathfrak{R}_{\mathrm{ev}}.\tag{4.1}$$

We denote $\langle\cdot|\cdot\rangle_{m|n}$ simply as $\langle\cdot|\cdot\rangle$. If we introduce the duality between $\mathfrak{R}^{m|n}$ as in (**??**), we may define the transposed operator as

$$\langle MX|\Xi\rangle_{m|n} = \langle X|M^t\,\Xi\rangle_{r|s} \quad \text{for any} \quad M\in \mathrm{Mat}_{\,\mathrm{ev}}((m|n)\times(r|s):\mathfrak{C}),$$

for $X=(x,\theta)\in\mathfrak{R}^{r|s}$ and $\Xi=(\xi,\omega)\in\mathfrak{R}^{m|n}$. More precisely, we have

$$M^t = \begin{pmatrix} A & C\\ D & B\end{pmatrix}^t = \begin{pmatrix} A^t & D^t\\ -C^t & B^t\end{pmatrix} \quad\text{and}\quad M^{tttt}=M.$$

Analogously, if we define the duality between $\mathfrak{C}_Z^{m|n}$ and $\mathfrak{C}_\Upsilon^{m|n}$ by

$$\langle Z|\Upsilon\rangle_{m|n} = \sum_{j=1}^m \langle z_j|\eta_j^*\rangle + \sum_{k=1}^n\langle\theta_k|\rho_k^*\rangle, \quad \text{or} \quad = \sum_{j=1}^m \langle z_j^*|\eta_j\rangle + \sum_{k=1}^n\langle\theta_k^*|\rho_k\rangle,$$

for $Z=(z,\theta)\in\mathfrak{C}^{r|s}$, $\Upsilon=(\eta,\rho)\in\mathfrak{C}^{m|n}$, we may introduce M^*, the adjoint of matrix M, by

$$\langle MZ|\Upsilon\rangle_{m|n} = \langle Z|M^*\Upsilon\rangle_{r|s}.$$

Therefore, we have

$$M^* = \begin{pmatrix} A & C\\ D & B\end{pmatrix}^* = \begin{pmatrix} A^* & D^*\\ -C^* & B^*\end{pmatrix} \quad\text{and}\quad M^{****}=M.$$

Lemma 4.1 *For $M\in\mathrm{Mat}\,((m|n)\times(r|s):\mathfrak{C})$ and $N\in\mathrm{Mat}\,((r|s)\times(p|q):\mathfrak{C})$, we have*

$$(MN)^t = N^t\,M^t, \quad (MN)^* = N^*M^*, \quad (M^t)^t = \Lambda M\Lambda, \quad \textit{where} \quad \Lambda = \begin{pmatrix}\mathbb{I}_m & 0\\ 0 & -\mathbb{I}_n\end{pmatrix}.$$

If $M\in\mathrm{Mat}\,[m|n:\mathfrak{C}]$ is even, denoted by $M\in\mathrm{Mat}_{\,\mathrm{ev}}[m|n:\mathfrak{C}]$, then M acts on $\mathfrak{R}^{m|n}$ linearly. Denoting this by T_M, we call it super linear transformation on $\mathfrak{R}^{m|n}$ and M is called the representative matrix of T_M.

Porposition 4.1 *Let $M\in\mathrm{Mat}_{\,\mathrm{ev}}[m|n:\mathfrak{C}]$ and assume* $\det M_{\mathrm{B}}\neq 0$. *Then, for given $Y\in\mathfrak{R}^{m|n}$,*

$$T_M X = Y \tag{4.2}$$

has the unique solution $X\in\mathfrak{R}^{m|n}$, which is denoted by $X=M^{-1}Y$.

Proof. Since M_{B} has the inverse matrix M_{B}^{-1}, (**??**) is reduced to

$$X + N_{\mathrm{S}}X = Y', \quad Y' = M_{\mathrm{B}}^{-1}Y$$

where $N_{\mathrm{S}} = M_{\mathrm{B}}^{-1}M_{\mathrm{S}}$. Remark that $N_{\mathrm{S}}X^{[j]}\in\sum_{k\geq j+1}^\infty \mathfrak{C}^{[k]}$ for $j\geq 0$. Decomposing by degree, we get

$$X^{[j]} = Y'^{[j]} - (N_{\mathrm{S}}X^{(j-1)})^{[j]} \quad \text{for} \quad j=1,2,\dots.$$

As $X^{(0)}=X^{[0]}=Y'^{[0]}$, we get $X^{[j]}$ from $X^{(j-1)}$ for $j\geq 1$ by induction. □

Exercise 4.1 *How about $M\in\mathrm{Mat}_{\,\mathrm{od}}((m|n)\times(n|m):\mathfrak{C})$?*

Definition 4.2 *$M\in\mathrm{Mat}_{\,\mathrm{ev}}[m|n:\mathfrak{C}]$ is called invertible or non-singular if M_{B} is invertible, i.e.* $\det A_{\mathrm{B}}\cdot\det B_{\mathrm{B}}\neq 0$, *and denoted by $M\in\mathrm{GL}_{\mathrm{ev}}[m|n:\mathfrak{C}]$.*

4.2 Supertrace, superdeterminant

Lemma 4.2 *Let V, W be two rectangular matrices with odd elements, $m\times n$, $n\times m$, respectively. We have*
(1) $\mathrm{tr}\,(VW)^k = -\,\mathrm{tr}\,(WV)^k$ *for any* $k=1,2,\cdots$.
(2) $\det(\mathbb{I}_m + VW) = \det(\mathbb{I}_n + WV)^{-1}$.

Proof. Let $V=(v_{ij})$, $W=(w_{jk})$ with $v_{ij}, w_{jk}\in\mathfrak{C}_{\mathrm{od}}$.

$$\begin{aligned}\mathrm{tr}\,(VW)^k &= \sum v_{ij_1}w_{j_1j_2}v_{j_2j_3}\cdots v_{j_{k-1}j_\ell}w_{j_ki}\\ &= -\sum w_{j_1j_2}v_{j_2j_3}\cdots v_{j_{\ell-1}j_k}w_{j_ki}v_{ij_1} = -\,\mathrm{tr}\,(WV)^k.\end{aligned}$$

Using this, we have $\mathrm{tr}\,((WV)^{\ell-1}WV) = -\,\mathrm{tr}\,(V(WV)^{\ell-1}W)$ which yields

$$\begin{aligned}\log\,\det(\mathbb{I}_n+WV) = \mathrm{tr}\,\log\,(\mathbb{I}_n+WV) &= \sum_\ell \frac{(-1)^{\ell+1}}{\ell}\,\mathrm{tr}\,((WV)^{\ell-1}WV)\\ &= \sum_\ell \frac{(-1)^{\ell+1}}{\ell}\big[-\,\mathrm{tr}\,(V(WV)^{\ell-1}W)\big] = -\sum_\ell\frac{(-1)^{\ell+1}}{\ell}\,\mathrm{tr}\,(VW)^\ell\\ &= -\log\,\det(\mathbb{I}_m+VW). \qquad \square\end{aligned}$$

If $A=(a_{ij})\in\mathrm{Mat}\,(m\times n:\mathfrak{C}_{\mathrm{ev}})$, $B=(b_{jk})\in\mathrm{Mat}\,(n\times m:\mathfrak{C}_{\mathrm{ev}})$, then we have
(1) $\mathrm{tr}\,(AB)^k = \mathrm{tr}\,(BA)^k$,
(2) $\det(\mathbb{I}_m+AB)=\det(\mathbb{I}_n+BA)$.

Definition 4.3 *Let* $M=\begin{bmatrix}A & C\\ D & B\end{bmatrix}\in\mathrm{Mat}\,[m|n:\mathfrak{C}]$. *We define the supertrace of M by*

$$\mathrm{str}\ M = \mathrm{tr}\ A - (-1)^{p(M)}\,\mathrm{tr}\ B.$$

Using Lemma **??**, we get readily

Porposition 4.2 *(a) Let* $M,N\in\mathrm{Mat}\,[m|n:\mathfrak{C}]$ *such that* $p(M)+p(N)\equiv 0 \bmod 2$. *Then, we have*

$$\mathrm{str}\,(M+N) = \mathrm{str}\ M + \mathrm{str}\ N.$$

(b) M is a matrix of size $(m+n)\times(r+s)$ and N is a matrix of size $(r+s)\times(m+n)$. Then,

$$\mathrm{str}\,(MN) = (-1)^{p(M)p(N)}\,\mathrm{str}\,(NM).$$

Definition 4.4 *Let $B=(B_{jk})$ be $(\ell\times\ell)$-matrix with elements in $\mathfrak{C}_{\mathrm{ev}}$, denoted by, $B\in\mathrm{Mat}\,[\ell:\mathfrak{C}_{\mathrm{ev}}]$. As $\mathfrak{C}_{\mathrm{ev}}$ is a commutative ring, we may define* $\det B$ *as usual:*

$$\det B = \sum_{\rho\in\wp_\ell}\mathrm{sgn}\,(\rho)B_{1\,\rho(1)}\cdots B_{\ell\,\rho(\ell)}.$$

Then, we have, as ordinary case,

$$\det(AB) = \det A\det B,\quad \det(\exp A) = \exp(\mathrm{tr}\ A)\quad \text{for}\quad A,B\in\mathrm{Mat}\,[\ell:\mathfrak{C}_{\mathrm{ev}}]. \tag{4.3}$$

Definition 4.5 *Let M be a supermatrix. When $\det B_{\rm B} \neq 0$, we put*

$$\text{sdet } M = (\det(A - CB^{-1}D))(\det B)^{-1}$$

and call it superdeterminant or Berezinian of M.

Let

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad M = \begin{pmatrix} \mathbb{I}_m & 0 \\ -A_{22}^{-1}A_{21} & \mathbb{I}_n \end{pmatrix},$$

be block matrices of even elements. Then, we have

$$\det A = \det AM = \det \begin{pmatrix} A_{11} - A_{12}A_{22}^{-1}A_{21} & A_{12} \\ 0 & A_{22} \end{pmatrix} = \det(A_{11} - A_{12}A_{22}^{-1}A_{21}) \det A_{22}.$$

Corollary 4.1 *When $\det B_{\rm B} \neq 0$ and $\text{sdet } M \neq 0$, then $\det A_{\rm B} \neq 0$.*

Exercise 4.2 *Prove the above corollary.*

Remark 4.2 *It seems meaningful to cite here the result of Dyson*[1]*:*

Theorem 4.1 (Dyson) *Let R be a ring with a unit element and without divisors of zero. Assume that on the matrix ring A with $n > 1$, a mapping D exists satisfying the following axioms:*

Axiom 1. For any $a \in A$, $D(a) = 0$ if and only if there is a non-zero $w \in W$ with $aw = 0$. Here, W is the set of single-column matrices with elements in R.

Axiom 2. $D(a)D(b) = D(ab)$.

Axiom 3. Let the elements of a be a_{ij} $i, j = 1, \cdots, n$, and similarly for b and c. If for some row-index k we have

$$\begin{cases} a_{ij} = b_{ij} = c_{ij}, & i \neq k \\ a_{ij} + b_{ij} = c_{ij}, & i = k, \end{cases}$$

then

$$D(a) + D(b) = D(c).$$

Then, R is commutative.

This theorem states that if the elements of matrix are taken from non-commutative algebra, then it is impossible to define the determinant having above three properties. But, he claims a certain 'determinant' is defined for some class of matrices with elements in 'quarternion' requiring only one or two properties above (By the way, Moore's point of view, is reconsidered significantly in that paper).
In fact, we may define "superdeterminant" for "supermatrix" as above which staisfies the properties below.

[1] F.J. Dyson, *Quaternion determinants*, Helvetica Physica Acta 45(1972), pp. 289-302.