Foundation of Super Analysis Lecture 3 – 08.10.17 A. Inoue

1 Necessity of the non-commutative analysis and its merit

2 Dirac and Weyl equations

2.1 The origin of Dirac and Weyl equations

2.2 The method of characteristics and Hamiltonian path-integral representation

2.3 The decomposition of 2×2-matrices – what is the meaning of matrix operation

3 Super number and Superspace

3.1 Super number

3.1.1 The Grassmann generators

For symbols $\{\sigma_j\}_{j=1}^{\infty}$ satisfying the Grassmann relation

$$\sigma_j\sigma_k + \sigma_k\sigma_j = 0, \quad j,k = 1,2,\cdots, \tag{3.1}$$

we put formally

$$\mathfrak{C} = \{X = \sum_{I\in\mathcal{I}} X_I\sigma^I \mid X_I \in \mathbb{C}\} \tag{3.2}$$

and

$$\begin{cases} \mathfrak{C}^{(0)} = \mathfrak{C}^{[0]} = \mathbb{C}, \\ \mathfrak{C}^{(j)} = \left\{X = \sum_{|I|\le j} X_I\sigma^I\right\} \quad \text{and} \\ \mathfrak{C}^{[j]} = \left\{X = \sum_{|I|=j} X_I\sigma^I\right\} = \mathfrak{C}^{(j)}/\mathfrak{C}^{(j-1)}, \end{cases}$$

where

$$\mathcal{I} = \{I = (i_k) \in \{0,1\}^{\mathbf{N}} \mid |I| = \sum_k i_k < \infty\},$$

$$\sigma^I = \sigma_1^{i_1}\sigma_2^{i_2}\cdots, \quad I = (i_1, i_2, \cdots), \quad \sigma^{\tilde{0}} = 1, \quad \tilde{0} = (0,0,\cdots) \in \mathcal{I}.$$

Remark 3.1 *How do we construct symbols $\{\sigma_j\}_{j=1}^{\infty}$ satisfying the Grassmann relation? What is the meaning of summation appeared above? These will be soon explained.*

In today's lecture, we prove the following Proposition which garantees that $\mathfrak{C}$ (or $\mathfrak{R}$, defined later) plays the alternative role of $\mathbb{C}$ (or $\mathfrak{R}$) in analysis.

Porposition 3.1 (Inoue and Maeda[1]) *$\mathfrak{C}$ forms an ∞-dimensional Fréchet-Grassmann algebra over $\mathbb{C}$, that is, an associative, distributive and non-commutative ring with degree, which is endowed with the Fréchet topology.*

Remark 3.2 *There exist some papers using $\mathfrak{C}$, for example, S. Matsumoto and K. Kakazu[2], Y. Choquet-Bruhat[3], P. Bryant[4]. But, seemingly, they didn't try to construct "elementary and real analysis" on this "fundamental field" $\mathfrak{C}$ (or $\mathfrak{R}$).*

3.1.2 Sequence spaces and their topologies

Following Köthe[5], we intorduce the sequence spaces ω and ϕ.

$$\begin{cases} \phi = \{\mathfrak{x} = (x_k) = (x_1, x_2, \cdots, x_k, \cdots) \mid x_k \in \mathbb{C} \text{ and } x_k = 0 \text{ except for finitely many } k\}, \\ \omega = \{\mathfrak{u} = (u_k) = (u_1, u_2, \cdots, u_k, \cdots) \mid u_k \in \mathbb{C}\}. \end{cases} \tag{3.3}$$

For any sequence space $\mathcal{X}$ containing ϕ, we define the space $\mathcal{X}^\times$ by

$$\mathcal{X}^\times = \left\{ \mathfrak{u} = (u_k) \mid \sum_k |u_k||x_k| < \infty \quad \text{for any} \quad \mathfrak{x} = (x_k) \in \mathcal{X} \right\},$$

then, we get

$$\phi^\times = \omega \quad \text{and} \quad \omega^\times = \phi.$$

We introduce the (normal) topology in $\mathcal{X}$ and $\mathcal{X}^\times$ by defining the seminorms

$$p_{\mathfrak{u}}(\mathfrak{x}) = \sum_k |u_k||x_k| = p_{\mathfrak{x}}(\mathfrak{u}) \quad \text{for } \mathfrak{x} \in \mathcal{X} \text{ and } \mathfrak{u} \in \mathcal{X}^\times. \tag{3.4}$$

Especially, $\mathfrak{x}^{(n)}$ converges to $\mathfrak{x}$ in ϕ, that is, $p_{\mathfrak{u}}(\mathfrak{x}^{(n)} - \mathfrak{x}) \to 0$ as $n \to \infty$ for each $\mathfrak{u} \in \omega$ if and only if for any $\epsilon > 0$, there exist L and n_0 such that

$$\begin{cases} \text{(i)} \quad x_k^{(n)} = x_k = 0 \quad \text{for} \quad k > L \quad \text{when} \quad n \geq n_0, \quad \text{and} \\ \text{(ii)} \quad |x_k^{(n)} - x_k| < \epsilon \quad \text{for} \quad k \leq L \quad \text{when} \quad n \geq n_0. \end{cases} \tag{3.5}$$

Analogously, $\mathfrak{u}^{(n)}$ converges to $\mathfrak{u}$ in ω, that is, $p_{\mathfrak{x}}(\mathfrak{u}^{(n)} - \mathfrak{u}) \to 0$ as $n \to \infty$ for each $\mathfrak{x} \in \phi$ if and only if for any $\epsilon > 0$ and each k, there exists $n_0 = n_0(\epsilon, k)$ such that

$$|u_k^{(n)} - u_k| < \epsilon \quad \text{when} \quad n \geq n_0. \tag{3.6}$$

Clearly, ω forms a Fréchet space because the above topology in ω is equivalent to the one defined by countable seminorms: $\{p_k(\mathfrak{u})\}_{k \in \mathbb{N}}$ where $p_k(\mathfrak{u}) = |u_k|$ for $\mathfrak{u} = (u_1, u_2, \cdots) = \sum_{j=1}^\infty u_j \mathfrak{e}_j \in \omega$ with $\mathfrak{e}_j = (\overbrace{0, \cdots, 0, 1}^{j}, 0, \cdots) \in \omega$.

[1] A. Inoue & Y. Maeda: *Foundations of calculus on super Euclidean space $\mathfrak{R}^{m|n}$ based on a Fréchet-Grassmann algebra*, Kodai Math.J.14(1991), pp. 72-112.

[2] S. Matsumoto and K. Kakazu, *A note on topology of supermanifolds*, J.Math.Phys.27(1986), pp. 2690-2692.

[3] Y. Choquet-Bruhat, *Supergravities and Kaluza-Klein theories*, in "Topological properties and global structure of space-time" (eds. P. Bergmann and V. de Sabbata), New York, Plenum Press, 1986, pp. 31-48.

[4] P. Bryant, *DeWitt supermanifolds and infinite dimensional ground rings*, J.London Math.Soc.39(1989), pp. 347-368.

[5] G. Köthe: Topological Linear Spaces I, Berlin-New York-Tokyo-Heidelberg, Springer-Verlag, 1969.

Now, we define the isomorphism (diadic-decomposition) from $\mathcal{I}$ onto $\mathbb{N}$ defined by

$$r: \mathcal{I} \ni I = (i_k) \to r(I) = 1 + \frac{1}{2}\sum_{k=1}^{\infty} 2^k i_k \in \mathbb{N} \quad \text{where} \quad i_k = 0 \text{ or } 1. \tag{3.7}$$

Using $r(I)$ in (3.7), we define a map

$$T: \ \sigma^I \to \mathfrak{e}_{r(I)} \quad \text{for} \quad I = (i_k) \in \mathcal{I}.$$

Extending this map linearly, we put

$$T(X) = \sum x_{r(I)}\mathfrak{e}_{r(I)} \in \omega \quad \text{for} \quad X = \sum_{|I| \le j} X_I \sigma^I \in \mathfrak{C}^{(j)}. \tag{3.8}$$

More explicitely, we have the following first few terms:

$$\sum x_{r(I)}\mathfrak{e}_{r(I)} = (X_{(0,0,0,\cdots)}, X_{(1,0,0,\cdots)}, X_{(0,1,0,\cdots)}, X_{(1,1,0,\cdots)}, X_{(0,0,1,\cdots)}, X_{(1,0,1,\cdots)}, X_{(0,1,1,\cdots)}, \cdots).$$

Then, since $T(\mathfrak{C}^{[j]})$ and $T(\mathfrak{C}^{[k]})$ are disjoint sets in ω if $j \neq k$, we have

$$\sum_{j=0}^{\infty} T(\mathfrak{C}^{[j]}) = \omega. \tag{3.9}$$

Therefore, it is reasonable to write as in (3.2) and more precisely,

$$\mathfrak{C} = \oplus_{j=0}^{\infty}\mathfrak{C}^{[j]}, \quad \text{that is,} \quad X = \sum_{j=0}^{\infty} X^{[j]} \quad \text{with} \quad X^{[j]} = \sum_{|I|=j} X_I \sigma^I. \tag{3.10}$$

Here, $X^{[j]}$ is called the j-th degree component of $X \in \mathfrak{C}$. By definition, we get

$$\begin{cases} \mathfrak{C}^{(j)} \subset \mathfrak{C}^{(k)} \quad \text{for} \quad j \le k, \\ \mathfrak{C} = \sum_{j=0}^{\infty} \mathfrak{C}^{[j]} \quad \text{with} \quad \cap_{j=0}^{\infty} \mathfrak{C}^{(j)} = \mathbb{C}, \end{cases} \tag{3.11}$$

$$\mathfrak{C}^{[j]} \cdot \mathfrak{C}^{[k]} \subset \mathfrak{C}^{[j+k]} \quad \text{and} \quad \mathfrak{C}^{(j)} \cdot \mathfrak{C}^{(k)} \subset \mathfrak{C}^{(j+k)}. \tag{3.12}$$

Remark 3.3 *The second relation with $\mathfrak{C}^{(*)}$ in* (3.12) *also holds for the Clifford algebras but the first one with $\mathfrak{C}^{[\cdot]}$ is specific to the Grassmann algebras satisfying* (3.1). *Here, the Clifford relation for $\{e_j\}$ is defined by*

$$e_i e_j + e_j e_i = 2\delta_{ij}\mathbb{I} \quad \textit{for any} \quad i, j = 1, 2, \cdots. \tag{3.13}$$

Typical examples, though not countably many but finitely many elements, are the 2×2-Pauli matrices $e_j = \{\boldsymbol{\sigma}_j\}_{j=1,2,3}$ and the 4×4-Dirac matrices $\{e_j\}_{j=0,1,2,3} = \{\boldsymbol{\beta}, \boldsymbol{\alpha}_j\}$.

3.1.3 Topology

We introduce the weakest topology in $\mathfrak{C}$ which makes the map T continuous from $\mathfrak{C}$ to ω, that is, $X = \sum_{I \in \mathcal{I}} X_I \sigma^I \to 0$ in $\mathfrak{C}$ if and only if $\mathrm{proj}_I(X) \to 0$ for each $I \in \mathcal{I}$ with $\mathrm{proj}_I(X) = X_I$; it is equivalent to the metric $\mathrm{dist}(X, Y) = \mathrm{dist}(X - Y)$ defined by

$$\mathrm{dist}(X) = \sum_{I \in \mathcal{I}} \frac{1}{2^{r(I)}} \frac{|\,\mathrm{proj}_I(X)|}{1 + |\,\mathrm{proj}_I(X)|} \quad \text{for any} \quad X \in \mathfrak{C}. \tag{3.14}$$

For example, $X^{(\ell)} = f(\ell)\sigma_1 \cdots \sigma_\ell \to 0$ in $\mathfrak{C}$ even if $f(\ell) \to \infty$ because $\mathrm{dist}(X^{(\ell)}) \le 2^{-2^\ell + 1}$.

3.1.4 Algebraic operations

For any $X, Y \in \mathfrak{C}$, we define

$$X+Y=\sum_{j=0}^{\infty}(X+Y)^{[j]} \quad \text{with} \quad (X+Y)^{[j]}=X^{[j]}+Y^{[j]} \quad \text{for} \quad j\geq 0 \tag{3.15}$$

and

$$XY=\sum_{j=0}^{\infty}(XY)^{[j]} \quad \text{where} \quad (XY)^{[j]}=\sum_{k=0}^{j}X^{[j-k]}Y^{[k]}=\sum_{|I|=j}(XY)_I\sigma^I. \tag{3.16}$$

Here, $(XY)_I=\sum_{I=J+K}(-1)^{\tau(I;J,K)}X_JY_K\in\mathbb{C}$ is well-defined because for any set $I\in\mathcal{I}$, there exist only finitely many decompositions by sets J, K satisfying $I=J\dot{+}K$ (*i.e.* $I=J\cup K,\ J\cap K=\emptyset$). Here, the indeces $\tau(I;J,K)$, or more generally $\tau(I;J_1,\cdots,J_k)$ are defined by

$$(-1)^{\tau(I;J_1,\cdots,J_k)}\sigma^{J_1}\cdots\sigma^{J_k}=\sigma^I \quad \text{with} \quad I=J_1\dot{+}J_2\dot{+}\cdots\dot{+}J_k. \tag{3.17}$$

But for notational simplicity, we will use $(-1)^{\tau(*)}$ without specifying the decomposition if there occurs no confusion.

Exercise 3.1 *For sets J, K satisfying $I=J+K$,*

$$(-1)^{|J||K|}(-1)^{\tau(I;J,K)}=(-1)^{\tau(I;K,J)}.$$

Moreover, we get

Lemma 3.1 *The product defined by* (3.16) *is continuous from $\mathfrak{C}\times\mathfrak{C}\to\mathfrak{C}$.*

Proof. It is simple by noting that there exist $2^{|I|}$ elements $J\in\mathcal{I}$ satisfying $J\subset I$ and that

$$|(XY)_I|\leq\sum_{I=J+K}|X_J||Y_K|\leq 2^{r(I)}(\max_{J\subset I}|X_J|)(\max_{K\subset I}|Y_K|) \quad \text{for any} \quad X,Y\in\mathfrak{C}. \qquad \square$$

Proof of Proposition 3.1. Clearly, we get

$$\begin{cases} X(YZ)=(XY)Z & (\textit{associativity}),\\ X(Y+Z)=XY+XZ & (\textit{distributivity}). \end{cases}$$

Other properties have been proved. □

Remark 3.4 *We may consider that an element of $X\in\mathfrak{C}$ stands for the 'state' such that the position labeled by σ^I is occupied by $X_I\in\mathbb{C}$. In other word, considering $\{\sigma_i\}$ as the countable indeterminate letters, it seems reasonable to regard $\mathfrak{C}$ as the set of certain formal power series* [6] *with simple topology. Therefore, it is permitted to reorder the terms freely under 'summation sign'. That is, the summation $\sum_{I\in\mathcal{I}}X_I\mathfrak{e}_{r(I)}$ is 'unconditionally (though not absolutely) convergent'* [7] *and so is $\sum_{I\in\mathcal{I}}X_I\sigma^I$. We use such a big space $\mathfrak{C}$ with rather weak topology because this algebra is considered as the ambient space for reordering the places. We feel such a big ambient space will be prefarable and tractable for our future use.*

[6] with the special property that same letter appears only once in each monomials

[7] diverting the terminology of the basis problem in the Banach spaces

3.1.5 The supernumber

The set $\mathfrak{C}$ defined by (3.2) is called the *(complex) supernumber algebra* over $\mathbb{C}$ and any element X of $\mathfrak{C}$ is called *(complex) supernumber.*

Parity: We introduce the parity in $\mathfrak{C}$ by setting

$$p(X)=\begin{cases}0 & \text{if } X=\sum_{I\in\mathcal{I},|I|=\mathrm{even}} X_I\sigma^I,\\ 1 & \text{if } X=\sum_{I\in\mathcal{I},|I|=\mathrm{odd}} X_I\sigma^I.\end{cases} \tag{3.18}$$

$X\in\mathfrak{C}$ is called homogeneous if it satisfies $p(X)=0$ or $=1$. We put also

$$\begin{cases}\mathfrak{C}_{\mathrm{ev}}=\oplus_{j=0}^{\infty}\mathfrak{C}^{[2j]}=\{X\in\mathfrak{C}\mid p(X)=0\},\\ \mathfrak{C}_{\mathrm{od}}=\oplus_{j=0}^{\infty}\mathfrak{C}^{[2j+1]}=\{X\in\mathfrak{C}\mid p(X)=1\},\\ \mathfrak{C}\cong\mathfrak{C}_{\mathrm{ev}}\oplus\mathfrak{C}_{\mathrm{od}}\cong\mathfrak{C}_{\mathrm{ev}}\times\mathfrak{C}_{\mathrm{od}}.\end{cases} \tag{3.19}$$

Moreover, it splits into its even and odd parts, called *(complex) even number* and *(complex) odd number*, respectively :

$$X=X_{\mathrm{ev}}+X_{\mathrm{od}}=\sum_{|a|=\mathrm{even}}X_a\sigma^a+\sum_{|a|=\mathrm{odd}}X_a\sigma^a=\sum_{j=\mathrm{even}}X^{[j]}+\sum_{j=\mathrm{odd}}X^{[j]}. \tag{3.20}$$

Using (3.20), we decompose

$$X=X_{\mathrm{B}}+X_{\mathrm{S}}\qquad\text{where}\quad X_{\mathrm{S}}=\sum_{1\le j<\infty}X^{[j]}\quad\text{and}\quad X_{\mathrm{B}}=X_{\tilde{0}}=X^{[0]} \tag{3.21}$$

and the number X_{B} is called *the body (part)* of X and the remainder X_{S} is called *the soul (part)* of X, respectively. We define the map π_{B} from $\mathfrak{C}$ to $\mathbb{C}$ by $\pi_{\mathrm{B}}(X)=X_{\mathrm{B}}$, called the body projection (or called the augmentation map).

Remark 3.5 (Important) *$\mathfrak{C}$ does not form a field because $X^2=0$ for any $X\in\mathfrak{C}_{\mathrm{od}}$. But, it is easily proved that*

(i) if X satisfies $XY=0$ for any $Y\in\mathfrak{C}_{\mathrm{od}}$, then $X=0$, and

(ii) the decomposition of X with respect to degree in (3.10) *is unique.*

These properties are shared only if the number of Grassmann generators is infinite. For example, if the number of Grassmann generators is finite, say n, then the number $\sigma_1\sigma_2\cdots\sigma_n$, which is not zero, is recognized 0 *for the multiplication of any odd number.*

Lemma 3.2 (the invertible elements) *Let $X\in\mathfrak{C}$ with $X_{\mathrm{B}}\neq 0$. Then there exists a unique element $Y\in\mathfrak{C}$ such that $XY=1=YX$.*

Proof. In fact, decomposing $X=X_{\mathrm{B}}+X_{\mathrm{S}}$ and $Y=Y_{\mathrm{B}}+Y_{\mathrm{S}}$, we should have

$$X_{\mathrm{B}}Y_{\mathrm{B}}=1,\quad X_{\mathrm{B}}Y_{\mathrm{S}}+X_{\mathrm{S}}Y_{\mathrm{B}}+X_{\mathrm{S}}Y_{\mathrm{S}}=0.$$

Therefore, putting $X_{\mathrm{S}}=\sum_{|I|>0}X_I\sigma^I$ and $Y_{\mathrm{S}}=\sum_{|J|>0}Y_J\sigma^J$ and noting that $\sigma^I\sigma^J=(-1)^{\tau(K;I,J)}\sigma^K$ for $K=I+J$, we have

$$Y_{\mathrm{B}}=X_{\mathrm{B}}^{-1},\quad Y_K=-X_{\mathrm{B}}^{-1}\sum_{K=I+J}(-1)^{\tau(K;I,J)}X_IY_J.$$

For example,

$$\text{for } |K|=1, \text{ then } \quad Y_K = -X_{\rm B}^{-1}X_K Y_{\rm B}, \cdots,$$
$$\text{for } |K|=\ell, \text{ then } \quad Y_K = -X_{\rm B}^{-1}\sum_{K=I+J}(-1)^{\tau(K;I,J)}X_I Y_J.$$

If $X_{\rm B}=0$, there exists no Y satisfying $XY=1$ or $YX=1$. □

Now, we define our *(real) supernumber algebra* by

$$\mathfrak{R} = \pi_{\rm B}^{-1}(\mathbb{R})\cap\mathfrak{C} = \left\{ X=\sum_{I\in\mathcal{I}} X_I\sigma^I \;\middle|\; X_{\rm B}\in\mathbb{R} \quad\text{and}\quad X_I\in\mathbb{C} \quad\text{for}\quad |I|\neq 0\right\}. \tag{3.22}$$

Defining as same as before, we have

$$\mathfrak{R} = \mathfrak{R}_{\rm ev}\oplus\mathfrak{R}_{\rm od}, \quad \mathfrak{R} = \oplus_{j=0}^{\infty}\mathfrak{R}^{[j]}. \tag{3.23}$$

Analogous to $\mathfrak{C}$, we put

$$\begin{cases} \mathfrak{R} = \{X\in\mathfrak{C} \mid \pi_{\rm B}X\in\mathbb{R}\}, \qquad \mathfrak{R}^{[j]} = \mathfrak{R}\cap\mathfrak{C}^{[j]}, \\ \mathfrak{R}_{\rm ev} = \mathfrak{R}\cap\mathfrak{C}_{\rm ev}, \quad \mathfrak{R}_{\rm od} = \mathfrak{R}\cap\mathfrak{C}_{\rm od} = \mathfrak{C}_{\rm od}, \\ \mathfrak{R}\cong\mathfrak{R}_{\rm ev}\oplus\mathfrak{R}_{\rm od}\cong\mathfrak{R}_{\rm ev}\times\mathfrak{R}_{\rm od}. \end{cases} \tag{3.24}$$

Here, we introduced the body (projection) map $\pi_{\rm B}$ by $\pi_{\rm B}X = {\rm proj}_{\rm B}(X) = X_{\tilde 0} = X_{\rm B}$.

$\mathfrak{R}^{(j)}$ and other terminologies are analogously introduced.

3.1.6 Conjugation

We define the operation * as follows: Denoting the complex conjugation of X_I by $\overline{X_I}$ and defining $\overline{\sigma^I} = \sigma_n^{i_n}\cdots\sigma_1^{i_1}$ for $I=(i_1,\cdots,i_n)$, we put

$$X^* = \sum_{I\in\mathcal{I}}\overline{X_I}\,\overline{\sigma^I} = \sum_{I\in\mathcal{I}}(-1)^{\frac{|I|(|I|-1)}{2}}\overline{X_I}\sigma^I. \tag{3.25}$$

Then,

Lemma 3.3 *For $X,Y\in\mathfrak{C}$ and $\lambda\in\mathbb{C}$, we have*

$$(X^*)^* = X, \quad (XY)^* = Y^*X^*, \quad (\lambda X)^* = \bar\lambda X^*. \tag{3.26}$$

Exercise 3.2 *Prove $\overline{\sigma^I\sigma^J} = \overline{\sigma^J}\,\overline{\sigma^I}$. (Hint: Use (3.17))*

Remark 3.6 *We may introduce "real" as $X^*=X$ for $X\in\mathfrak{C}$, or from purely aethetical point of view, the set of "reals" may be defined by*

$$\mathfrak{R}^{\mathbb{R}} = \{X=\sum_{I\in\mathcal{I}}X_I\sigma^I \mid X_I\in\mathbb{R}\},$$

but we don't use this "real" in the sequel. Because the analysis is really done for the body part and the soul part is used only for reordering the places, therefore, we imagine that the set

$$\mathcal{R}_K = \left\{ x=\sum_{I\in\mathcal{I}}x_I\sigma^I \;\middle|\; x_{\rm B}\in\mathbb{R} \quad\text{and}\quad x_I\in K\right\}$$

would be more natural as our 'supernumber algebra'. Here, K should be an associative algebra such that we may define seminorms analogously as before. This point of view will be discussed if necessity occurs.

Remark 3.7 *There is another way of defining the conjugation: We define $\bar\sigma_j$ as a linear mapping from to $\mathbb{C}$ such that $\langle\bar\sigma_j,\sigma_k\rangle=\delta_{jk}$, and by this, we may introduce the duality $\langle\cdot,\cdot\rangle$ between $\mathfrak{C}$ and $\bar{\mathfrak{C}}$ which is the Grassmann algebra generated by $\{\bar\sigma_j\}$, and whose Fréchet topology is compatible with the duality above. In this case, putting $\overline{\sigma^I}=\bar\sigma_n^{i_n}\cdots\bar\sigma_1^{i_1}$ for $I=(i_1,\cdots,i_n)$ and*

$$X^*=\sum_{I\in\mathcal{I}}\overline{X_I}\overline{\sigma^I}=\sum_{I\in\mathcal{I}}(-1)^{\frac{|I|(|I|-1)}{2}}\overline{X_I}\bar\sigma^I,$$

we have also (3.26).

3.2 The superspace

Definition 3.1 *The super Euclidean space or (real) superspace $\mathfrak{R}^{m|n}$ of dimension $m|n$ is defined by*

$$\begin{gathered}\mathfrak{R}^{m|n}=\mathfrak{R}^m_{\mathrm{ev}}\times\mathfrak{R}^n_{\mathrm{od}}\ni X={}^t({}^tx,{}^t\theta),\\ \textit{where}\quad x={}^t(x_1,\cdots,x_m)\quad\textit{and}\quad\theta={}^t(\theta_1,\cdots,\theta_n)\quad\textit{with}\quad x_j\in\mathfrak{R}_{\mathrm{ev}},\,\theta_s\in\mathfrak{R}_{\mathrm{od}}.\end{gathered}\tag{3.27}$$

Notation: In the following, we abbreviate the symbol 'transposed' ${}^t(x_1,\cdots,x_m)$ and denote $x=(x_1,\cdots,x_m)$, etc. unless there occurs confusion.

The topology of $\mathfrak{R}^{m|n}$ is induced from the metric defined by $\mathrm{dist}_{m|n}(X,Y)=\mathrm{dist}_{m|n}(X-Y)$ for $X,Y\in\mathfrak{R}^{m|n}$, where we put

$$\mathrm{dist}_{m|n}(X)=\sum_{j=1}^m\bigg(\sum_{I\in\mathcal{I}}\frac{1}{2^{r(I)}}\frac{|\,\mathrm{proj}_I(x_j)|}{1+|\,\mathrm{proj}_I(x_j)|}\bigg)+\sum_{s=1}^n\bigg(\sum_{I\in\mathcal{I}}\frac{1}{2^{r(I)}}\frac{|\,\mathrm{proj}_I(\theta_s)|}{1+|\,\mathrm{proj}_I(\theta_s)|}\bigg).\tag{3.28}$$

Clearly, $\mathrm{dist}_{1|1}(X)=\mathrm{dist}(X)$ for $X\in\mathfrak{R}^{1|1}\cong\mathfrak{R}\subset\mathfrak{C}$. Analogously, the complex superspace of dimension $m|n$ is defined by

$$\mathfrak{C}^{m|n}=\mathfrak{C}^m_{\mathrm{ev}}\times\mathfrak{C}^n_{\mathrm{od}}.\tag{3.29}$$

We generalize the body map $\pi_{\mathrm B}$ from $\mathfrak{R}^{m|n}$ or to $\mathbb{R}^m$ by $\pi_{\mathrm B}X=\pi_{\mathrm B}x=(\pi_{\mathrm B}x_1,\cdots,\pi_{\mathrm B}x_m)\in\mathbb{R}^m$ for $X=(x,\theta)\in\mathfrak{R}^{m|n}$. The (complex) superspace $\mathfrak{C}^{m|n}$ is defined analogously.

Dual superspace. We denote the superspace $\mathfrak{R}^{m|n}$ by $\mathfrak{R}^{m|n}_X$ whose point is presented by $X=(x,\theta)=(x_1,\cdots,x_m,\theta_1,\cdots,\theta_n)$. We prepare another superspace $\mathfrak{R}^{m|n}_\Xi$ whose point is denoted by $\Xi=(\xi,\pi)=(\xi_1,\cdots,\xi_m,\pi_1,\cdots,\pi_n)$, such that they are "dual" each other by

$$\langle X|\Xi\rangle_{m|n}=\sum_{j=1}^m\langle x_j|\xi_j\rangle+\sum_{k=1}^n\langle\theta_k|\pi_k\rangle\in\mathfrak{R}_{\mathrm{ev}}.\tag{3.30}$$

We abbreviate $\langle\cdot|\cdot\rangle_{m|n}$ above by $\langle\cdot|\cdot\rangle$ unless there occurs confusion.

3.3 Banach-Grassmann algebra

Denote by $\mathcal{M}_L$ the set of integer sequences given by

$$\mathcal{M}_L=\{\mu\mid\mu=(\mu_1,\mu_2,\cdots,\mu_k),\,1\le\mu_1<\mu_2<\cdots<\mu_k\le L\}\quad\text{and}\quad\mathcal{M}_\infty=\cup_{L=1}^\infty\mathcal{M}_L.$$

We regard $\emptyset \in \mathcal{M}_L$ and for any $j \in \mathbb{N}$, we put $(j) \in \mathcal{M}_\infty$. For each $r \in \mathbb{N}$, we may correspond a member $\mu \in \mathcal{M}_\infty$ by using

$$r = \frac{1}{2}(2^{\mu_1} + 2^{\mu_2} + \cdots + 2^{\mu_k}). \tag{3.31}$$

Conversely, for each $\mu \in \mathcal{M}_\infty$, we define e_μ as $e_\mu = (\overbrace{0, \cdots, 0, 1}^{r}, 0, \cdots)$ where r and μ are related by (3.31). Then, $w = \sum_\mu w_\mu e_\mu$. Now, we introduce the multiplication by

$$\begin{cases} e_\mu e_\emptyset = e_\emptyset e_\mu = e_\mu & \text{for} \quad \mu \in \mathcal{M}_\infty, \\ e_{(i)} e_{(j)} = -e_{(j)} e_{(i)} & \text{for} \quad i, j \in \mathbb{N}, \\ e_\mu = e_{(\mu_1)} e_{(\mu_2)} \cdots e_{(\mu_k)} & \text{where} \quad \mu = (\mu_1, \mu_2, \cdots, \mu_k). \end{cases} \tag{3.32}$$

That is, we identify

$$\omega \ni w = (w_1, w_2, w_3, w_4, \cdots) = \sum_{j=1} w_j e_{(j)} \longleftrightarrow (w_{(1)}, w_{(2)}, w_{(1,2)}, w_{(3)}, \cdots) = \sum_\mu w_\mu e_\mu$$

where

$$e_{(j)} \leftrightarrow \sigma_j, \; e_{(1)} e_{(2)} = e_{(1,2)} \leftrightarrow \sigma_1 \sigma_2 = \sigma^I, \; I_{(1,2)} = (1, 1, 0, \cdots),$$

$$e_\mu = e_{(\mu_1)} e_{(\mu_2)} \cdots e_{(\mu_k)} \leftrightarrow \sigma_{\mu_1} \sigma_{\mu_2} \cdots \sigma_{\mu_k} = \sigma^I, \; I_\mu = (\underbrace{\overbrace{0, \cdots, 0, 1}^{\mu_1}, 0, \cdots, 0, 1}_{\mu_k}, 0, \cdots).$$

In stead of the sequence space ω, Rogers uses ℓ^1 to construct the real Banach-Grassmann algebra, which is the set of absolutely convergent sequences

$$\|X\| = \sum_{I \in \mathcal{I}} |X_I| < \infty \quad \text{for } X = \textstyle\sum_{I \in \mathcal{I}} X_I \sigma^I \text{ with } X_I \in \mathbb{R}, \text{ such that } \|XY\| \leq \|X\| \|Y\|.$$

Porposition 3.2 (Roger) *ℓ^1 with the above multiplication forms a Banach-Grassmann algebra with countably infinite generators.*

3.3.1 The set of formal power series

We follow the description in p.25 of

F. Treves, "Topological vector spaces, Distributions and Kernels", Academic Press, 1967.

Filter and filter base :

Definition 3.2 *Let E be a set. A filter $\mathcal{F}$ is a family of subsets E, submitted to three conditions:*

(F_1) The empty set $\emptyset$ should not belong to the family $\mathcal{F}$.

(F_2) The intersection of any two sets, belonging to the family, also belongs to the family $\mathcal{F}$.

(F_3) Any set, which contains a set belonging to $\mathcal{F}$, should also belong to $\mathcal{F}$.

Definition 3.3 *Let E be a set. A family $\mathcal{B}$ of subsets E is a basis of a filter $\mathcal{F}$ on E if the following two conditions are satisfied:*

(BF_1) $\mathcal{B} \subset \mathcal{F}$, i.e., any subset which belongs to $\mathcal{B}$ must belong to $\mathcal{F}$.

(BF_2) Every subset of E belonging to $\mathcal{F}$ contains some subset of E which belongs to $\mathcal{B}$.

Claim 3.1 *A family $\mathcal{A}$ of subsets E is a basis of a filter $\mathcal{F}$ on E if it is a family of non-empty subsets of E sastifying*

(BF) The intersection of any two sets, belonging to $\mathcal{A}$, contains a set which belongs to $\mathcal{A}$.

Topology compatible with the linear structure : Let E be a vector space over $\mathbb{C}$. Defining

$$A_v : E \times E \to E, \quad (x, y) \to x + y,$$
$$M_s : \mathbb{C} \times E \to E, \quad (\lambda, x) \to \lambda x,$$

we say that a topology $\mathcal{T}$ of E is compatible with the linear structure of E if A_v and M_s are continuous when we provide E with the topology $\mathcal{T}$, $E \times E$ with the product topology $\mathcal{T} \times \mathcal{T}$, and $\mathbb{C} \times E$ with $\mathcal{C} \times \mathcal{T}$ where $\mathcal{C}$ is the ordinary topology of $\mathbb{C}$.

Definition 3.4 *(i) A subset A of a vector space E is said to be absorbing if to every $x \in E$ there exists a number $c_x > 0$ such that, for all $\lambda \in \mathbb{C}$, $|\lambda| \leq c_x$, we have $\lambda x \in A$.*

(ii) A subset A of a vector space E is said to be balanced if for avery $x \in A$ and every $\lambda \in \mathbb{C}$, $|\lambda| \leq 1$, we have $\lambda x \in A$.

Theorem 3.1 (Theorem 3.1 of Treves) *A filter $\mathcal{F}$ of a vector space E is the filter of neighborhoods of the origin in a topology compatible with the linear structure of E iff it has the following properties:*

(1) The origin belongs to every subst U belonging to $\mathcal{F}$.

(2) To every $U \in \mathcal{F}$, there is $V \in \mathcal{F}$ such that $V + V \subset U$.

(3) For every $U \in \mathcal{F}$ and every $\lambda \in \mathbb{C}$, $\lambda \neq 0$, we have $\lambda U \in \mathcal{F}$

(4) Every $U \in \mathcal{F}$ is absorbing.

(5) Every $U \in \mathcal{F}$ contains some $V \in \mathcal{F}$ which is balanced.

The set of formal power series : We denote by $\mathbb{C}[[X]]$ the ring of formal power series in one variable, X, with complex coefficients:

$$\mathbb{C}[[X]] = \{u = u(X) = \sum_{n=0}^{\infty} u_n X^n \mid u_n \in \mathbb{C}\}.$$

Clearly, u is essentially a sequence of complex numbers $(u_0, u_1, \cdots, u_n, \cdots)$. For another one $v = v(X) = \sum_{n=0}^{\infty} v_n X^n$ and a complex number λ, we define

$$u + v = \sum_{n=0}^{\infty} (u_n + v_n) X^n,$$
$$uv = \sum_{n,p=0}^{\infty} u_n v_p X^{n+p} = \sum_{n=0}^{\infty} \big(\sum_{p=0}^{n} u_{n-p} v_p \big) X^n,$$
$$\lambda u = \sum_{n=0}^{\infty} (\lambda u_n) X^n.$$

By these operations, $\mathbb{C}[[X]]$ forms an algebra, moreover it has the unit $1 = 1 + \sum_{n=1}^{\infty} 0X^n$.

Lemma 3.4 *For a formal power series u to have an inverse, it is necessary and sufficient that its first coefficient, u_0, is different from zero.*

Topolgy in algebra : We introduce also sets $\mathfrak{M}^k$ for $k \geq 0$ as

$$\mathfrak{M}^k = \{u = u(X) = \sum_{n=0}^{\infty} u_n X^n \mid u_n \in \mathbb{C}, \ \ u_p = 0 \text{ if } p < k\}.$$

Then, we have the sequences of sets

$$\mathfrak{M}^0 = \mathbb{C}[[X]] \supset \mathfrak{M}^1 \supset \mathfrak{M}^2 \supset \cdots \supset \mathfrak{M}^n \supset \cdots$$

which is totally ordered for inclusion and satisfies Axiom (BF) for filter bases.

Let $\{\mathcal{M}^n\}_{n=0}^{\infty}$ generate the filter $\mathcal{F}$, i.e.

$$\mathcal{F} = \{U \subset \mathbb{C}[[X]] \mid \exists n \gg 1 \text{ s.t. } \mathcal{M}^n \subset U\}.$$

Let u be an arbitrary formal power series, and let $\mathcal{M}^n + u = \{v + u \mid v \in \mathcal{M}^n\}$. Denoting by $\mathcal{F}(u)$, the filter generated by the basis $\mathcal{M}^n + u (n = 0, 1, 2, \cdots)$, we know that $\mathbb{C}[[X]]$ forms a topological ring, but

Claim 3.2 $\mathbb{C}[[X]]$ *with this topology does not form a topological vector space.*

Topology in analysis : The topology of simple convergence of the coefficients is defined by sets

$$V_{m,n} = \{u = \sum_{p=0}^{\infty} u_p X^p \in \mathbb{C}[[X]] \mid \forall p \leq n, \ \ |u_p| \leq \frac{1}{m}\}.$$

The filter generated by the bases $\{V_{m,n}\}$ satisfies the conditions of Theorem 3.1 above.