Foundation of Super Analysis Lecture 2– 08.10.10 A. Inoue

1 Necessity of the non-commutative analysis and its merit

1.1 Feynman's path-integral representation of the solution for Schrödinger equation

1.2 Non-existence of Feynman measure

1.3 Fujiwara's procedure

Since there doen't exist the so-called Feynman measure which garantees the beautiful path-integral expression, how do we treat the Schrödinger equation?

As the operator

$$\hat{H} = \hat{H}(q, -i\hbar\partial_q) = -\frac{\hbar^2}{2}\Delta + V(q)$$

is essentially self-adjoint on $L^2(\mathbb{R}^m)$ under certain conditions on V, there exists a solution $e^{i\hbar^{-1}t\hat{H}}\underline{u}$ (by Stone's theorem) of the initial value problem

$$i\hbar\frac{\partial u(t,q)}{\partial t} = \hat{H}u(t,q) \quad \text{with} \quad u(0,q) = \underline{u}(q).$$

Moreover, by L. Schwartz's distributional kernel theorem, we have a kernel $E(t,q,q') \in \mathcal{D}'(\mathbb{R}\times\mathbb{R}^m\times\mathbb{R}^m)$ such that

$$\langle e^{i\hbar^{-1}t\hat{H}}\underline{u}, \varphi\rangle = \langle E(t,q,q')\underline{u}(q'), \varphi(q)\rangle = \langle E(t,q,q'), \underline{u}(q')\varphi(q)\rangle = \langle E(t,\cdot,\cdot), \underline{u}\otimes\varphi\rangle.$$

On the other hand, for the heat case $e^{t\hat{H}}\underline{v}$, the distributional kernel $H(t,q,q')$ has the representation by the "classical quantity".

Method of Fujiwara : About 30 years before, there doesn't exist a paper on the construction of a fundamental solution for the initial value problem of Schrödinger equation. He adopts the argument of Feynman modifying mathematically.

(1) For given Lagrangian $L(\gamma,\dot\gamma) = \frac{1}{2}|\dot\gamma|^2 - V(\gamma) \in C^\infty(TM)(M=\mathbb{R}^m)$, by Legendre transform, we have the Hamilton function $H(q,p) = \inf_{\dot q}[\dot q p - L(q,\dot q)] \in C^\infty(T^*M)$.

(2) For the Hamilton function $H(q,p) = \frac{1}{2}|p|^2 + V(q)$, we construct a solution $S(t,q,\underline{q})$ of the Hamilton-Jacobi equation

$$S_t(t,q,\underline{q}) + H(q, S_q(t,q,\underline{q})) = 0 \quad \text{with} \quad \lim_{t\to 0} tS(t,q,\underline{q}) = \frac{1}{2}|q-\underline{q}|^2.$$

(3) For the action function $S(t,q,\underline{q})$ obtained above, the amplitude function defined by

$$D(t,q,\underline{q}) = \det\left(\frac{\partial^2 S(t,q,\underline{q})}{\partial q\partial\underline{q}}\right) \quad \text{(van Vleck determinant)}$$

satisfies the continuity equation

$$D_t(t,q,\underline{q}) + \partial_q(D(t,q,\underline{q})H_p(q,S_q(t,q,\underline{q}))) = 0 \quad \text{with} \quad \lim_{t\to 0} D(t,q,\underline{q}) = 1.$$

(4) Then we define the integral transformation

$$F(t)\underline{u}(q) = (2\pi i\hbar)^{-m/2}\int_{\mathbb{R}^m} d\underline{q}\, D(t,q,\underline{q})^{1/2} e^{i\hbar^{-1}S(t,q,\underline{q})}\underline{u}(\underline{q}). \tag{1.1}$$

Theorem 1.1 (Theorem 2.2 of Fujiwara) *Assume* $\sup_{q\in\mathbb{R}^m} |D^\alpha V(q)| \le C_\alpha\ (|\alpha| \ge 2)$. *Fix* $0 < T < \infty$ *arbitrarily. Put* $\mathbb{H} = L^2(\mathbb{R}^m : \mathbb{C})$, $\mathcal{B}(\mathbb{H})$=*the set of bounded linear operators on* $\mathbb{H}$.

(1) $F(t)$ *defines a bounded linear operator in* $\mathbb{H}$

$$\|F(t)u\| \le C\|u\| \quad \textit{by Cotlar's lemma.}$$

(2) For any $u \in L^2(\mathbb{R}^m : \mathbb{C})$, t, s, $t+s \in [-T,T]$,

$$\begin{gathered}\lim_{t\to 0}\|F(t)u - u\| = 0,\\ i\hbar\,\frac{\partial}{\partial t}(F(t)u)(q)\bigg|_{t=0} = \hat{H}(q,-i\hbar\partial_q)u(q),\\ \|F(t+s) - F(t)F(s)\| \le C(t^2+s^2).\end{gathered}$$

(3) Moreover, there exists a limit $\lim_{k\to\infty}(F(t/k))^k = E(t)$ *in* $\mathcal{B}(\mathbb{H})$, *i.e. in the operator norm of* $L^2(\mathbb{R}^m : \mathbb{C})$, *which satisfies the initial value problem below:*

$$\begin{cases} i\hbar\dfrac{\partial}{\partial t}(E(t)u)(q) = \hat{H}(q,-i\hbar\partial_q)(E(t)u)(q),\\ (E(0)u)(q) = \underline{u}(q).\end{cases}$$

Outline of the proof: In (2), for the construction of a solution of the Hamilton-Jacobi equation, he takes the Jacobi's method.

(a) For the given $H(q,p)$ and the initial data $(\underline{q},\underline{p})$, there exists a unique Hamilton flow $(q(s),p(s)) = (q(s,\underline{q},\underline{p}),p(s,\underline{q},\underline{p}))$.

(b) For the given time interval t which is sufficiently small, and for any given terminal position $\bar{q}$, applying the implicit function theorem to $\bar{q} = q(t,\underline{q},\underline{p})$, we get the unique $\underline{p}$ denoted by $\underline{p} = \xi(t,\underline{q},\bar{q})$.

(c) Using this, we put

$$S(t,\bar{q},\underline{q}) = S_0(t,\underline{q},\underline{p})\big|_{\underline{p}=\xi(t,\underline{q},\bar{q})}.$$

That is, there exists a unique path γ_c in $C_{t,\underline{q},\bar{q}}$ such that

$$\inf_{\gamma\in C_{t,\underline{q},\bar{q}}} S_t(\gamma) = S_t(\gamma_c) = S(t,\underline{q},\bar{q}) \quad \text{with} \quad S_t(\gamma) = \int_0^t d\tau\, L(\gamma(\tau),\dot{\gamma}(\tau)).$$

Moreover, this function $S(t,\bar{q},\underline{q})$ is a solution of the Hamiltonian-Jacobi equation.

(3) is proved from (2) algebraically.

(4) Since we have estimates of $S(t,\bar{q},\underline{q})$ or $D(t,\bar{q},\underline{q})$ w.r.t. $(\bar{q},\underline{q})$, we may prove the L^2- boundedness of the operator (**??**) applying Cotlar's lemma. Since we take $D(t,\bar{q},\underline{q})^{1/2}$ as the amplitude, the operator

(**??**) is considered as acting on the half-density bundle (or the intrinsic Hilbert space) "$L^2(\mathbb{R}^m : \mathbb{C})$". I regard this fact as admitting Copenhagen interpretation.

(5) Though above theorem is sufficient concerning the convergence of parametrix (**??**), but this convergence is not sufficient for the Feynman's expression. Concerning this or the construction of the fundamental solution, there exists another paper [1] by Fujiwara which isn't discussed in this lecture.

Problem 1: In the above theorem, the momentum energy is given by the flat Riemannian metric $\frac{1}{2}|p|^2$ on $\mathbb{R}^m$. Whether this procedure works for the Riemannian metric is studied by physicist[2]. In general, to prove the L^2-boundedness of the pseudo-differential operator with symbol of order 0, we formulate it in flat space. Therefore, it is an open problem to associate a quantum mechanics for given Riemann metric $g_{ij}(q)$ on $\mathbb{R}^m$ following Feynman's procedure.

On the other hand, above procedure of Fujiwara was used also by Inoue-Maeda[3] to explain mathematically the origin of the term $(1/12)R$, $R =$the scalar curvature of the configuration manifold, which appeared when one wants to "quantize" the Lagrangian on a curved manifold.

Problem 2: Feynman or Fujiwara used Lagrangian function. How do we connect the above procedure directly to the Hamiltonian without using Lagrangian?

Problem 3: How do we proceed when
(1) V has singularities like Coulomb potentials? and
(2) there exists many paths connecting points q and q' like the dynamics on the circle[4]?
(3) V is not order 2, for example $V(q) = q^4$?

1.4 Feynman's murmur

2 Dirac and Weyl equations

2.1 The origin of Dirac and Weyl equations

Why and how does P. Dirac introduce, now so-called, Dirac equation? We follow the description of

K. Nishijima:*Relativistic Quantum Mechanics (in Japanese)* Baifu-kan, 1973.

Energy E and momentum p of the free particle with mass m satisfy the Einstein relation[5]

$$E^2 = c^2|p|^2 + c^4m^2.$$

Following the canonical quantization procedure of substitution

$$p_j \longrightarrow \frac{\hbar}{i}\frac{\partial}{\partial q_j}, \quad E \longrightarrow i\hbar\frac{\partial}{\partial t}$$

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[1] D. Fujiwara, *Remarks on convergence of the Feynman path integrals,*Duke Math. J.47(1980), pp. 559-600.

[2] For example, B. DeWitt, *Dynamical theory in curved spaces I. A review of the classical and quantum action principles*, Reviews of modern physics 29(1984), pp. 377-397.

[3] A. Inoue and Y. Maeda, *On integral transformations associated with a certain Lagrangian– as a prototype of quantization*, J.Math.Soc.Japan 37(1985), pp. 219-244.

[4] L. Schulman, *A path integral for spin*, Physical Review 176(1968), pp. 1558-1142.

[5] For $p = 0$, this gives the theoretical foundation of the possibility of atomic bomb!

as we did to get the Schrödinger equation, we have the Klein-Gordon equation

$$\hbar^2 \frac{\partial^2}{\partial t^2} u - c^2\hbar^2 \Delta u + c^4 m^2 u = 0.$$

Unfortunately, the solution u of this equation does not permit the Copenhagen interpretation, that is, the qunatity $\rho = |u|^2$ is not interpreted as the probability density.

In order to have the equation which stems from Einstein relation and admits probabilistic interpretation, we need to have

$$\big(i\hbar \frac{\partial}{\partial t} - \hat{H}\big)\psi = 0$$

which satisfies

$$\big(\hbar^2 \frac{\partial^2}{\partial t^2} + \hat{H}^2\big)\psi = 0.$$

Assuming that this equation coincide with Klein-Gordon equation, we need that "the symbol corresponding to the operator $\hat{H}$" should satisfy

$$H^2 = c^2|p|^2 + c^4 m^2.$$

Supposing that the state vector ψ which satisfies the desired equation has multicomponents, then we may have the option such that

$$H = \sum_{j=1}^{3} \boldsymbol{\alpha}_j\, p_j + mc^2 \boldsymbol{\beta}.$$

Here, above appeared letters $\{\boldsymbol{\alpha}_j, \boldsymbol{\beta}\}$ satisfy

$$\boldsymbol{\alpha}_j\, \boldsymbol{\alpha}_k + \boldsymbol{\alpha}_k\, \boldsymbol{\alpha}_j = 2\delta_{jk}\mathbb{I}, \quad \boldsymbol{\alpha}_j\, \boldsymbol{\beta} + \boldsymbol{\beta}\, \boldsymbol{\alpha}_j = 0, \quad \boldsymbol{\beta}^2 = \mathbb{I} \tag{2.1}$$

Dirac gave an example of 4×4 matrices satisfying the relation (**??**), which is no called Dirac matrices:

$$\boldsymbol{\alpha}_j = \begin{pmatrix} 0 & \boldsymbol{\sigma}_j \\ \boldsymbol{\sigma}_j & 0 \end{pmatrix} = \boldsymbol{\sigma}_1 \otimes \boldsymbol{\sigma}_j, \quad \boldsymbol{\beta} = \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{pmatrix} = \boldsymbol{\sigma}_3 \otimes \mathbb{I}_2.$$

Here, Pauli matrices $\{\boldsymbol{\sigma}_j\}_{j=1}^3$ are gievn by

$$\boldsymbol{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.2}$$

[Report problem 2-1]: There are many representations satisfying (**??**), named Majonara rep, chiral rep, etc. Seek such representations as many as possible and check the relationship between them. By the way of checking these, study also the Lorentz invariant.

For a given external electro-magnetic field, the initial value problem for the Dirac equation is as follows: Find $\psi(t,q) : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}^4$, for the given initial data $\underline{\psi}(q) \in C_0^\infty(\mathbb{R}^3 : \mathbb{C}^4)$, satisfying

$$\begin{cases} i\hbar \dfrac{\partial}{\partial t}\psi(t,q) = \mathbb{H}(t)\psi(t,q), \\ \psi(\underline{t},q) = \underline{\psi}(q) \end{cases} \tag{2.3}$$

Here,

$$\mathbb{H}(t) = c \sum_{k=1}^{3} \boldsymbol{\alpha}_k \Big(\frac{\hbar}{i} \frac{\partial}{\partial q_k} - \frac{e}{c} A_k(t,q) \Big) + mc^2 \boldsymbol{\beta} + eA_0(t,q). \tag{2.4}$$

As had seemingly been an audience of Dirac's talk, Weyl proposed 2 × 2-matrix representation for the relation (**??**) in stead of 4 × 4-one when the mass $m = 0$. From this, he derived the initial value problem of the free Weyl equation: For $\psi(t,q) = \begin{pmatrix} \psi_1(t,q) \\ \psi_2(t,q) \end{pmatrix}$,

$$\begin{cases} i\hbar \dfrac{\partial}{\partial t}\psi(t,q) = \mathbb{H}\psi(t,q), \quad \mathbb{H} = -ic\hbar \displaystyle\sum_{j=1}^{3} \boldsymbol{\sigma}_j \frac{\partial}{\partial q_j}, \\ \psi(0,q) = \underline{\psi}(q). \end{cases} \tag{2.5}$$

In spite of the beauty of this equation, since the parity is not preserved by this one, it is rejected by physics society until Lee-Yang's theory and Wu's experiment exhibits in weak interaction, shows that the parity is not necessarily preserved.

Since Neutrino has been considered as the particle with mass 0, Weyl equation is believed to be the governing equation of Neutrino untill by the recent experiment of Kamiokande which suggests that Neutrino has non-zero mass.

[Report Problem 2-2]: Search "Weyl equation" in internet to check whether the usage of this equation in condensed matter physics, etc. Report what you appreciate interesting.

Problem: Haw may we represent the solution of (**??**) using the "classical quantities"?

Ordinary procedure . Though the equation (**??**) is a system but with constant coefficients, applying Fourier transform, we may have the solution rather by algebraic operation. In fact, defining Fourier transform as

$$\hat{u}(p) = (2\pi\hbar)^{-m/2} \int_{\mathbb{R}^m} dq\, e^{-i\hbar^{-1}qp} u(q), \quad u(q) = (2\pi\hbar)^{-m/2} \int_{\mathbb{R}^m} dp\, e^{i\hbar^{-1}qp} \hat{u}(p),$$

and applying this to $q \in \mathbb{R}^3$ of (**??**), we get

$$i\hbar \frac{\partial}{\partial t}\hat{\psi}(t,p) = \hat{\mathbb{H}}\hat{\psi}(t,p) \tag{2.6}$$

Here,

$$\hat{\mathbb{H}} = c\sum_{j=1}^{3} \boldsymbol{\sigma}_j p_j = c\begin{pmatrix} p_3 & p_1 - ip_2 \\ p_1 + ip_2 & -p_3 \end{pmatrix} \quad \text{and} \quad \hat{\mathbb{H}}^2 = c^2|p|^2\mathbb{I}_2.$$

From this, we have

Porposition 2.1 *For any $t \in \mathbb{R}$ and $\underline{\psi} \in L^2(\mathbb{R}^3 : \mathbb{C}^2)$, we have*

$$e^{-i\hbar^{-1}t\mathbb{H}}\underline{\psi}(q) = (2\pi\hbar)^{-3/2} \int_{\mathbb{R}^3} dp\, e^{i\hbar^{-1}qp} e^{-i\hbar^{-1}t\hat{\mathbb{H}}}\hat{\underline{\psi}}(p). \tag{2.7}$$

If $\underline{\psi} \in \mathcal{S}(\mathbb{R}^3 : \mathbb{C}^2)$, then

$$\mathbb{E}(t,q) = q(2\pi\hbar)^{-3} \int_{\mathbb{R}^3} dp\, e^{i\hbar^{-1}qp} \big[\cos(c\hbar^{-1}t|p|)\mathbb{I}_2 - i\frac{\sin(c\hbar^{-1}t|p|)}{c|p|}\hat{\mathbb{H}}\big] \in \mathcal{S}'(\mathbb{R}^3 : \mathbb{C}^2) \tag{2.8}$$

and

$$e^{-i\hbar^{-1}t\mathbb{H}}\underline{\psi}(q) = \mathbb{E} * \underline{\psi}(t,q) = \int_{\mathbb{R}^3} dq'\, \mathbb{E}(t,q-q')\underline{\psi}(q'), \tag{2.9}$$

Remark: Pauli said one day that "There exists no classical counter-part corresponding to quantum spinning particle", so I saw somewhere but I can't remember where exactly. Therefore, such saying didn't exist? Please give a look to the splendid book written in Japanese, (title is translated)

S. Tomonaga, *Spin rotates – Quantum Mechanics in the matured time*, Chuokoron Publ. 1974,

In any way, it seems difficult to imagine the classical mechanics corresponding to the equation (**??**) from the formula (**??**). This is the one reason why I denote Feynman's murmur as Feynman's problem.

Claim 2.1 *In spite of above, I claim that I may construct the classical mechanics corresponding to* (**??**)*, which gives a path-integral representation of it!*

2.2 The method of characteristics and Hamiltonian path-integral representation

Here, I want to give a device to resolve Feynman's problem called "Hamiltonian path-integral representation", by exhibiting a simple example.

The point is that, though Schrödinger equation has 2-times partial derivativesbut there exists only 1-time partial derivatives w.r.t. the space variables in Dirac or Weyl equations, this is the very reason why we need Hamiltonian path-integral representation.

We may solve the following equation readily:

$$\begin{cases} i\hbar\dfrac{\partial}{\partial t}u(t,q) = a\dfrac{\hbar}{i}\dfrac{\partial}{\partial q}u(t,q) + bqu(t,q), \\ u(0,q) = \underline{u}(q). \end{cases} \tag{2.10}$$

From the right-hand side of above, we get a Hamiltonian function

$$H(q,p) = e^{-i\hbar^{-1}qp}\Big(a\frac{\hbar}{i}\frac{\partial}{\partial q} + bq\Big)e^{i\hbar^{-1}qp}\Big|_{\hbar=0} = ap + bq,$$

then, the corresponding classical orbit is obtained easily from the Hamilton equation

$$\begin{cases} \dot{q}(t) = H_p = a, \\ \dot{p}(t) = -H_q = -b \end{cases} \quad \text{with} \quad \begin{pmatrix} q(0) \\ p(0) \end{pmatrix} = \begin{pmatrix} \underline{q} \\ \underline{p} \end{pmatrix} \tag{2.11}$$

such as

$$q(s) = \underline{q} + as, \quad p(s) = \underline{p} - bs. \tag{2.12}$$

Using these, by applying the method of characteristics, we get

$$U(t,\underline{q}) = \underline{u}(\underline{q})e^{-i\hbar^{-1}(b\overline{q}t+2^{-1}abt^2)}.$$

Using the inverse function $\underline{q} = y(t,\bar{q}) = \bar{q} - at$ of $\overline{q} = q(t,\underline{q})$, the solution of (**??**) is given as

$$u(t,\overline{q}) = U(t,\underline{q})\big|_{\underline{q}=y(t,\bar{q})} = \underline{u}(\overline{q} - at)e^{-i\hbar^{-1}(b\overline{q}t-2^{-1}abt^2)}.$$

Remark: In the above method, the information from $p(t)$ is not used.

[Report problem 2-3]: Study the method of characteristics for the first order PDE. Since from the information obtained from ODE(such as (non-linear) Hamilton equation), we get a solution of PDE(such as (linear)

Louville equation), this is the core of the method of characteristics. What is the linear Liouville equation corresponding to the non-linear field equation, for example, the Hopf equation represented by functional derivatives is the Liouville equation corresponding to the Navier-Stokes equation.

Another point of view from **Hamiltonian path-integral method**: Put

$$S_0(t,\underline{q},\underline{p}) = \int_0^t ds\,[\dot{q}(s)p(s) - H(q(s),p(s))] = -b\underline{q}t - 2^{-1}abt^2,$$

$$S(t,\overline{q},\underline{p}) = \Big(\underline{q}\,\underline{p} + S_0(t,\underline{q},\underline{p})\Big)\Big|_{\underline{q}=y(t,\overline{q})} = \overline{q}\underline{p} - a\underline{p}t - b\overline{q}t + 2^{-1}abt^2.$$

Then, the classical action $S(t,\bar{q},\underline{p})$ satisfies the Hamilton-Jacobi equation.

$$\begin{cases} \dfrac{\partial}{\partial t}S + H(\bar{q},\partial_{\bar{q}}S) = 0, \\ S(0,\bar{q},\underline{p}) = \bar{q}\,\underline{p}. \end{cases}$$

On the other hand, the van Vleck determinant (though scalar in this case) is calculated as

$$D(t,\bar{q},\underline{p}) = \frac{\partial^2 S(t,\overline{q},\underline{p})}{\partial\overline{q}\partial\underline{p}} = 1.$$

This quantity satisfies the continuity equation:

$$\begin{cases} \dfrac{\partial}{\partial t}D + \dfrac{1}{2}\partial_{\bar{q}}(DH_p) = 0 \quad \text{where } H_p = \dfrac{\partial H}{\partial p}\big(\bar{q},\dfrac{\partial S}{\partial\bar{q}}\big), \\ D(0,\bar{q},\underline{p}) = 1. \end{cases}$$

As an interpretation of Feynman's idea, we regard that **the transition from classical to quantum** is to study the following quantity or the one represented by this (the term "quantization" is not so well-defined mathematically):

$$u(t,\overline{q}) = (2\pi\hbar)^{-1/2}\int_{\mathbb{R}} d\underline{p}\,D^{1/2}(t,\overline{q},\underline{p})e^{i\hbar^{-1}S(t,\overline{q},\underline{p})}\underline{\hat{u}}(\underline{p}).$$

That is, in our case at hand, we should study the quantity defined by

$$\begin{aligned} u(t,\overline{q}) &= (2\pi\hbar)^{-1/2}\int_{\mathbb{R}} d\underline{p}\,e^{i\hbar^{-1}S(t,\overline{q},\underline{p})}\underline{\hat{u}}(\underline{p}) \\ &= (2\pi\hbar)^{-1}\iint_{\mathbb{R}^2} d\underline{p}d\underline{q}\,e^{i\hbar^{-1}(S(t,\overline{q},\underline{p})-\underline{q}\,\underline{p})}\underline{u}(\underline{q})\big(=\underline{u}(\overline{q}-at)e^{i\hbar^{-1}(-b\overline{q}t+2^{-1}abt^2)}\big). \end{aligned}$$

Therefore, we may say that this second construction gives the explicit connection between the solution (**??**) and the classical mechanics given by (**??**). We feel the above expression "good" because there appear two classical quantities S and D explicitly.

Claim: Applying superanalysis, we may extend the second argument above to a system of PDOs e.g. quantum mechanical equations with spin such as Dirac, Weyl or Pauli equations, (and if possible, any other system of PDOs), after interpreting these equations as those on superspaces.

2.3 Decomposition of 2×2 matrix by Clifford algebra

では、一般の縦ベクトルに 2×2 行列を施す演算の別解釈ができないものか？上の複素数で述べた事は、「事柄」には色々の見方があるということで、それを「一般化」できないのか？

Guided by the following

Theorem 2.1 (C. Chevalley) *Any Clifford algebra has the representation on Grassmann algebra,*

we decompose a 2×2-matrix as follows:

(I) For any 2×2 matrix, we have

$$\begin{aligned}\begin{pmatrix} a & c \\ d & b\end{pmatrix} &= \frac{a+b}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \frac{a-b}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + \frac{c+d}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + \frac{c-d}{2}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\\ &= \frac{a+b}{2}\mathbb{I}_2 + \frac{a-b}{2}\boldsymbol{\sigma}_3 + \frac{c+d}{2}\boldsymbol{\sigma}_1 + i\frac{c-d}{2}\boldsymbol{\sigma}_2.\end{aligned}$$

Here, $\{\boldsymbol{\sigma}_j\}$ satisfies not only (**??**) but also the following relation where (j,k,ℓ) is a even permutation of $(1,2,3)$,

$$\boldsymbol{\sigma}_j\boldsymbol{\sigma}_k = i\boldsymbol{\sigma}_\ell.$$

This decomposition stands for that a set of all 2×2 matrices is spanned by Pauli matrices $\{\boldsymbol{\sigma}_k\}$ having Clifford structure.

(II-1) Now, preparing a letter θ satisfying $\theta^2=0$, we identify Pauli matrices with differential operators acting on Grassmann algebra $\Lambda=\{u(\theta)=u_0+u_1\theta \mid u_0,\ u_1\in\mathbb{C}\}$.

For

$$u_0+u_1\theta \sim \begin{pmatrix}u_0\\ u_1\end{pmatrix},$$

define the action as

$$\theta u(\theta)=u_0\theta\sim\begin{pmatrix}0\\ u_0\end{pmatrix}=\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\begin{pmatrix}u_0\\ u_1\end{pmatrix},\quad \frac{\partial}{\partial\theta}u(\theta)=u_1\sim\begin{pmatrix}u_1\\ 0\end{pmatrix}=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}u_0\\ u_1\end{pmatrix}.$$

Then, we have

$$\begin{aligned}(\theta+\frac{\partial}{\partial\theta})u(\theta)&=u_0\theta+u_1\sim\begin{pmatrix}u_1\\ u_0\end{pmatrix}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}u_0\\ u_1\end{pmatrix},\\ (\theta-\frac{\partial}{\partial\theta})u(\theta)&=u_0\theta-u_1\sim\begin{pmatrix}-u_1\\ u_0\end{pmatrix}=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}u_0\\ u_1\end{pmatrix},\\ (1-2\theta\frac{\partial}{\partial\theta})u(\theta)&=u_0-u_1\theta\sim\begin{pmatrix}u_0\\ -u_1\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}u_0\\ u_1\end{pmatrix}.\end{aligned}$$

This means that Pauli matrices are represented as differential operators acting on Λ.
But such a representation is not unique !

(II-2) Here is another representation: Preparing 2 letters $\theta_1,\ \theta_2$ satisfying $\theta_i\theta_j+\theta_j\theta_i=0$, we put

$$\Lambda_{\text{ev}}=\{u=u_0+u_1\theta_1\theta_2\mid u_0,u_1\in\mathbb{C}\},\quad \Lambda_{\text{od}}=\{v=v_1\theta_1+v_2\theta_2\mid v_1,v_2\in\mathbb{C}\},$$

and define differential operators acting on Λ_{ev} as

$$\begin{aligned}\sigma_1(\theta,\partial_\theta)&=\left(\theta_1\theta_2-\frac{\partial^2}{\partial\theta_1\partial\theta_2}\right)u(\theta)=u_0\theta_1\theta_2+u_1\sim\begin{pmatrix}u_1\\ u_0\end{pmatrix}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}u_0\\ u_1\end{pmatrix},\\ -i\sigma_2(\theta,\partial_\theta)&=\left(\theta_1\theta_2+\frac{\partial^2}{\partial\theta_1\partial\theta_2}\right)u(\theta)=u_0\theta_1\theta_2-u_1\sim\begin{pmatrix}-u_1\\ u_0\end{pmatrix}=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}u_0\\ u_1\end{pmatrix},\\ \sigma_3(\theta,\partial_\theta)&=\left(1-\theta_1\frac{\partial}{\partial\theta_1}-\theta_2\frac{\partial}{\partial\theta_2}\right)u(\theta)=u_0-u_1\theta_1\theta_2\sim\begin{pmatrix}u_0\\ -u_1\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}u_0\\ u_1\end{pmatrix}.\end{aligned}$$

Remark 2.1 *Above defined differential operators $\sigma_j(\theta,\partial_\theta)$ annihilate Λ_{od}. Moreover, the symbols corresponding to them are "even". This evenness is crucial to derive Hamilton flow corresponding to Weyl or Dirac equations.*