

1 Necessity of the non-commutative analysis and its merit

1.1 Feynman’s path-integral representation of the solution for Schrödinger equation

More than sixty years ago, as a graduate student, R. Feynman has a primitive question why Schrödinger equation may be considered as the governing equation of quantum mechanics¹? In other word, though Bohr’s correspondence principle which is derived after many experiments and thoughts, should be essential in Quantum mechanics, but it seems difficult to derive that principle from the Schrödinger equation itself.

Mathematically, this question is interpreted as follows: How does the solution $u(t, q) : \mathbb{R} \times \mathbb{R}^m \rightarrow \mathbb{C}$ of the initial value problem for the Schrödinger equation

$$\begin{cases} i\hbar \frac{\partial}{\partial t} u(t, q) = -\frac{\hbar^2}{2} \Delta u(t, q) + V(q)u(t, q), \\ u(0, q) = \underline{u}(q). \end{cases} \quad (1.1)$$

depend on \hbar ? Especially, can we deduce the Bohr’s correspondence principle from this?

On the other hand, about fourty years before when I was a student, main research subjects of linear PDE are the existence, uniqueness and regularity of solutions for the given equation². Essential ingredients of these subjects is almost exhausted³ and one of the recent problems is to represent the solution as explicit as possible by using known objects⁴. From this point of view, to make clear the dependence on Planck’s constant \hbar for the solution of Schrödinger equation and to explain mathematically the appearance of Bohr’s correspondence principle is a good starting problem.

Therefore, we begin with retracing the heuristic procedure taken in Feynman’s doctor thesis⁵ where he introduced his path-integral representation.

For the right-hand side of (1.1), we define the Hamiltonian operator on $C_0^\infty(\mathbb{R}^m)$ as

$$\hat{H} = -\frac{\hbar^2}{2} \Delta + V(\cdot) = \hat{H}_0 + V, \quad \hat{H}_0 = \Delta = \sum_{j=1}^m \frac{\partial^2}{\partial q_j^2}.$$

If above \hat{H} is essentially self-adjoint in $L^2(\mathbb{R}^m)$, applying Stone’s theorem, solution of (1.1) is written by

$$u(t, q) = (e^{-i\hbar^{-1}t\hat{H}} \underline{u})(q).$$

¹R. Feynman, *Space-time approach to non-relativistic quantum mechanics*, Rev. Modern Phys. 20 (1948) pp. 367-387.

²Before advent of functional analytic approach to PDE, it is too hard to obtain a solution for a generally given PDE.

³L. Hörmander, “The Analysis of Linear Partial Differential Operators, I–IV” Springer, 1983-85

⁴R. Beals, *Exact fundamental solutions*, Journées Équations aux dérivées partielles, Saint-Jean-de-Monts, 2-5 juin 1998.

⁵See also, S.A. Albeverio and R.J. Hoegh-Krohn, *Mathematical Theory of Feynman Integrals*, Lec.Notes in Math. 523, Heidelberg-New York, Springer-Verlag, 1976.

Or generalizing a little, when and how the exponential function of a given operator A

$$e^{tA} = \sum_{k=0}^{\infty} \frac{(tA)^k}{k!} \quad (t \in \mathbb{R}^+, \text{ or } t \in i\mathbb{R})$$

is well-defined? Guiding this problem, Hille-Yosida theory of semigroups is established.

[Report problem 1-1]: Check what is the Stone's theorem. If the Hilbert space is finite-dimensional, what is the corresponding theorem in elementary linear algebra? It is also preferable to check what is the theory of Hille-Yosida.

On the other hand, Lie-Trotter-Kato's product formula says that if $\hat{H} = \hat{H}_0 + V$, $e^{-i\hbar^{-1}t\hat{H}}$ is given by

$$e^{-i\hbar^{-1}t\hat{H}} = \text{s-lim}_{k \rightarrow \infty} \left(e^{-i\hbar^{-1}\frac{t}{k}V} e^{-i\hbar^{-1}\frac{t}{k}\hat{H}_0} \right)^k \quad \text{even if } [\hat{H}_0, V] \neq 0.$$

Remark 1.1 (i) In the above, if $[\hat{H}_0, V] = 0$, then $(\hat{H}_0 + V)^k = \sum_{j=0}^k \binom{k}{j} \hat{H}_0^j V^{k-j}$ and we have $e^{s(\hat{H}_0+V)} = e^{s\hat{H}_0} e^{sV}$. That means, it isn't necessary to take the limit $\text{s-lim}_{k \rightarrow \infty}$.

(ii) There doesn't exist the difference between strong and weak convergence in finite-dimensional vector spaces. Check the difference between the convergence of operators in "strong" or "uniform" sense in infinite-dimensional Banach space,

If the initial data \underline{u} belongs to $\mathcal{S}(\mathbb{R}^m)$ (=a space of Schwartz' rapidly decreasing functions), where

$$\begin{aligned} \mathcal{S}(\mathbb{R}^m) &= \{u \in C^\infty(\mathbb{R}^m : \mathbb{C}) \mid p_{k,\mathcal{S}}(u) < \infty \quad \forall k \in \mathbb{N}\} \\ \text{with } p_{k,\mathcal{S}}(u) &= \sup_{q \in \mathbb{R}^m, \ell + |\beta| \leq k} \langle q \rangle^\ell |\partial_q^\beta u(q)|, \quad \langle q \rangle = (1 + |q|^2)^{1/2}, \end{aligned}$$

since we know

$$(e^{-i\hbar^{-1}t\hat{H}_0}\underline{u})(\bar{q}) = (2\pi i\hbar t)^{-m/2} \int_{\mathbb{R}^m} d\underline{q} e^{i\hbar^{-1}(\bar{q}-\underline{q})^2/(2t)} \underline{u}(\underline{q}),$$

we have

$$\begin{aligned} (e^{-i\hbar^{-1}t\hat{H}}\underline{u})(\bar{q}) &\sim (e^{-i\hbar^{-1}tV}(e^{-i\hbar^{-1}t\hat{H}_0}\underline{u}))(\bar{q}) \\ &\sim (2\pi i\hbar t)^{-m/2} e^{-i\hbar^{-1}tV(\bar{q})} \int_{\mathbb{R}^m} d\underline{q} e^{i\hbar^{-1}(\bar{q}-\underline{q})^2/(2t)} \underline{u}(\underline{q}). \end{aligned}$$

Therefore, we get

$$\begin{aligned} (e^{-i\hbar^{-1}s\hat{H}}(e^{-i\hbar^{-1}t\hat{H}}\underline{u}))(\bar{q}) &\sim (2\pi i\hbar s)^{-m/2} e^{-i\hbar^{-1}sV(\bar{q})} \int_{\mathbb{R}^m} dq^{(1)} e^{i\hbar^{-1}(\bar{q}-q^{(1)})^2/(2s)} (e^{-i\hbar^{-1}t\hat{H}}\underline{u})(q^{(1)}) \\ &\sim (2\pi i\hbar)^{-m} (ts)^{-m/2} e^{-i\hbar^{-1}sV(\bar{q})} \int_{\mathbb{R}^m} dq^{(1)} e^{i\hbar^{-1}(\bar{q}-q^{(1)})^2/(2s)} \\ &\quad \times \left[e^{-i\hbar^{-1}tV(q^{(1)})} \int_{\mathbb{R}^m} d\underline{q} e^{i\hbar^{-1}(q^{(1)}-\underline{q})^2/(2t)} \underline{u}(\underline{q}) \right] \\ &= (2\pi i\hbar)^{-m} (ts)^{-m/2} \int_{\mathbb{R}^m} d\underline{q} \\ &\quad \times \left[\int_{\mathbb{R}^m} dq^{(1)} e^{-i\hbar^{-1}(sV(\bar{q})+tV(q^{(1)}))} e^{i\hbar^{-1}(\bar{q}-q^{(1)})^2/(2s)+i\hbar^{-1}(q^{(1)}-\underline{q})^2/(2t)} \right] \underline{u}(\underline{q}) \end{aligned}$$

Putting $t = s$ in the above, we have

$$\begin{aligned} &-i\hbar^{-1}t(V(\bar{q}) + V(q^{(1)})) + i\hbar^{-1}[(\bar{q} - q^{(1)})^2 + (q^{(1)} - \underline{q})^2]/(2t) \\ &= i\hbar^{-1}t \left[\frac{1}{2} \left(\frac{\bar{q} - q^{(1)}}{t} \right)^2 - V(\bar{q}) + \frac{1}{2} \left(\frac{q^{(1)} - \underline{q}}{t} \right)^2 - V(q^{(1)}) \right] \end{aligned}$$

Repeating this procedure k -times and denoting $q^{(k)} = \bar{q}$, $q^{(0)} = \underline{q}$, we get

$$\left(e^{-i\hbar^{-1} \frac{t}{k} V} e^{-i\hbar^{-1} \frac{t}{k} \hat{H}_0} \right)^k \underline{u}(\underline{q}) \sim \int d\underline{q} F_k(t, \bar{q}, q^{(1)}, \dots, q^{(1)}, \underline{q}) \underline{u}(\bar{q}).$$

Here, we put

$$F_k(t, \bar{q}, q^{(1)}, \dots, q^{(1)}, \underline{q}) = (2\pi i\hbar(t/k))^{-km/2} \int \dots \int dq^{(1)} \dots dq^{(k-1)} e^{i\hbar^{-1} S_t(\bar{q}, q^{(k-1)}, \dots, q^{(1)}, \underline{q})},$$

$$S_t(q^{(k)}, \dots, q^{(0)}) = \sum_{j=1}^k \left[\frac{1}{2} \left(\frac{q^{(j)} - q^{(j-1)}}{t/k} \right)^2 - V(q^{(j)}) \right] \frac{t}{k}.$$

Making $k \rightarrow \infty$ formally, we have

$$F(t, \bar{q}, \underline{q}) = \text{s-lim}_{k \rightarrow \infty} (2\pi i\hbar(t/k))^{-km/2} \int \dots \int dq^{(1)} \dots dq^{(k-1)} e^{i\hbar^{-1} S_t(\bar{q}, q^{(k-1)}, \dots, q^{(1)}, \underline{q})}. \quad (1.2)$$

and

$$(e^{-i\hbar^{-1} t \hat{H}} \underline{u})(\bar{q}) = \int d\underline{q} F(t, \bar{q}, \underline{q}) \underline{u}(\underline{q}).$$

Denoting $q^{(k)} = \bar{q}$, $q^{(0)} = \underline{q}$ and putting

$$S_t(q^{(k)}, \dots, q^{(0)}) = \sum_{j=1}^k \left[\frac{1}{2} \left(\frac{q^{(j)} - q^{(j-1)}}{t/k} \right)^2 - V(q^{(j)}) \right] \frac{t}{k}$$

and

$$F(t, \bar{q}, \underline{q}) = \text{s-lim}_{k \rightarrow \infty} (2\pi i\hbar t)^{-km/2} \int \dots \int dq^{(1)} \dots dq^{(k-1)} e^{i\hbar^{-1} S_t(\bar{q}, q^{(k-1)}, \dots, q^{(1)}, \underline{q})}, \quad (1.3)$$

we have

$$(e^{-i\hbar^{-1} t \hat{H}} \underline{u})(\bar{q}) \sim \int d\underline{q} F(t, \bar{q}, \underline{q}) \underline{u}(\underline{q}).$$

[Report Problem 1-2]: Show that the function space $\mathcal{S}(\mathbb{R}^m)$ forms a Fréchet space.

Feynman's interpretation: The set of "paths" is denoted by

$$C_{t, \bar{q}, \underline{q}} = \{\gamma(\cdot) \in AC([0, t] : \mathbb{R}^m) \mid \gamma(0) = \underline{q}, \gamma(t) = \bar{q}\},$$

$$C_{t, loop} = \{\phi(\cdot) \in AC([0, t] : \mathbb{R}^m) \mid \phi(0) = \phi(t)\},$$

where AC stands for absolute continuity. In this case, for any $\gamma \in C_{t, \bar{q}, \underline{q}}$, we have

$$C_{t, \bar{q}, \underline{q}} = \gamma + C_{t, loop}.$$

For example, take as γ the straight line combining \underline{q} and \bar{q} such that $\gamma = \gamma(s) = (1-s)\underline{q} + s\bar{q}$. By connecting two paths, we may define the sum operation in $C_{t, loop}$ which makes it linear space.

We get a Lagrange function $L(\gamma, \dot{\gamma})$ from a Hamilton function $H(q, p)$ by Legendre transform;

$$L(\gamma, \dot{\gamma}) = \frac{1}{2} \dot{\gamma}^2 - V(\gamma) \in C^\infty(T\mathbb{R}^m).$$

For any path $\gamma \in C_{t, \bar{q}, \underline{q}}$, regarding $S_t(q^{(k)}, \dots, q^{(0)})$ as a Riemann sum of an action function $S_t(\gamma)$, we get

$$S_t(\gamma) = \int_0^t L(\gamma(\tau), \dot{\gamma}(\tau)) d\tau = \lim_{k \rightarrow \infty} S_t(q^{(k)}, \dots, q^{(0)}).$$

Making $k \rightarrow \infty$, we "construct" a limit of measures $dq^{(1)} \dots dq^{(k-1)}$

$$d_F \gamma = \prod_{0 < \tau < t} d\gamma(\tau)$$

which is regarded as “the measure” on the path space $C_{t,\bar{q},\underline{q}}$:

$$F(t, \bar{q}, \underline{q}) = \int_{C_{t,\bar{q},\underline{q}}} d_F \gamma e^{i\hbar^{-1} \int_0^t L(\gamma(\tau), \dot{\gamma}(\tau)) d\tau}.$$

Then, if we could apply the stationary phase method to this representation when $\hbar \rightarrow 0$, we got the main term which is obtained from the classical path γ_c , i.e.

$$\delta \int_0^t L(\gamma(\tau), \dot{\gamma}(\tau)) d\tau = \frac{d}{d\epsilon} \int_0^t L((\gamma_c + \epsilon\phi)(\tau), (\dot{\gamma}_c + \epsilon\dot{\phi})(\tau)) d\tau \Big|_{\epsilon=0} = 0 \quad \forall \phi \in C_{t,loop}$$

In this sense, Bohr’s correspondence principle is now derived! The obstruction of this beautiful expression is the claim “There doesn’t exist a non trivial Lebesgue-like measure on any infinite-dimensional barreled locally convex vector space”⁶.

[Report Problem 1-3 (Campbell-Hausdorff’s formula and its application)] :

- (1) Search “Campbell Hausdorff” in Google and check what it is.
- (2) Apply that formula to e^{tX} where

$$X = \begin{pmatrix} 0 & \mu & 1 & 0 \\ -\mu & 0 & 0 & 1 \\ \Omega^2 - \mu^2 & 0 & 0 & \mu \\ 0 & \Omega^2 - \mu^2 & -\mu & 0 \end{pmatrix}$$

and get the concrete expression. Don’t use the diagonalization procedure but apply Campbell-Hausdorff formula to the suitable decomposition of X .

- (3) Search also “Lie-Trotter-Kato formula”.

[Report Problem 1-4]: What is the meaning of AC function, what property it shares?

1.2 Non-existence of Feynman measure

To “feel” the reason why there doesn’t exist Lebesgue-like measure (called Feynman measure), we give a simple theorem due to Kuo⁷. Since that theorem is formulated in Hilbert space and the path space $C_{t,loop}$ is not Hilbert one, those who don’t satisfy this explanation, consult the paper by Smolyanov-Fomin⁸.

For the sake of those who forget terminology, we recall the following:

Definition 1.1 (Complete σ -algebra) For a given space X , a subset \mathcal{B} of all subsets \mathcal{P}^X satisfying

- $\emptyset \in \mathcal{B}$,
- $A \in \mathcal{B} \implies A^c = X \setminus A \in \mathcal{B}$,
- $A_n \in \mathcal{B} \implies \sum_{n=1}^{\infty} A_n \in \mathcal{B}$

is called complete σ -algebra.

Definition 1.2 (measure) A set function μ defined on a complete σ -algebra \mathcal{B} of a space X is called a measure if it satisfies

⁶Though to construct Lebesgue’s integration theory, we are taught to prepare measure theory but is it truely necessary to do so? For example, Berezin integral below works without measure.

⁷H.H. Kuo, Gaussian Measures in Banach Spaces, Lecture Notes in Mathematics 463, Heidelberg-New York, Springer-Verlag, 1975.

⁸O.G. Smolyanov and S.V. Fomin, *Measures on linear topological spaces*, Russian Math.Surveys 31(1976), pp. 1-53.

- $0 \leq \mu(A) \leq \infty, \quad \mu(\emptyset) = 0,$
- $A_n \in \mathcal{B}, A_j \cap A_k = \emptyset (j \neq k) \implies \mu(\sum_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n).$

Definition 1.3 (Borel-algebra) A family \mathcal{B} of sets of a topological space X is called a Borel-algebra and denoted $\mathcal{B} = \mathcal{B}(X)$ if it satisfies

- $A \in \mathcal{B} \implies A^c = X \setminus A \in \mathcal{B},$
- $A_n \in \mathcal{B} \implies \sum_{n=1}^{\infty} A_n \in \mathcal{B},$
- $\mathcal{O}(X) \subset \mathcal{B}$

and is the minimum in \mathcal{P}^X for the ordering by the set inclusion.

Definition 1.4 A Borel measure⁹ satisfying below is called Lebesgue-like:

- (1) For any bounded Borel set, its measure is not only finite, but also positive if a set is not empty.
- (2) That measure is translation invariant.

Theorem 1.1 There exists no non-trivial Lebesgue-like Borel measure on a infinite dimensional separable Hilbert space.

Proof. Since H is separable, there exists a countable orthonormal base $\{e_1, e_2, \dots\}$ ¹⁰.

Assume that there exists a non-trivial Lebesgue-like Borel measure μ on $\mathcal{B}(H)$. Define open sets as

$$B_n = \{u \in H \mid \|u - e_n\| < \frac{1}{2}\} \quad \text{and} \quad B = \{u \in H \mid \|u\| < 2\},$$

then they satisfy

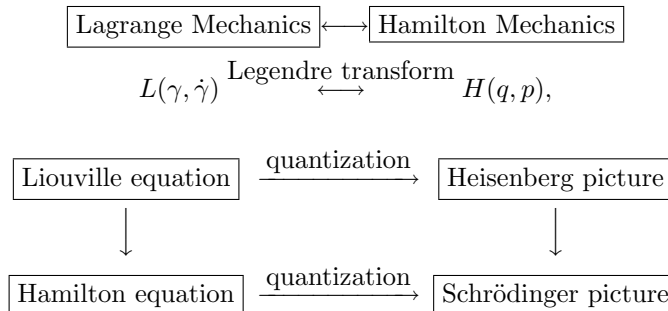
$$B_n \cap B_m = \emptyset \quad \text{and} \quad \cup_{n=1}^{\infty} B_n \subset B.$$

Since the measure is Lebesgue-like, we have

$$0 < \mu(B_1) = \mu(B_2) = \dots < \infty, \quad \infty = \sum_{n=1}^{\infty} \mu(B_n) \leq \mu(B) < \infty. \quad \text{Contradiction!} \quad \square$$

1.3 Resume of known procedures

Assuming a certain convexity to apply Legendre transform, we have



⁹measure defined on Borel algebra

¹⁰Hilbert-Schmidt's procedure of orthogonalization holds for countable number of bases. What occurs if bases has continuous cardinality? By the way, check whether there exist non-separable Hilbert space. Check also the basis problem in general Banach space.

Classical Mechanics

Hamilton equation $\begin{cases} \dot{q} = H_p(q, p), \\ \dot{p} = -H_q(q, p), \end{cases}$ with $\begin{pmatrix} q(0) \\ p(0) \end{pmatrix} = \begin{pmatrix} \underline{q} \\ \underline{p} \end{pmatrix},$

i.e. $\frac{d}{dt} \begin{pmatrix} q \\ p \end{pmatrix} = \mathbb{J} \begin{pmatrix} H_q \\ H_p \end{pmatrix}$ with $\mathbb{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$

Liouville equation $\dot{\phi} = \{\phi, H\} = \sum_{j=1}^m \left(\frac{\partial \phi}{\partial q_j} \frac{\partial H}{\partial p_j} - \frac{\partial \phi}{\partial p_j} \frac{\partial H}{\partial q_j} \right)$ with $\phi(0, q, p) = \underline{\phi}(q, p).$

Quantum Mechanics

“ $L(\gamma, \dot{\gamma})$ or $H(q, p) \rightarrow \hat{H} = \hat{H}(q, -i\hbar\partial_q)$ ”

(S) A description of the time change of the state vector $u(t)$:

Schrödinger picture $i\hbar \frac{\partial u(t)}{\partial t} = \hat{H}u(t)$ with $u(0) = \underline{u},$

i.e. $u(t) = e^{-i\hbar^{-1}t\hat{H}}\underline{u}.$

(H) A description of the time change of the kinetic operator $\hat{F}(t)$:

Heisenberg picture $i\hbar \frac{d}{dt} \hat{F}(t) = [\hat{F}(t), \hat{H}]$ with $\hat{F}(0) = \underline{\hat{F}}.$

(F) Path Integral method, clarifying Bohr’s correspondence principle:

Feynman picture $u(t, q) = \int d\underline{q} E(t, 0, q, \underline{q})u(\underline{q})$

with

$$E(t, 0, q, \underline{q}) = \int_{C_{t,q,\underline{q}}} \exp(i\hbar^{-1} \int_0^t L(\gamma(s), \dot{\gamma}(s)) ds) d_F \gamma(\cdot)$$

$$C_{t,q,\underline{q}} = \{\gamma \in C([0, t] : \mathbb{R}^d) \mid \gamma(0) = \underline{q}, \gamma(t) = q\}$$

$$E(t, 0, q, \underline{q}) \sim D(t, 0, q, \underline{q})^{1/2} e^{i\hbar^{-1}S(t,0,q,\underline{q})}$$

Problem 1: Give a meaning to the symbolic representation

$$\int d_F \gamma e^{i\hbar^{-1} \int_0^t L(\gamma(\tau), \dot{\gamma}(\tau)) d\tau}$$

for wider class of Lagrangian $L.$

(0) Concerning this question, D. Fujiwara¹¹ gives a rigorous meaning when the potential V satisfies $|\partial_x^\alpha V(x)| \leq C_\alpha (|\alpha| \geq 2).$

(i) For the Coulomb potential $V(q) = 1/|q|,$ i.e. hydrogen atom, because of the singularity, we have not yet established the analogous result as Fujiwara.

(a) I propose to calculate this by replacing $1/|q|$ with $1/(|q|^2 + \epsilon)^{1/2}$ for any $\epsilon > 0$ and finally making $\epsilon \rightarrow 0,$ or

(b) Use the fact that Schrödinger equation with 3-dimensional Coulomb potential is obtained from 4-dimensional harmonic oscillator¹².

¹¹D. Fujiwara, *A construction of the fundamental solution for the Schrödinger equation,* J. D’Analyse Math. 35 (1979), pp. 41-96.

¹²See, for example, N.E. Hurt, *Geometric Quantization in Action,* Reidel Pub.Co., 1983.

(ii) At least in dimension 1, the essential selfadjointness of $-\Delta + |q|^4$ is proved by many methods. But we might not apply the procedure used by Fujiwara to construct a parametrix using classical quantities¹³.
 (iii) When $|\partial_q^\alpha V(q)| \leq C_\alpha$ ($|\alpha| \geq 2$), the above constructed parametrix converges in uniform operator norm. On the other hand, Lie-Trotter-Kato product formula assures only for the strong convergence. How can one express the reason for this difference? In case of using polygonal line approximation for classical path to the harmonic oscillator, we get the strong but non-uniform convergence of parametrices. One possibility may be to use non-standard analysis to check why there exists the difference of the convergence.

[Report Problem 1-5]: What is the meaning of essential adjointness? Check Reed-Simon vol I!

Problem 2: Fujiwara formulated his procedure in Lagrangian manner, or without using Fourier transform. Does there exist the Hamiltonian object^{14 15} corresponding to this parametrix?:

$$\iint d_F x d_F \xi e^{i\hbar^{-1} \int_0^t H(x(\tau), \xi(\tau)) d\tau} ?$$

1.4 Feynman's murmur

In p. 355 of their book¹⁶ Feynman wrote as follows (Underlined by atlon):

... path integrals suffer grievously from a serious defect. They do not permit a discussion of spin operators or other such operators in a simple and lucid way. They find their greatest use in systems for which coordinates and their conjugate momenta are adequate. Nevertheless, spin is a simple and vital part of real quantum-mechanical systems. It is a serious limitation that the half-integral spin of the electron does not find a simple and ready representation. It can be handled if the amplitudes and quantities are considered as quaternions instead of ordinary complex numbers, but the lack of commutativity of such numbers is a serious complication.

Main Problem : How do we treat this murmur as a mathematical problem?

Though for a given Schrödinger equation, we may associate a corresponding classical mechanics, but how do we define the classical mechanics corresponding to Dirac or Weyl equations? In other words, since Schrödinger equations are obtained from Lagrangian or Hamiltonian function by quantization, can we define a Hamiltonian function from which we get Dirac equation after quantization?

Our answer is “yes, it is possible not only using superspace formulation” but also re-interpreting the method of characteristics by Hamilton flow and Fourier transform.

¹³But, see, S. Albeverio and S. Mazzucchi, *Feynman path integrals for polynomially growing potentials*, J. Functional Analysis 221(2005), 83-121.

¹⁴A. Intissar: *A Remark on the convergence of Feynman path integrals for Weyl pseudo-differential operators on \mathbb{R}^n* , Commun. in Partial Differential Equations 7 (1982) pp. 1403-1437.

¹⁵A. Inoue: *On a “Hamiltonian path-integral” derivation of the Schrödinger equation*, Osaka J. Math. 36(1999), pp. 111-150.

¹⁶R. Feynman and A.R. Hibbs, *Quantum Mechanics and Path Integrals*, New York, McGraw-Hill Book Co. 1965.