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WHAT IS SUPERANALYSIS?

AN EXAMPLE OF NON-COMMUTATIVE ANALYSIS

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- 1 MOTIVATIONS FOR “NEW VARIABLES”
- 2 A NEW LOOK AT MATRIX STRUCTURE
- 3 SCHRÖDINGER EQUATION AS “REAL”
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 - Harmonic oscillator and its super-extension
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MOTIVATIONS:

Why we need new “field” besides \mathbb{R} or \mathbb{C} .

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- **New treatise of matrix structures in PDE**

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- New treatise of matrix structures in PDE
- New “slowness variables?” in RMT

FEYNMAN'S PROBLEM

How to describe the dependence of \hbar for the solution of the Schrödinger equation?

$$u(t, q) = \int_{\mathbb{R}^m} E(t, q, q') \underline{u}(q') dq'$$

$$E(t, q, q') \sim \int_{C_{t,q,q'}} \exp\left(i\hbar^{-1} \int_0^t L(q(s), \dot{q}(s)) ds\right) d_F q(\cdot)$$

How about the Dirac equation?

WITTEN'S LAPLACIAN

(M, g) , d , d^* and $\phi : M \rightarrow \mathbb{R}$

$$d_\lambda = e^{-\lambda\phi} d e^{\lambda\phi}, \quad d_\lambda^* = e^{\lambda\phi} d^* e^{-\lambda\phi},$$

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Witten's Laplacian:

$$\begin{aligned} H_\lambda &= d_\lambda d_\lambda^* + d_\lambda^* d_\lambda \\ &= dd^* + d^*d + \lambda^2 (d\phi)^2 + \sum_{i,j=1}^d \lambda \frac{D^2\phi}{Dq^i Dq^j} [a^{i*}, a^j]_- \end{aligned}$$

a^j and a^{j*} annihilation and creation operators

WITTEN'S LAPLACIAN 2

H_λ is obtained after quantization from the Lagrangian formulated action:

$$\begin{aligned} \mathcal{S}_\lambda = & \frac{1}{2} \int dt \left[g_{ij} \left(\frac{dq^i}{dt} \frac{dq^j}{dt} + i\bar{\psi}^i \frac{D\psi^j}{Dt} \right) \right. \\ & + \frac{1}{4} R_{ijkl} \bar{\psi}^i \psi^k \bar{\psi}^j \psi^l - \lambda^2 g^{ij} \frac{\partial \phi}{\partial q^i} \frac{\partial \phi}{\partial q^j} \\ & \left. - \lambda \frac{D^2 \phi}{Dq^i Dq^j} \bar{\psi}^i \psi^j \right]. \end{aligned}$$

Here, ψ^i and $\bar{\psi}^i$ are anti-commuting fields tangent to M , which becomes the creation and annihilation.

MATRICES AS DIFFERENTIAL OP'S:

THEOREM

Any Clifford algebra has a representation in Grassmann algebra.

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$$\begin{pmatrix} a & c \\ d & b \end{pmatrix} = \frac{a+b}{2}\mathbb{I}_2 + \frac{a-b}{2}\sigma_3 + \frac{c+d}{2}\sigma_1 + \frac{c-d}{2}i\sigma_2,$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_j\sigma_k + \sigma_k\sigma_j = 2\delta_{jk}\mathbb{I}_2 \quad (\text{Clifford relation})$$

MATRICES AS DIFFERENTIAL OP'S1:

Prepare 1-forms dy_1, dy_2 and wedge and inner products: Define

$$dy_1 \wedge (u_0 + u_1 dy_1 \wedge dy_2) = u_0 dy_1 \quad (\text{multiplication})$$

$$\frac{\partial}{\partial(dy_1)} dy_1 \sim \frac{\partial}{\partial y_1} \lrcorner dy_1 = 1 \quad (\text{1-differentiation})$$

$$\frac{\partial}{\partial y_1} \left(\frac{\partial}{\partial y_2} \lrcorner (dy_1 \wedge dy_2) \right) = -1 \quad (\text{2-differentiation})$$

MATRICES AS DIFFERENTIAL OP'S2:

Identifying $\theta_1 = dy_1, \theta_2 = dy_2$, we prepare Grassmann algebra as $\Gamma_0 = \{u_0 + u_1\theta_1\theta_2\}$. Define

$$\left(\theta_1\theta_2 - \frac{\partial^2}{\partial\theta_1\partial\theta_2}\right)(u_0 + u_1\theta_1\theta_2) = u_0\theta_1\theta_2 + u_1 \\ \sim \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix},$$

That is, we may identify

$$\theta_1\theta_2 - \frac{\partial^2}{\partial\theta_1\partial\theta_2} \sim \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

MATRICES AS DIFFERENTIAL OP'S3:

Analogously

$$\left(1 - \theta_1 \frac{\partial}{\partial \theta_1} - \theta_2 \frac{\partial}{\partial \theta_2}\right)(u_0 + u_1 \theta_1 \theta_2) = u_0 - u_1 \theta_1 \theta_2,$$

that is,

$$1 - \theta_1 \frac{\partial}{\partial \theta_1} - \theta_2 \frac{\partial}{\partial \theta_2} \sim \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

These operators annihilate $\Gamma_1 = \{v_1 \theta_1 + v_2 \theta_2\}$.

MATRICES AS DIFFERENTIAL OP'S4:

These representations are not unique!

For $\Gamma = \{w = w_0 + w_1\theta\}$, we have

$$\left(1 - 2\theta \frac{\partial}{\partial \theta}\right)(w_0 + w_1\theta) = w_0 - w_1\theta,$$

$$\left(\theta + \frac{\partial}{\partial \theta}\right)(w_0 + w_1\theta) = w_1 + w_0\theta,$$

$$\left(\theta - \frac{\partial}{\partial \theta}\right)(w_0 + w_1\theta) = -w_1 + w_0\theta.$$

AS A REAL SYSTEM?

Consider

$$i\hbar \frac{\partial}{\partial t} u(t, q) = H(q, D_q) u(t, q), \quad H(q, D_q) = D_q^2 + V(q),$$

with

$$V(q) \in C^\infty(\mathbb{R} : \mathbb{R}).$$

For $u = u_0 + iu_1$ with $u_j = u_j(t, q) \in \mathbb{R}$,

$$\hbar \frac{\partial}{\partial t} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = H(q, D_q) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}.$$

AS A REAL SYSTEM? (CONTINUED)

For $u(t, x, \theta) = u_0(t, x) + u_1(t, x)\theta_1\theta_2$, we rewrite

$$i\hbar \frac{\partial}{\partial t} u = -iH(x, D_x)(\theta_1\theta_2 + \frac{\partial^2}{\partial\theta_1\partial\theta_2})u$$

with symbol(or Hamiltonian function)

$$H(x, \xi) = \xi^2 + V(x),$$

$$\mathcal{H}(x, \xi, \theta, \pi) = -iH(x, \xi)(\theta_1\theta_2 - \hbar^{-2}\pi_1\pi_2),$$

FOUNDATION OF SUPERANALYSIS:

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- **Fourier transformation.**

FRÉCHET-GRASSMANN ALGEBRA:

$\{\sigma_j\}_{j=1}^{\infty} : \sigma_j\sigma_k + \sigma_k\sigma_j = 0$ (Grassmann generators),

$$\mathfrak{C} = \left\{ X = \sum_{l \in \mathcal{I}} X_l \sigma^l \mid X_l \in \mathbb{C} \right\}, \quad \mathfrak{R} = \{ X \in \mathfrak{C} \mid X_{\tilde{0}} \in \mathbb{R} \}.$$

$$\mathcal{I} = \left\{ l = (i_1, i_2, \dots) \in \{0, 1\}^{\mathbb{N}} \mid |l| = \sum_{j=1}^{\infty} i_j < \infty \right\},$$

$$\tilde{0} = (0, 0, \dots), \quad \sigma^l = \sigma_1^{i_1} \sigma_2^{i_2} \dots$$

THEOREM

\mathfrak{C} forms a Fréchet-Grassmann algebra.

SUPERSPACES:

$$\pi_B X = X_B = X_{\tilde{0}}, \quad X = X_B + X_S$$

$$\mathfrak{R}^{[\ell]} = \left\{ X = \sum_{|I|=\ell} X_I \sigma^I \right\}, \quad \mathfrak{R} = \bigcup_{\ell=0}^{\infty} \mathfrak{R}^{[\ell]}$$

$$\mathfrak{R}_{\text{ev}} = \bigcup_{\ell=0}^{\infty} \mathfrak{R}^{[2\ell]}, \quad \mathfrak{R}_{\text{od}} = \bigcup_{\ell=0}^{\infty} \mathfrak{R}^{[2\ell+1]}$$

$$\mathfrak{R}^{m|n} = \{(x, \theta)\}$$

$$x = (x_1, \dots, x_m), \quad x_j \in \mathfrak{R}_{\text{ev}}, \quad \theta = (\theta_1, \dots, \theta_n), \quad \theta_k \in \mathfrak{R}_{\text{od}}$$

SUPERSMOOTH FUNCTIONS:

For $x = x_B + x_S$ with $q = x_B$, and f ,

$$f(q) \rightarrow \tilde{f}(x) = \sum_{|\alpha|=0}^{\infty} \frac{\partial_q^\alpha f(x_B)}{\alpha!} x_S^\alpha,$$

Supersmooth function : $u(x, \theta) = \sum_{|a| \leq n} \tilde{u}_a(x) \theta^a$

$$a = (a_1, \dots, a_n) \in \{0, 1\}^n, \quad \theta^a = \theta_1^{a_1} \dots \theta_n^{a_n}$$

$\tilde{u}_a(x)$ is the Grassmann continuation of $u_a(q)$.

DIFFERENTIATION:

$$\partial_{x_j} \tilde{u}(x) = \widetilde{\partial_{q_j} u}(x), \quad \partial_x^\alpha u(x) = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_m}^{\alpha_m}$$

$$\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{N}^m, \quad |\alpha| = \sum_{j=1}^m \alpha_j$$

$$\partial_{\theta_k} \theta^a = (-1)^{\ell(a,k)} \theta_1^{a_1} \cdots \theta_{k-1}^{a_{k-1}} \theta_{k+1}^{a_{k+1}} \cdots \theta_n^{a_n}$$

$$\partial_{\theta_k} \theta_j = \delta_{jk}, \quad \theta_j^0 = 1, \quad \theta_j^{-1} = 0,$$

$$\ell(a, k) = \sum_{j=1}^{k-1} a_j, \quad |a| = \sum_{j=1}^n a_j.$$

INTEGRATION:

Even case: $\int_{\mathbb{R}^m} dq f(q) = \int_{\mathfrak{R}^{m|0}} dx \tilde{f}(x)$

Odd case: $v(\theta) = \sum_{|a|=0}^n v_a \theta^a \implies \int d\theta v(\theta) = v_{\tilde{\mathbf{1}}}$

$$v_{\tilde{\mathbf{1}}} = \left. \frac{\partial^n}{\partial \theta_n \cdots \partial \theta_1} v(\theta) \right|_{\theta=0}, \quad \tilde{\mathbf{1}} = (1, \dots, 1)$$

$$\int f(\lambda x) dx = \frac{1}{\lambda} \int f(y) dy, \quad \int d\theta v(\mu\theta) = \mu \int d\omega v(\omega).$$

INTEGRATION (GAUSSIAN TYPE) :

A : $N \times N$ -positive definite matrix,

$$\text{Even case: } \int_{\mathbb{C}^{2N}} d[\bar{z}, z] e^{-\bar{z} \cdot Az} = \frac{1}{\det A},$$

$$d[\bar{z}, z] = (-2\pi i)^{-N} d\bar{z}_1 dz_1 \cdots d\bar{z}_N dz_N,$$

$$z = {}^t(z_1, \dots, z_N) \in \mathbb{C}^N, \quad \bar{z} = (\bar{z}_1, \dots, \bar{z}_N) \in \mathbb{C}^N.$$

$$\text{Odd case: } \int_{\mathfrak{R}_\theta^{0|N} \times \mathfrak{R}_{\bar{\theta}}^{0|N}} d[\bar{\theta}, \theta] e^{-\bar{\theta} \cdot A\theta} = \det A,$$

$$d[\bar{\theta}, \theta] = d\bar{\theta}_N d\theta_N \cdots d\bar{\theta}_1 d\theta_1,$$

$$\theta = (\theta_1, \dots, \theta_N) \in \mathfrak{R}_\theta^{0|N}, \quad \bar{\theta} = (\bar{\theta}_1, \dots, \bar{\theta}_N) \in \mathfrak{R}_{\bar{\theta}}^{0|N}.$$

FOURIER TRANSFORMATION:

$$(F_e v)(\xi) = (2\pi\hbar)^{-m/2} \int_{\mathfrak{R}^{m|0}} dx e^{-i\hbar^{-1}\langle x|\xi\rangle} v(x),$$

$$(\bar{F}_e w)(x) = (2\pi\hbar)^{-m/2} \int_{\mathfrak{R}^{m|0}} d\xi e^{i\hbar^{-1}\langle x|\xi\rangle} w(\xi),$$

$$(F_o v)(\pi) = \mathfrak{k}^{n/2} \iota_n \int_{\mathfrak{R}^{0|n}} d\theta e^{-i\mathfrak{k}^{-1}\langle \theta|\pi\rangle} v(\theta),$$

$$(\bar{F}_o w)(\theta) = \mathfrak{k}^{n/2} \iota_n \int_{\mathfrak{R}^{0|n}} d\pi e^{i\mathfrak{k}^{-1}\langle \theta|\pi\rangle} w(\pi).$$

$$\langle x|\xi\rangle = \sum_{j=1}^m x_j \xi_j, \quad \langle \theta|\pi\rangle = \sum_{k=1}^n \theta_k \pi_k, \quad \iota_n = e^{-\frac{\pi i}{4} n(n-2)}.$$

SEMI-CIRCLE LAW IN RMT:

RMT=Random matrix theory,

$$\mathfrak{U}_N = \{H = (H_{jk}) : \text{Hermitian } N \times N \text{ matrices}\} \sim \mathbb{R}^{N^2}$$

$$d\mu_N(H) = \prod_{k=1}^N d(\Re H_{kk}) \prod_{j < k}^N d(\Re H_{jk}) d(\Im H_{jk}) P_{N,J}(H)$$

$$P_{N,J}(H) = Z_{N,J}^{-1} \exp \left[-\frac{N}{2J^2} H^* H \right]$$

$$E_\alpha = E_\alpha(H) \in \mathbb{R} \quad (\alpha = 1, \dots, N) : \text{e.v. of } H \in \mathfrak{U}_N$$

SEMI-CIRCLE LAW:

$$\rho_N(\lambda) = \rho_N(\lambda; H) = N^{-1} \sum_{\alpha=1}^N \delta(\lambda - E_{\alpha}(H)),$$

$$\langle f \rangle_N = \langle f(\cdot) \rangle_N = \int_{\mathfrak{A}_N} d\mu_N(H) f(H),$$

THEOREM (WIGNER)

$$\begin{aligned} \lim_{N \rightarrow \infty} \langle \rho_N(\lambda) \rangle_N &= w_{sc}(\lambda) \\ &= \begin{cases} (2\pi J^2)^{-1} \sqrt{4J^2 - \lambda^2} & |\lambda| < 2J, \\ 0 & |\lambda| > 2J. \end{cases} \end{aligned}$$

EFETOV'S REPRESENTATION0:

$$\langle \rho_N(\lambda) \rangle_N = \pi^{-1} \Im \int_{\Omega} dQ \left(\{(\lambda - i0)\mathbb{I}_2 - Q\}^{-1} \right)_{bb} \\ \times \exp[-N\mathcal{L}(Q)],$$

$$\Omega = \left\{ Q = \begin{pmatrix} x_1 & \rho_1 \\ \rho_2 & ix_2 \end{pmatrix} \mid x_1, x_2 \in \mathfrak{R}_{\text{ev}}, \rho_1, \rho_2 \in \mathfrak{R}_{\text{od}} \right\}$$

$$\cong \mathfrak{R}^{2|2}, \quad dQ = \frac{dx_1 dx_2}{2\pi} d\rho_1 d\rho_2,$$

$$\mathcal{L}(Q) = \text{str} \left[(2J^2)^{-1} Q^2 + \log((\lambda - i0)\mathbb{I}_2 - Q) \right],$$

EFETOV'S REPRESENTATION 00:

$$\begin{aligned} & \left(((\lambda - i0)I_2 - Q)^{-1} \right)_{bb} \\ &= \frac{(\lambda - i0 - x_1)(\lambda - i0 - ix_2) + \rho_1\rho_2}{(\lambda - i0 - x_1)^2(\lambda - i0 - ix_2)}. \end{aligned}$$

- This formula is justified only for $\lambda - i\epsilon$, but not $\lambda - i0$, mathematically.

EFETOV'S REPRESENTATION 00:

$$\begin{aligned} & \left(((\lambda - i0)I_2 - Q)^{-1} \right)_{bb} \\ &= \frac{(\lambda - i0 - x_1)(\lambda - i0 - ix_2) + \rho_1\rho_2}{(\lambda - i0 - x_1)^2(\lambda - i0 - ix_2)}. \end{aligned}$$

- 1 This formula is justified only for $\lambda - i\epsilon$, but not $\lambda - i0$, mathematically.
- 2 Perhaps, we need more delicate integration theory than Lebesgue's one, which admits making $\epsilon \rightarrow 0$ under integral sign and also permits saddle point analysis.

EFETOV'S REPRESENTATION-1:

$$\begin{aligned}\delta(q) &= \frac{1}{\pi} \lim_{\epsilon \rightarrow 0} \Im \frac{1}{q - i\epsilon} = \frac{1}{2\pi i} \lim_{\epsilon \rightarrow 0} \left[\frac{1}{q - i\epsilon} - \frac{1}{q + i\epsilon} \right] \\ &= \frac{1}{\pi} \lim_{\epsilon \rightarrow 0} \frac{\epsilon}{q^2 + \epsilon^2} \quad \text{in } \mathcal{D}'(\mathbb{R}),\end{aligned}$$

$$g(\lambda - i\epsilon, N, H) = \frac{1}{\pi N} \Im \sum_{\alpha=1}^N \frac{1}{\lambda - i\epsilon - E_{\alpha}(H)}.$$

EFETOV'S REPRESENTATION-2:

$$\begin{aligned} & \int_{\mathfrak{U}_N} d\mu_N(H) \langle \phi(\cdot), \frac{1}{N} \sum_{\alpha=1}^N \delta(\cdot - E_\alpha(H)) \rangle \\ & \stackrel{\text{def}}{=} \int_{\mathfrak{U}_N} d\mu_N(H) \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}} d\lambda \phi(\lambda) g(\lambda - i\epsilon, N, H) \\ & = \lim_{\epsilon \rightarrow 0} \int_{\mathfrak{U}_N} d\mu_N(H) \int_{\mathbb{R}} d\lambda \phi(\lambda) g(\lambda - i\epsilon, N, H) \\ & = \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}} d\lambda \phi(\lambda) G(\lambda - i\epsilon, N). \end{aligned}$$
$$G(\lambda - i\epsilon, N) = \int_{\mathfrak{U}_N} d\mu_N(H) g(\lambda - i\epsilon, N, H).$$

EFETOV'S

REPRESENTATION-NOTATION:

$$z_j = x_j + iy_j, \bar{z}_j = x_j - iy_j, x_j, y_j \in \mathfrak{R}_{\text{ev}},$$

$$\theta_k, \bar{\theta}_k \in \mathfrak{R}_{\text{od}} = \mathfrak{C}_{\text{od}},$$

$$X = {}^t(z, \theta), z = {}^t(z_1, \dots, z_N), \theta = {}^t(\theta_1, \dots, \theta_N),$$

$$X^* = (z^*, \theta^*), z^* = (\bar{z}_1, \dots, \bar{z}_N), \theta^* = (\bar{\theta}_1, \dots, \bar{\theta}_N).$$

Here, θ_k and $\bar{\theta}_k$ are considered as two different odd variables.

EFETOV'S REPRESENTATION-3:

LEMMA

Put $\mu = \lambda - i\epsilon$ ($\epsilon > 0$).

$$\begin{aligned} \frac{1}{\mu I_N - H} &= \sum_{\alpha=1}^N \frac{1}{\mu - E_{\alpha}(H)} \\ &= i \int_{N|2N} \prod_{j=1}^N \frac{d\bar{z}_j dz_j}{2\pi i} \prod_{k=1}^N d\bar{\theta}_k d\theta_k \\ &\quad \times (z^* \cdot z) \exp[-iX^*(I_2 \otimes (\mu I_N - H))X]. \end{aligned}$$

EFETOV'S REPRESENTATION-3BIS1:

LEMMA

For $\mu = \lambda - i\epsilon$ ($\epsilon > 0$),

$$\begin{aligned} \left\langle \frac{1}{\mu I_N - H} \right\rangle_N &= i \int \prod_{j=1}^N \frac{d\bar{z}_j dz_j}{2\pi i} \prod_{k=1}^N d\bar{\theta}_k d\theta_k \\ &\quad \times (z^* \cdot z) \exp[-iX^*(I_2 \otimes \mu I_N)X] \\ &\quad \times \exp\left[-\frac{J^2}{2N} \sum_{j,k=1}^N (\bar{z}_j z_k + \bar{\theta}_j \theta_k)\right. \\ &\quad \left. \times (\bar{z}_k z_j + \bar{\theta}_k \theta_j)\right]. \end{aligned}$$

EFETOV'S REPRESENTATION-3BIS2:

Since $X^*(I_2 \otimes H)X = H_{jk}(\bar{z}_j z_k + \bar{\theta}_j \theta_k)$, we get

$$\begin{aligned} & \langle \exp [\pm i \sum_{j,k=1}^N H_{jk}(\bar{z}_j z_k + \bar{\theta}_j \theta_k)] \rangle_N \\ &= \exp \left[-\frac{J^2}{2N} \sum_{j,k=1}^N (\bar{z}_j z_k + \bar{\theta}_j \theta_k)(\bar{z}_k z_j + \bar{\theta}_k \theta_j) \right]. \end{aligned}$$

HUBBARD-STRATONOVICH FORMULA:

LEMMA

Let A be any even 2×2 supermatrix. Let

$$\Omega = \left\{ Q = \begin{pmatrix} x_1 & \rho_1 \\ \rho_2 & ix_2 \end{pmatrix} \mid x_1, x_2 \in \mathfrak{R}_{\text{ev}}, \rho_1, \rho_2 \in \mathfrak{R}_{\text{od}} \right\}$$
$$\cong \mathfrak{R}^{2|2}, \quad dQ = \frac{dx_1 dx_2}{2\pi} d\rho_1 d\rho_2.$$

$$\exp\left[-\frac{J^2}{2N} \text{str } A^2\right]$$
$$= \int_{\Omega} dQ \exp\left[-\frac{N}{2J^2} \text{str } Q^2 \pm i \text{str } (QA)\right].$$

PROOF OF H-S FORMULA 1

Proof:

$$Q = \begin{pmatrix} x_1 & \rho_1 \\ \rho_2 & ix_2 \end{pmatrix}, \quad x_1, x_2 \in \mathfrak{R}_{\text{ev}}, \quad \rho_1, \rho_2 \in \mathfrak{R}_{\text{od}},$$

$$\forall A = \begin{pmatrix} a & \theta_1 \\ \theta_2 & b \end{pmatrix}, \quad a, b \in \mathfrak{R}_{\text{ev}}, \quad \theta_1, \theta_2 \in \mathfrak{R}_{\text{od}}$$

$$\int_{\Omega} dQ \exp \left[-\frac{1}{2} \text{str} (\gamma Q \pm i\gamma^{-1}A)^2 \right] = 1 \quad (\forall \gamma > 0).$$

PROOF OF H-S FORMULA 2

$$\begin{aligned} & \text{str} (\gamma Q \pm i\gamma^{-1}A)^2 \\ &= \gamma^2(x_1^2 + x_2^2 + 2\rho_1\rho_2) \\ & \quad \pm 2i(x_1a + \rho_1\theta_2 - \rho_2\theta_1 - ix_2b) \\ & \quad - \gamma^{-2}(a^2 - b^2 + 2\theta_1\theta_2), \end{aligned}$$

$$\begin{aligned} & \int d\rho_1 d\rho_2 \exp[-\gamma^2\rho_1\rho_2 \mp i(\rho_1\theta_2 - \rho_2\theta_1) + \gamma^{-2}\theta_1\theta_2] \\ &= (\gamma^2 - \theta_1\theta_2)(1 + \gamma^{-2}\theta_1\theta_2) = \gamma^2, \end{aligned}$$

PROOF OF H-S FORMULA 3

$$\begin{aligned} & -\frac{\gamma^2}{2}(x_1^2 + x_2^2) \mp i(x_1 a - ix_2 b) + \frac{\gamma^{-2}}{2}(a^2 - b^2) \\ & = -\frac{1}{2}(\gamma x_1 \pm i\gamma^{-1}a)^2 - \frac{1}{2}(\gamma x_2 \pm \gamma^{-1}b)^2 \end{aligned}$$

$$\begin{aligned} & \int \frac{dx_1 dx_2}{2\pi} \exp \left[-\frac{\gamma^2}{2}(x_1^2 + x_2^2) \right. \\ & \quad \left. \times \mp i(x_1 a - ix_2 b) + \frac{\gamma^{-2}}{2}(a^2 - b^2) \right] = \gamma^{-2}. \end{aligned}$$

THE SPECTRUM EDGE PROBLEM:

THEOREM

Let $z \in [-1, 1]$. When $N \rightarrow \infty$, we have

$$\begin{aligned}\langle \rho_N(2J - zN^{-2/3}) \rangle_N &= N^{-1/3} f(z/J) + O(N^{-2/3}), \\ \langle \rho_N(-2J + zN^{-2/3}) \rangle_N &= -N^{-1/3} f(z/J) + O(N^{-2/3}),\end{aligned}$$

where

$$\begin{aligned}f(w) &= \frac{1}{4\pi^2 J} (\text{Ai}'(w)^2 - \text{Ai}''(w) \text{Ai}(w)), \\ \text{Ai}(w) &= \int_{\mathbb{R}} dx \exp\left[-\frac{i}{3}x^3 + iw x\right].\end{aligned}$$

HARMONIC OSCILLATOR:

$$i\hbar \frac{\partial u}{\partial t} = H(q, D_q)u, \quad u(0, q) = \underline{u}(q), \quad D_q = \frac{\hbar}{i} \frac{\partial}{\partial q}.$$

$$H(q, D_q) = \frac{1}{2m} D_q^2 - \frac{m\omega^2}{2} q^2, \quad H(q, p) = \frac{1}{2m} p^2 - \frac{m\omega^2}{2} q^2.$$

$$(CM) \quad \dot{q} = \frac{p}{m}, \quad \dot{p} = m\omega^2 q, \quad (q(0), p(0)) = (\underline{q}, \underline{p}),$$

$$\begin{aligned} S_0(t, \underline{q}, \underline{p}) &= \int_0^t ds [\dot{q}(s)p(s) - H(q(s), p(s))] \\ &= \frac{\mathcal{C}_t \mathcal{S}_t}{\omega} H(\underline{q}, \underline{p}) + \mathcal{S}_t^2 \underline{q} \underline{p}. \end{aligned}$$

$$\text{notation: } \cosh \omega s = \mathcal{C}_s, \quad \sinh \omega s = \mathcal{S}_s$$

HARMONIC OSCILLATOR(SL1):

LPIM=Lagrangian path-integral method for Schrödinger type

$$\underline{p} = \frac{m\omega}{\mathcal{F}_t}(\bar{q} - \mathcal{C}_t \underline{q}) = y(t, \bar{q}, \underline{q})$$

$$\begin{aligned} S(t, \bar{q}, \underline{q}) &= S_0(t, \underline{q}, \underline{p}) \Big|_{\underline{p}=y(t, \bar{q}, \underline{q})} \\ &= \frac{m\omega \mathcal{C}_t}{2\mathcal{F}_t} \left(\frac{m}{2} \bar{q}^2 + \frac{m\omega^2}{2} \underline{q}^2 \right) - \frac{m\omega}{\mathcal{F}_t} \bar{q} \underline{q} \end{aligned}$$

$$D(t, \bar{q}, \underline{q}) = - \frac{\partial^2 S(t, \bar{q}, \underline{q})}{\partial \bar{q} \partial \underline{q}} = \frac{m\omega}{\mathcal{F}_t}.$$

HARMONIC OSCILLATOR(SL2):

$$S_t + H(\bar{q}, S_{\bar{q}}) = 0, \quad \lim_{t \rightarrow 0} tS(t, \bar{q}, \underline{q}) = \frac{m(\bar{q} - \underline{q})^2}{2},$$

$$D_t + \partial_{\bar{q}}(D\tilde{H}_p) = 0, \quad \lim_{t \rightarrow 0} tD(t, \bar{q}, \underline{q}) = m, \quad \tilde{H}_p = H_p(\bar{q}, S_{\bar{q}}).$$

$$\begin{aligned} U_t^L \underline{u}(\bar{q}) &= \frac{1}{\sqrt{2\pi i\hbar}} \int_{\mathbb{R}} d\underline{q} D^{1/2}(t, \bar{q}, \underline{q}) e^{i\hbar^{-1}S(t, \bar{q}, \underline{q})} \underline{u}(\underline{q}) \\ &= \sqrt{\frac{m\omega}{2\pi i\hbar} \mathcal{S}_t} \int_{\mathbb{R}} d\underline{q} e^{i\hbar^{-1}S(t, \bar{q}, \underline{q})} \underline{u}(\underline{q}). \end{aligned}$$

HARMONIC OSCILLATOR(SH):

HPIM=Hamiltonian path-integral method

$$\underline{q} = \frac{1}{\mathcal{L}_t} (\bar{q} - \frac{\mathcal{S}_t}{m\omega} \underline{p}) = \eta(t, \bar{q}, \underline{p}),$$

$$\begin{aligned} S(t, \bar{q}, \underline{p}) &= [\underline{q}\underline{p} + S_0(t, \underline{q}, \underline{p})] \Big|_{\underline{q}=\eta(t, \bar{q}, \underline{p})} \\ &= -\frac{\mathcal{S}_t}{\omega \mathcal{L}_t} \left(\frac{\underline{p}^2}{2m} - \frac{m\omega^2 \bar{q}^2}{2} \right) + \frac{1}{\mathcal{L}_t} \bar{q} \underline{p}, \end{aligned}$$

$$D(t, \bar{q}, \underline{p}) = \frac{\partial^2 S(t, \bar{q}, \underline{p})}{\partial \bar{q} \partial \underline{p}} = \frac{1}{\mathcal{L}_t}.$$

HARMONIC OSCILLATOR(SH1):

$$S_t + H(\bar{q}, S_{\bar{q}}) = 0, \quad \lim_{t \rightarrow 0} S(t, \bar{q}, \underline{p}) = \bar{q}\underline{p},$$

$$D_t + \partial_{\bar{q}}(D\tilde{H}_p) = 0, \quad \lim_{t \rightarrow 0} D(t, \bar{q}, \underline{p}) = 1, \quad \tilde{H}_p = H_p(\bar{q}, S_{\bar{q}}).$$

$$\begin{aligned} U_t^H \underline{u}(\bar{q}) &= \frac{1}{\sqrt{2\pi\hbar}} \int_{\mathbb{R}} d\underline{p} D(t, \bar{q}, \underline{p})^{1/2} e^{i\hbar^{-1}S(t, \bar{q}, \underline{p})} \hat{\underline{u}}(\underline{p}) \\ &= \frac{1}{\sqrt{2\pi\hbar}} \int_{\mathbb{R}} d\underline{p} \sqrt{\frac{1}{\cosh(\omega t)}} e^{i\hbar^{-1}S(t, \bar{q}, \underline{p})} \hat{\underline{u}}(\underline{p}) \\ &= \frac{1}{2\pi\hbar} \iint_{\mathbb{R}^2} d\underline{p} d\underline{q} \sqrt{\frac{1}{\cosh(\omega t)}} e^{i\hbar^{-1}(S(t, \bar{q}, \underline{p}) - \underline{q}\underline{p})} \underline{u}(\underline{q}). \end{aligned}$$

HARMONIC OSCILLATOR(HL):

$$\frac{\partial v}{\partial t} = H(q, D_q)v, \quad v(0, q) = \underline{v}(q), \quad D_q = \frac{\hbar}{i} \frac{\partial}{\partial q}.$$

Lagrangian Path Integral Method for heat type:

$$\begin{aligned} V_t^L \underline{v}(\bar{q}) &= \frac{1}{2\pi} \int_{\mathbb{R}} d\underline{q} D^{1/2}(t, \bar{q}, \underline{q}) e^{-S(t, \bar{q}, \underline{q})} \underline{v}(\underline{q}) \\ &= \sqrt{\frac{\omega}{2\pi \sinh(\omega t)}} \int_{\mathbb{R}} d\underline{q} e^{-S(t, \bar{q}, \underline{q})} \underline{v}(\underline{q}). \end{aligned}$$

HARMONIC OSCILLATOR(HH):

$$V_t^H \underline{v}(\bar{q}) \sim \frac{1}{\sqrt{2\pi\hbar}} \int_{\mathbb{R}} d\underline{q} D^{1/2}(t, \bar{q}, \underline{p}) e^{i\hbar^{-1}S(t, \bar{q}, \underline{p})} \hat{\underline{v}}(\underline{p}) \Big|_{\hbar=-i}$$

But, we don't have Hamiltonian Path Integral Representation for heat equation directly.

$$\tilde{V}_t^H \underline{v}(\bar{q}) = \frac{1}{\sqrt{2\pi\hbar}} \int_{\mathbb{R}} d\underline{q} D^{1/2}(t, \bar{q}, \underline{p}) e^{-S(t, \bar{q}, \underline{p})} \hat{\underline{v}}(\underline{p}).$$

SUPER-EXTENSION:

Spin addition:

$$\mathbb{Q}_- = \frac{1}{\sqrt{2m}} \left(\frac{\hbar}{i} \partial_q - m\omega q \right) \mathbf{b},$$

$$\mathbb{Q}_+ = \frac{1}{\sqrt{2m}} \left(\frac{\hbar}{i} \partial_q + m\omega q \right) \mathbf{b}^*,$$

$$\mathbf{b}^2 = \mathbf{b}^{*2} = 0, \quad \{\mathbf{b}^*, \mathbf{b}\} = 1, \quad \mathbb{Q} = \mathbb{Q}_+ + \mathbb{Q}_-,$$

$$\mathbb{H} = \mathbb{Q}^2 = \mathbb{Q}_- \mathbb{Q}_+ + \mathbb{Q}_+ \mathbb{Q}_- = H(q, D_q) \mathbb{I} + \frac{\hbar}{2i} \omega [\mathbf{b}, \mathbf{b}^*]$$

$$\mathcal{H}(x, \xi, \theta, \pi) = \frac{1}{2m} \xi^2 - \frac{m\omega^2}{2} x^2 + \hbar\omega\theta\pi.$$

[Q]: HH-FORMULATION

$$\begin{aligned} & \mathcal{S}(t, \bar{x}, \bar{\theta}, \underline{\xi}, \underline{\pi}) \\ &= [\underline{x}\underline{\xi} + \underline{\theta}\underline{\pi} + \mathcal{S}_0(t, \underline{x}, \underline{\xi}, \underline{\theta}, \underline{\pi})] \Big|_{\substack{\underline{x}=y(t, \bar{x}, \underline{\xi}, \bar{\theta}, \underline{\pi}) \\ \underline{\theta}=\omega(t, \bar{x}, \underline{\xi}, \bar{\theta}, \underline{\pi})}} \\ &= e^{-\omega t} \bar{\theta} \underline{\pi} + \mathcal{S}(t, \bar{x}, \underline{\xi}), \\ & \mathcal{S}_t + \mathcal{H}(\bar{x}, \mathcal{S}_{\bar{x}}, \bar{\theta}, \mathcal{S}_{\bar{\theta}}) = 0, \quad \lim_{t \rightarrow 0} \mathcal{S} = \bar{x} \underline{\xi} + \bar{\theta} \underline{\pi}. \end{aligned}$$

[Q]: LH-FORMULATION

$$\begin{aligned} \mathcal{S}(t, \bar{x}, \bar{\theta}, \underline{x}, \underline{\pi}) &= [\underline{\theta}\underline{\pi} + \mathcal{S}_0(t, \underline{x}, \underline{\xi}, \underline{\theta}, \underline{\pi})] \Big|_{\substack{\underline{\xi}=\eta(t, \bar{x}, \underline{x}, \bar{\theta}, \underline{\pi}) \\ \underline{\theta}=\omega(t, \bar{x}, \underline{x}, \bar{\theta}, \underline{\pi})}} \\ &= e^{-\omega t} \bar{\theta} \underline{\pi} + \mathcal{S}(t, \bar{x}, \underline{x}), \end{aligned}$$

$$\mathcal{S}_t + \mathcal{H}(\bar{x}, \mathcal{S}_{\bar{x}}, \bar{\theta}, \mathcal{S}_{\bar{\theta}}) = 0, \quad \lim_{t \rightarrow 0} \left(\mathcal{S} - \frac{(\bar{x} - \underline{x})^2}{2t} - \bar{\theta} \underline{\pi} \right) = 0.$$

[H]: LH-FORMULATION

$$\begin{aligned} \mathcal{S}(t, \bar{x}, \bar{\theta}, \underline{x}, \underline{\pi}) &= [\underline{\theta} \underline{\pi} + S_0(t, \underline{x}, \underline{\xi})] \Big|_{\substack{\underline{\xi} = \eta(t, \bar{x}, \bar{\theta}, \underline{x}, \underline{\pi}) \\ \underline{\theta} = \omega(t, \bar{x}, \bar{\theta}, \underline{x}, \underline{\pi})}} \\ &= e^{-\omega t} \bar{\theta} \underline{\pi} + S(t, \bar{x}, \underline{x}), \end{aligned}$$

$$\mathcal{D}(t, \bar{x}, \bar{\theta}, \underline{\xi}, \underline{\pi}) = \begin{pmatrix} -\frac{\partial^2 \mathcal{S}}{\partial \bar{x} \partial \underline{x}} & 0 \\ 0 & -\frac{\partial^2 \mathcal{S}}{\partial \theta \partial \underline{\pi}} \end{pmatrix} = \frac{\omega}{\sinh(\omega t)} e^{\omega t}$$

[H]: LH-FORMULATION CONTINUED

$$\begin{aligned}\mathcal{V}_t^{LH} \underline{v}(\bar{q}, \bar{\theta}) &= \sqrt{\frac{1}{2\pi}} \int d\underline{x} d\underline{\pi} \mathcal{D}^{1/2} e^{-S}(\underline{v}_1(\underline{x}) - \underline{v}_0(\underline{x})\underline{\pi}) \\ &= \sqrt{\frac{\omega}{2\pi \sinh(\omega t)}} \int d\underline{x} e^{-S(t, \bar{x}, \underline{x})} \\ &\quad \times (e^{\omega t/2} \underline{v}_0(\underline{x}) - e^{-\omega t/2} \underline{v}_1(\underline{x}) \bar{\theta}).\end{aligned}$$

[H]: WITTEN-INDEX

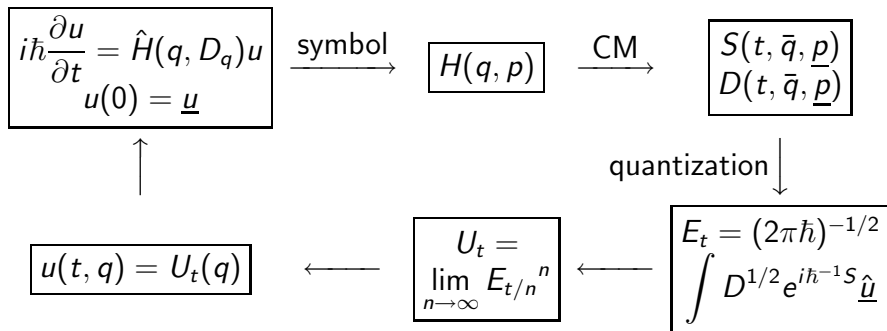
$$\begin{aligned} & [\text{Ker}(Pe^{-t\hat{H}})](\bar{x}, \underline{x}, \bar{\theta}, \underline{\theta}) \\ &= (1 - 2\bar{\theta}\partial_{\bar{\theta}})(e^{\omega t/2}\underline{\theta} + e^{-\omega t/2}\bar{\theta})\sqrt{\frac{\omega}{2\pi\sinh(\omega t)}}e^{-S(t, \bar{x}, \underline{x})}, \end{aligned}$$

$$\text{str } e^{-t\hat{H}} = \int d\underline{x}d\underline{\theta} [\text{Ker}(Pe^{-tH})](\underline{x}, \underline{x}, \underline{\theta}, \underline{\theta})$$

$$= (e^{\omega t/2} - e^{-\omega t/2})\sqrt{\frac{\omega}{2\pi\sinh(\omega t)}}\sqrt{\frac{\pi\sinh(\omega t)}{\omega(\cosh(\omega t) - 1)}}$$

$$= \frac{e^{\omega t/2} - e^{-\omega t/2}}{\sqrt{2(\cosh(\omega t) - 1)}} = 1.$$

BOSONIC RESUME:



COMPLEX TO REAL:

$$\boxed{\begin{aligned} i\hbar \frac{\partial u}{\partial t} &= \hat{H}(q, D_q)u \\ u(0) &= \underline{u} \end{aligned}}$$

b, &C ↑

$$\begin{array}{c} \times (-i) \\ \hline -i\sigma_2 \end{array}$$

$$\boxed{\hbar \frac{\partial}{\partial t} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \hat{H} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}}$$

↓ × i

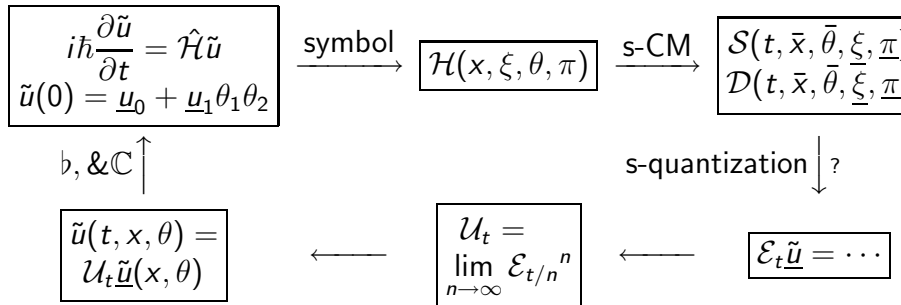
$$\boxed{\begin{aligned} i\hbar \frac{\partial \tilde{u}}{\partial t} &= \mathcal{H}(x, D_x, \theta, \partial_\theta) \tilde{u} \\ \tilde{u}(0) &= \underline{u}_0 + \underline{u}_1 \theta_1 \theta_2 \end{aligned}}$$

$$\begin{array}{c} \longleftarrow \\ \# \end{array}$$

$$\boxed{i\hbar \frac{\partial}{\partial t} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = i\hat{H} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}}$$

$$u = u_0 + iu_1 \longrightarrow \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \longrightarrow \tilde{u} = u_0 + u_1 \theta_1 \theta_2$$

NEW APPROACH:



$$\mathcal{E}_t \underline{u}(t, x, \theta) = (2\pi\hbar)^{-1/2} \int d\xi d\pi \mathcal{D}^{1/2}(\dots) e^{i\hbar^{-1} \mathcal{S}(\dots)} \underline{\tilde{u}}(\dots)$$

$$(\dots) = (t, x, \xi, \theta, \pi) \quad \text{or} \quad (t, \xi, \pi)$$

CM FOR $\mathcal{H}(x, \xi, \theta, \pi)$:

$$\begin{cases} \dot{x} = \mathcal{H}_\xi = -iH_\xi(x, \xi)(\theta_1\theta_2 - \hbar^{-2}\pi_1\pi_2), \\ \dot{\xi} = -\mathcal{H}_x = iH_x(x, \xi)(\theta_1\theta_2 - \hbar^{-2}\pi_1\pi_2), \end{cases}$$

$$\begin{cases} \dot{\theta}_1 = -\mathcal{H}_{\pi_1} = -i\hbar^{-2}H(x, \xi)\pi_2, \\ \dot{\theta}_2 = -\mathcal{H}_{\pi_2} = i\hbar^{-2}H(x, \xi)\pi_1, \\ \dot{\pi}_1 = -\mathcal{H}_{\theta_1} = iH(x, \xi)\theta_2, \\ \dot{\pi}_2 = -\mathcal{H}_{\theta_2} = -iH(x, \xi)\theta_1 \end{cases}$$

CM FOR $\mathcal{H}(x, \xi, \theta, \pi)$ -1:

$$x(t) = \sum_{j=0}^{\infty} x^{[2j]}(t), \quad x^{[2j]}(t) = \sum_{|l|=2j} x_l(t) \sigma^l, \text{ etc.}$$

$$\theta(t) = \sum_{j=0}^{\infty} \theta^{[2j+1]}(t), \quad \theta^{[2j+1]}(t) = \sum_{|l|=2j+1} \theta_l(t) \sigma^l, \text{ etc.}$$

$$\dot{x}^{[0]} = 0, \quad \dot{\xi}^{[0]} = 0 \implies x^{[0]}(t) = \underline{x}^{[0]}, \quad \xi^{[0]}(t) = \underline{\xi}^{[0]}$$

CM FOR $\mathcal{H}(\underline{x}, \underline{\xi}, \theta, \pi)$ -2:

$$\begin{cases} \dot{\theta}_1^{[1]} = -i\mathbf{k}^{-2}H(\underline{x}^{[0]}, \underline{\xi}^{[0]})\pi_2^{[1]}, \\ \dot{\theta}_2^{[1]} = i\mathbf{k}^{-2}H(\underline{x}^{[0]}, \underline{\xi}^{[0]})\pi_1^{[1]}, \\ \dot{\pi}_1^{[1]} = iH(\underline{x}^{[0]}, \underline{\xi}^{[0]})\theta_2^{[1]}, \\ \dot{\pi}_2^{[1]} = -iH(\underline{x}^{[0]}, \underline{\xi}^{[0]})\theta_1^{[1]} \end{cases}$$
$$\begin{cases} \dot{x}^{[2]} = -iH_{\xi}^{[0]}(\theta_1\theta_2 - \mathbf{k}^{-2}\pi_1\pi_2)^{[2]}, \\ \dot{\xi}^{[2]} = iH_x^{[0]}(\theta_1\theta_2 - \mathbf{k}^{-2}\pi_1\pi_2)^{[2]}. \end{cases}$$

CM FOR $\mathcal{H}(x, \xi, \theta, \pi)$ -2EV:

$$\left\{ \begin{array}{l} \dot{x}^{[2j]} = -i \sum_{\ell=0}^j H_{\xi}(x, \xi)^{[2\ell]} (\theta_1 \theta_2 - \hbar^{-2} \pi_1 \pi_2)^{[2j-2\ell]}, \\ \dot{\xi}^{[2j]} = i \sum_{\ell=0}^j H_x(x, \xi)^{[2\ell]} (\theta_1 \theta_2 - \hbar^{-2} \pi_1 \pi_2)^{[2j-2\ell]}. \end{array} \right.$$

CM FOR $\mathcal{H}(x, \xi, \theta, \pi)$ -2OD:

$$\left\{ \begin{array}{l} \dot{\theta}_1^{[2j+1]} = -ik^{-2} \sum_{\ell=0}^j H(x, \xi)^{[2\ell]} \pi_2^{[2j+1-2\ell]}, \\ \dot{\theta}_2^{[2j+1]} = ik^{-2} \sum_{\ell=0}^j H(x, \xi)^{[2\ell]} \pi_1^{[2j+1-2\ell]}, \\ \dot{\pi}_1^{[2j+1]} = i \sum_{\ell=0}^j H(x, \xi)^{[2\ell]} \theta_2^{[2j+1-2\ell]}, \\ \dot{\pi}_2^{[2j+1]} = -i \sum_{\ell=0}^j H(x, \xi)^{[2\ell]} \theta_1^{[2j+1-2\ell]} \end{array} \right.$$

CM FOR $\mathcal{H}(x, \xi, \theta, \pi)$ -3:

$$\begin{cases} \frac{d}{dt}H(x(t), \xi(t)) = 0, \\ \frac{d}{dt}\mathcal{M}(t) = 0, \end{cases}$$

$$H(x, \xi) = \frac{1}{2}|\xi|^2 + V(x), \quad H_x(x, \xi) = V_x(x), \quad H_\xi(x, \xi) = \xi,$$

$$\mathcal{M}(t) = \mathcal{M}(0) = \underline{\theta}_1 \underline{\theta}_2 - \kappa^{-2} \underline{\pi}_1 \underline{\pi}_2 = \underline{\mathcal{M}},$$

$$H(x(t), \xi(t)) = H(\underline{x}, \underline{\xi}) = \frac{\xi^2}{2} + V(\underline{x}) = \underline{H}$$

CM FOR $\mathcal{H}(x, \xi, \theta, \pi)$ -4:

$$x(t) = \underline{x} - i\underline{\mathcal{M}} \int_0^t ds H_\xi(x(s), \xi(s)) = \underline{x} - i\underline{\mathcal{M}}\phi(t),$$

$$\xi(t) = \underline{\xi} + i\underline{\mathcal{M}} \int_0^t ds H_x(x(s), \xi(s)).$$

$$\phi(t) = \int_0^t ds \xi(s), \quad H_\xi(x, \xi) = \xi, \quad H_x(x, \xi) = V_x.$$

CM FOR $\mathcal{H}(x, \xi, \theta, \pi)$ -5:

$$\begin{aligned}V_x(x(t)) &= V_x(\underline{x} - i\underline{\mathcal{M}}\phi(t)) \\ &= V_x(\underline{x}) - iV_{xx}(\underline{x})\underline{\mathcal{M}}\phi(t) - \frac{1}{2!}V_{xxx}(\underline{x})\underline{\mathcal{M}}^2\phi^2(t) + \dots\end{aligned}$$

CM FOR $\mathcal{H}(\underline{x}, \underline{\xi}, \theta, \pi)$ -51:

$$\dot{\phi}(s) = \xi(t) = \underline{\xi} - i \int_0^t ds (V_x(\underline{x}) - i V_{xx}(\underline{x}) \phi(s))$$

with $\phi(0) = \dot{\phi}(0) = 0$.

$$\phi(t) = \underline{\xi} t + i \underline{\mathcal{M}} V_x(\underline{x}) \frac{t^2}{2} + \underline{\mathcal{M}}^2 V_{xx}(\underline{x}) \frac{t^3}{3!},$$

$$\xi(t) = \underline{\xi} + i \underline{\mathcal{M}} V_x(\underline{x}) t + \underline{\mathcal{M}}^2 \underline{\xi} V_{xx}(\underline{x}) \frac{t^2}{2},$$

$$x(t) = \underline{x} - i \underline{\mathcal{M}} \underline{\xi} t + \underline{\mathcal{M}}^2 V_x(\underline{x}) \frac{t^2}{2}.$$

CM FOR $\mathcal{H}(x, \xi, \theta, \pi)$ -6:

$$\begin{cases} \mathcal{S}_t + \mathcal{H}(x, \mathcal{S}_x, \theta, \mathcal{S}_\theta) = 0, \\ \mathcal{S}(0, x, \xi, \theta, \pi) = \langle x | \xi \rangle + \langle \theta | \pi \rangle. \end{cases}$$

$$\mathcal{S}(t, x, \xi, \theta, \pi)$$

$$= \begin{cases} \langle x | \xi \rangle + i\hbar A(\theta_1\theta_2 - \hbar^{-2}\pi_1\pi_2) \\ \quad + B\langle \theta | \pi \rangle + \Phi\theta_1\theta_2\pi_1\pi_2 & \text{if } (t, x, \xi) \notin \mathfrak{B}, \\ \text{undefined} & \text{if } (t, x, \xi) \in \mathfrak{B}, \end{cases}$$

$$A = \frac{\sin(t\hbar^{-1}H)}{\cos(t\hbar^{-1}H)}, \quad B = \frac{1}{\cos(t\hbar^{-1}H)}, \quad \Phi = \frac{t^2\hbar^{-2}H_\xi H_x}{\cos^2(t\hbar^{-1}H)}$$

$$\mathfrak{B} = \{(t, x, \xi) \mid \exists n \in \mathbb{Z} \text{ s.t. } t\hbar^{-1}H(x, \xi) = (2n \pm \frac{1}{2})\pi\}$$

CM FOR $\mathcal{H}(x, \xi, \theta, \pi)$ -7:

$$\begin{cases} \mathcal{D} + \frac{\partial}{\partial x}(\mathcal{D}\tilde{\mathcal{H}}_\xi) + \frac{\partial}{\partial \theta}(\mathcal{D}\tilde{\mathcal{H}}_\pi) = 0, \\ \mathcal{D}(0, x, \xi, \theta, \pi) = 1. \end{cases}$$

$$\mathcal{D} = \begin{cases} \cos^2(t\hbar^{-1}H) + t^2\hbar^{-2}H_x H_\xi \cos(t\hbar^{-1}H) \langle \theta | \pi \rangle \\ \quad + t^2\hbar^{-2}H_{xx} H_{\xi\xi} \theta_1 \theta_2 \pi_1 \pi_2 \quad \text{if } (t, x, \xi) \notin \mathfrak{B}, \\ \text{undefined} \quad \text{if } (t, x, \xi) \in \mathfrak{B}. \end{cases}$$

HALF-DENSITY:

$$\mathcal{D} = \mathcal{A}^2,$$

$$\mathcal{A} = \begin{cases} \cos(t\hbar^{-1}H) + \frac{t^2\hbar^{-2}H_x H_\xi}{2} \langle \theta | \pi \rangle \\ + \left(\frac{t^2\hbar^{-2}H_{xx} H_{\xi\xi}}{2 \cos(t\hbar^{-1}H)} + \frac{t^4\hbar^{-4}H_x^2 H_\xi^2}{4 \cos(t\hbar^{-1}H)} \right) \theta_1 \theta_2 \pi_1 \pi_2, \\ \text{undefined} \end{cases}$$

for $(t, x, \xi) \notin \mathfrak{V}$ or $(t, x, \xi) \in \mathfrak{V}$, respectively.

PIM FOR $\mathcal{H}(x, \xi, \theta, \pi)$:

$$\mathcal{T}_t u(x, \theta) = (2\pi\hbar)^{-1/2} \hbar \iint_{\mathfrak{R}^{1|2}} d\xi d\pi \mathcal{A} e^{i\hbar^{-1}\mathcal{S}} \mathcal{F}u(\xi, \pi),$$

$$\begin{aligned} \mathcal{F}u(\xi, \pi) &= (2\pi\hbar)^{-1/2} \hbar \iint_{\mathfrak{R}^{1|2}} dx d\theta e^{i\hbar^{-1}(\langle x|\xi\rangle + \langle \theta|\pi\rangle)} \\ &\quad \times (u_0(x) + u_1(x)\theta_1\theta_2). \end{aligned}$$

PIM FOR $\mathcal{H}(x, \xi, \theta, \pi)1$:

Feynman's quantization:

$$\mathcal{I}_t u(x, \theta) \Big|_{t=0} = \mathcal{H}^F(x, -i\hbar\partial_x, \theta, \partial_\theta) u(x, \theta)$$

with

$$\mathcal{H}^F(x, -i\hbar\partial_x, \theta, \partial_\theta) u = \mathcal{H}^W(x, -i\hbar\partial_x, \theta, \partial_\theta) u$$

PIM FOR $\mathcal{H}(x, \xi, \theta, \pi)2$:

$$\begin{aligned} \mathcal{T}_t u(x, \theta) &\sim (2\pi\hbar)^{-1} \int_{1|0} d\xi e^{i\hbar^{-1}\langle x|\xi\rangle} (\mathcal{C} - i\mathcal{S})(\hat{u}_0(\xi) + i\hat{u}_1(\xi)) \\ &\quad + Ru_1(x) \\ &= (2\pi\hbar)^{-1} \int_{\mathbb{R}} dp e^{i\hbar^{-1}\langle q|p\rangle} \\ &\quad \times \left[\begin{pmatrix} \cos(\hbar^{-1}tH) & \sin(\hbar^{-1}tH) \\ -\sin(\hbar^{-1}tH) & \cos(\hbar^{-1}tH) \end{pmatrix} \begin{pmatrix} \hat{u}_0 \\ \hat{u}_1 \end{pmatrix} + \begin{pmatrix} 0 \\ R\hat{u}_1 \end{pmatrix} \right]. \end{aligned}$$

MAIN AND REMAINDER TERMS:

$$e^{i\hbar^{-1}\langle q|p\rangle} \begin{pmatrix} \cos(\hbar^{-1}tH) & \sin(\hbar^{-1}tH) \\ -\sin(\hbar^{-1}tH) & \cos(\hbar^{-1}tH) \end{pmatrix} \begin{pmatrix} \hat{u}_0 \\ \hat{u}_1 \end{pmatrix} \\ \sim e^{i\hbar^{-1}\langle q|p\rangle - i\hbar^{-1}tH(q,p)} \hat{u}(p)$$

$$Rw(x) = (2\pi\hbar)^{-1} \int_{\mathfrak{R}^{1|0}} d\xi e^{i\hbar^{-1}\langle x|\xi\rangle} \\ \times \left(\frac{t^2 H_{xx} H_{\xi\xi}}{2\mathcal{C}} + \frac{t^4 \hbar^{-2} H_x^2 H_\xi^2}{4\mathcal{C}} \right) \hat{w}(\xi).$$