A remark on algebraic dimension of twistor spaces

based on a joint work with Bernd Kreussler (Ireland)

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Twistor spaces of anti-self-dual manifolds

- $(M, g, +)$: oriented Riem. 4-mfd
- $\pi : Z \to M$: fiber bdle of compatible cx str's, i.e.

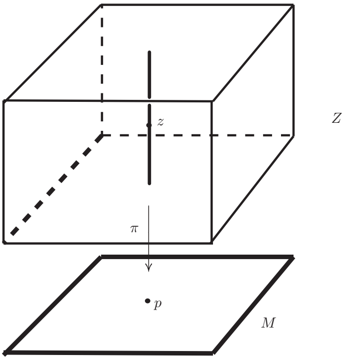
$$\pi^{-1}(p) = \left\{ J : T_pM \to T_pM \,|\, J^2 = -1,\ J \in \mathrm{O}(T_pM),\ \mathrm{ori}_J = + \right\}$$
$$\simeq S^2$$

- $\exists$ natural alm. cx str. on Z, say $\mathcal{J}$.
- $(Z, \mathcal{J})$ is conformally invariant, called the **twistor space** of $(M, [g])$.

The following is a fundamental result, due to Atiyah-Hitchin-Singer:

$\mathcal{J}$: integrable $\Longleftrightarrow W_+([g]) = 0$, i.e. g is ASD.

Basic structure of twistor spaces



- $\ell := \pi^{-1}(p) \simeq \mathbb{CP}^1$ is a cx submfd of Z, called the **twistor line**,
- $N_{\ell/Z} \simeq \mathcal{O}(1) \oplus \mathcal{O}(1) \quad (\therefore\ K_Z|_\ell \simeq \mathcal{O}(-4))$
- $J \mapsto -J$ defines an anti-hol. invol., denoted by σ, called the **real str.**

Basic structure of twistor spaces

- Reverse construction (i.e. twistor space $\Rightarrow$ ASD str.) is known.

Basic examples:

- $(S^4, g_{\text{round}}) \longleftrightarrow Z = \mathbb{CP}^3$
- $(\overline{\mathbb{CP}}^2, g_{\text{F.S.}}) \longleftrightarrow Z = \mathbb{F}$ (flag variety)

Further important properties of twistor spaces:

- K_Z always admits a natural root $K_Z^{1/2}$ as a hol. line bdle,
- $K_Z^{-1/2} =: F$ is called the **fundamental line bundle**. $F|_l \simeq \mathcal{O}(2)$
- $\sigma^* F \simeq \overline{F}$ (i.e. F is real)
- F is most fundamental for studying alg. str. of Z.

Effect of the presence of the real str. σ

- If $\pi : Z \to M$ and F are as above,

$$H^2(Z, \mathbb{R}) \simeq \pi^* H^2(M, \mathbb{R}) \oplus \langle c_1(F) \rangle.$$

- $\sigma^* = \mathrm{id}$ on $\pi^* H^2(M, \mathbb{R})$ while $\sigma^* = -\mathrm{id}$ on $\langle c_1(F) \rangle$

On the other hand,

- ω: Kähler form $\Rightarrow -\sigma^*\omega$: Kähler form $\therefore \omega - \sigma^*\omega$: Kähler form
- $\sigma^*(\omega - \sigma^*\omega) = -(\omega - \sigma^*\omega)$

Therefore, taking cohomology class,

$$[\omega - \sigma^*\omega] = t \cdot c_1(F), \quad \exists t \in \mathbb{R}$$

Since $F.\ell = 2$, we get $t > 0$. $\therefore$ $F > 0$.
So if a compact twistor space Z admits a Kähler metric, Z is Fano.

Basic theorems on cpt twistor spaces

- A cpt twistor space Z admits a Kähler metric $\Rightarrow Z = \mathbb{CP}^3$ or $\mathbb{F}$. (Hitchin '81)
- Z is a Moishezon twistor space $\Rightarrow M \simeq n\overline{\mathbb{CP}}^2$ $(0\,\overline{\mathbb{CP}}^2 := S^4)$ (Campana '91)
- $a(Z) = 2 \Rightarrow M \simeq n\overline{\mathbb{CP}}^2$ or $\widetilde{M} \simeq (S^3 \times S^1)\# n\overline{\mathbb{CP}}^2$, $\widetilde{M}$: unramified finite cover of M. (Fujiki '02)

Our interest here is twistor spaces on $n\overline{\mathbb{CP}}^2$ s.t. $a(Z) = 2$.

Twistor spaces on $n\overline{\mathbb{CP}}^2$

- If Z is a twistor spaces on $n\overline{\mathbb{CP}}^2$ (again as an effect of the real str.),

$$a(Z) = \kappa(Z, F) \quad \left(= \kappa(Z, K_Z^{-1}) = \kappa^{-1}(Z) \right)$$

- Riemann-Roch means

$$\chi(mF) = \frac{1}{6} F^3 m^3 + O(m^2).$$

- From topology, we have $F^3 = 2(4 - n)$.
- If the ASD metric is of pos. scal. curv., $H^2(mF) = 0 \ \forall m \geq 0$. (Hitchin's vanishing theorem)

As a consequence of these,

$$n < 4 \ \& \ \textit{pos. scal. curv.} \Rightarrow \kappa(Z, F) = 3 \ \therefore Z\textit{: Moishezon}$$

Fundamental linear system

- $Z \to n\overline{\mathbb{CP}}^2$, $S \in |F|$: real & irred. $\Rightarrow$ S: **smooth** rational surf. (Pedersen-Poon '94)
- By adjuntion formula,

$$K_S \simeq K_Z + S|_S \simeq -2F + F|_S \simeq -F|_S \qquad \therefore F|_S \simeq K_S^{-1}$$
$$\therefore K_S^2 = (F|_S)^2 = F^3 = 2(4-n).$$

- Riem. Roch: $\quad \chi(F) = 2(5-n)$
- If $n \geq 5$ and $h^0(F) \geq 3$, then $a(Z) = 3$ (Kreussler '99)

Twistor spaces whose $|F|$ is a pencil

So in the sequel, we consider Z on $n\overline{\mathbb{CP}}^2$ which satisfies

$$h^0(F) = 2.$$

This implies

- $n \geq 4$, and a general member of the pencil $|F|$ is non-singular,
- For any non-singular $S \in |F|$, we have $h^0(K_S^{-1}) = 1$,
- If C is the unique member of $|K_S^{-1}|$, C is either
  - a smooth elliptic curve, or
  - a cycle of smooth rational curves
- $\mathrm{Bs}\,|F| = C$

Remark. The assumption $h^0(F) = 2$ is satisfied by generic twistor spaces on $4\overline{\mathbb{CP}}^2$, but not for those on $n\overline{\mathbb{CP}}^2$ if $n > 4$.

Algebraic dimension of twistor spaces with $\dim|F| = 1$

Proposition

If $Z \to n\overline{\mathbb{CP}}^2$ satisfies $\dim|F| = 1$ and if $\kappa^{-1}(S) = j$ for **general** *member S of the pencil $|F|$, we have*

$$a(Z) = 1 + j.$$

pf) Recalling $F|_S \simeq K_S^{-1}$, the ineq. $a(Z) \le 1 + \kappa^{-1}(S)$ follows from $a(Z) = \kappa(Z, F)$ and an exa. seq. $0 \to (m-1)F \to mF \to mK_S^{-1} \to 0$, $m \in \mathbb{N}$. The reverse ineq. $a(Z) \ge 1 + j$ follows from the fiber str. $\widehat{f}$ below:

Hence in order to construct Z that satisfies $a(Z) = 2$, it is enough to find Z (with $\dim|F| = 1$) whose j is 1.

Rational surfaces satisfying $\kappa^{-1} = 1$

Suppose: S is a rational surf. with $|K_S^{-1}| = \{C\}$, with C being either

1) a smooth elliptic curve, or
2) a cycle of smooth rational curves.

Then

- In case 1),

  $$\kappa^{-1}(S) = 1 \Longleftrightarrow \tau.K_S^{-1}|_C \simeq \mathcal{O}_C, \ \exists\tau \in \mathbb{N}$$

  In particular, $K_S^2 = 0$.

- In case 2), let

  $$C = P + N$$

  be the Zariski decomp. of the cycle C, and $m_0 \in \mathbb{N}$ the smallest one s.t. m_0P & m_0N is integral. Then

  $$\kappa^{-1}(S) = 1 \Longleftrightarrow \tau.m_0P|_C \simeq \mathcal{O}_C, \ \exists\tau \in \mathbb{N}$$

Some twistor spaces with $\dim |F| = 1$

For any $n \geq 4$, it is possible to show the existence of $Z \to n\overline{\mathbb{CP}}^2$ which satisfies

$$\dim |F| = 1, \quad \text{and} \quad \kappa^{-1}(S) = 1 \text{ for } \textbf{some} \text{ smooth } \ S \in |F|.$$

These are good candidates for twistor spaces satisfying $a(Z) = 2$.

Actually, if $n = 4$, we have

Proposition

Let $Z \to 4\overline{\mathbb{CP}}^2$ and suppose $\dim |F| = 1$ and $\kappa^{-1}(S) = 1$ for some $S \in |F|$. Then $a(Z) = 2$. □

pf) Recall $F|_S \simeq K_S^{-1}$ and $|K_S^{-1}| = \{C\}$. Let $S' \in |F|$ be any smooth member. Then $F|_{S'} \simeq K_{S'}^{-1}$ and $|K_{S'}^{-1}| = \{C\}$ as well. So

$$K_{S'}^{-1}|_C \simeq F|_C \simeq K_S^{-1}|_C.$$

The last one is of finite order as $\kappa^{-1}(S) = 1$. Hence so is the first one. Therefore $\kappa^{-1}(S') = 1$. This means $j = 1$. So $a(Z) = 1 + 1 = 2$. □

Remark. More concrete result on $Z \to 4\overline{\mathbb{CP}}^2$ with $a(Z) = 2$ was obtained by Campana-Kreussler ('99).

Alg. dim. of twistor spaces with $\dim|F| = 1$ when $n > 4$

In contrast, when $n > 4$, we have

Theorem (Kreussler-H.)

Suppose $n \geq 5$, and let $Z \to n\overline{\mathbb{CP}}^2$ be a twistor space that satisfies $\dim|F| = 1$. Then $a(Z) \neq 2$. □

So even if $\kappa^{-1}(S) = 1$ for some $S \in |F|$, $\kappa^{-1}(S') = 0$ for generic $S' \in |F|$.

Corollary

Suppose $n \geq 5$ and let $Z \to n\overline{\mathbb{CP}}^2$ be a twistor space that satisfies $a(Z) = 2$. Then $h^0(F) \leq 1$. □

pf) By a result of Kreussler, $h^0(Z) \geq 3$ implies $a(Z) = 3$. □

Remark. No such a twistor space in the Cor. is known.

Outline of a proof of Theorem

As in Theorem, assume that a twistor space $Z \to n\overline{\mathbb{CP}}^2$, $n > 4$, satisfies

- $\dim |F| = 1$
- $\exists S \in |F|$: a smooth member satisfying $\kappa^{-1}(S) = 1$.

Let C be the unique member of $|K_S^{-1}|$, which is necessarily a cycle of rational curves. Recall $\mathrm{Bs}\,|F| = C$.

As before let

- $C = P + N$: the Zariski dcp. of the cycle C,
- $m_0 > 0$: the smallest integer for which $m_0 P$ & $m_0 N$ are integral.

It's easy to see, these are independent of the choice of a smooth $S \in |F|$. For simplicity of presentation, we suppose

$$m_0 P|_C \simeq \mathcal{O}_C \quad \text{i.e. } \tau = 1.$$

Outline of a proof of Theorem

For proving Theorem, recalling $a(Z) = \kappa(Z, F)$, it suffices to show

$|mF|$ is generated by the pencil $|F|$ for any $m > 0$.

But since there is an injection

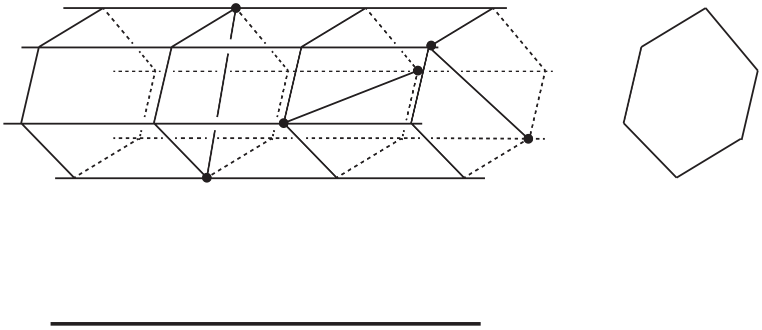
$$H^0((m-1)F) \subset H^0(mF), \quad \forall m \in \mathbb{N}$$

by the multiplication of a non-zero $s \in H^0(F)$, it suffices to show

$\left|km_0F\right|$ is generated by the pencil $|F|$ for any $k > 0$.

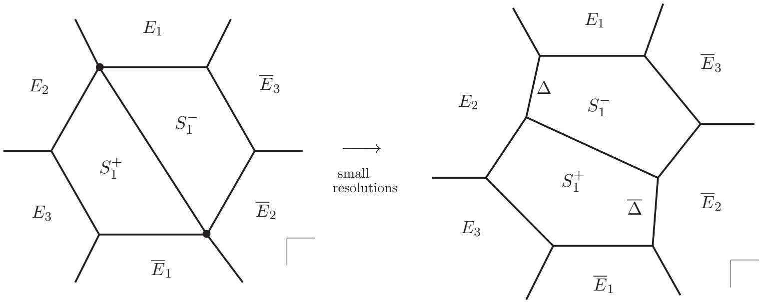
Outline of a proof of Theorem

To show this, consider the fiber space $\widehat{f} : \widehat{Z} \to \mathbb{CP}^1$ obtained before:



Outline of a proof of Theorem

We take small resolutions of all nodes like this:



- Δ, $\overline{\Delta}$: exceptional curves.
- Let $\widetilde{E}$ be the exceptional divisor in $\widetilde{Z}$.

Outline of a proof of Theorem

From the diagram, we obtain a basic relation

$$(\mu \circ \epsilon)^* F \simeq \widetilde{f}^* \mathcal{O}(1) + \widetilde{E}$$
$$\therefore (\mu \circ \epsilon)^* F^{km_0} \simeq \widetilde{f}^* \mathcal{O}(km_0) + km_0 \widetilde{E}, \quad \forall k \in \mathbb{N}$$

Introducing another parameter r, it is enough to show

The system $\left|\widetilde{f}^* \mathcal{O}(km_0) + rm_0 \widetilde{E}\right|$ always has $rm_0 \widetilde{E}$ as a fixed compo. for any $k, r \in \mathbb{N}$.

Outline of a proof of Theorem

- We define $\mathbb{Q}$-divisors $\mathbf{P}$ and $\mathbf{N}$ on $\widetilde{Z}$ by

$$m_0\widetilde{E} = m_0\mathbf{P} + m_0\mathbf{N}.$$

- Restriction to a general fiber of $\widetilde{f} : \widetilde{Z} \to \mathbb{CP}^1$ gives the integral Zariski dcp. $m_0C = m_0P + m_0N$.

- Clearly,

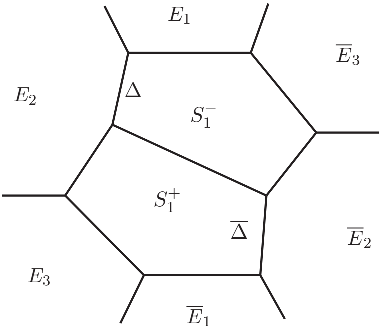
$$\widetilde{f}^*\mathcal{O}(km_0) + rm_0\widetilde{E} \simeq \widetilde{f}^*\mathcal{O}(km_0) + rm_0\mathbf{P} + rm_0\mathbf{N}.$$

- First, by a property of Zariski dcp., $rm_0\mathbf{N}$ is a fixed compo., so that

$$\left|\widetilde{f}^*\mathcal{O}(km_0) + rm_0\mathbf{P} + rm_0\mathbf{N}\right| \simeq \left|\widetilde{f}^*\mathcal{O}(km_0) + rm_0\mathbf{P}\right|$$

Outline of a proof of Theorem

In order to show that $rm_0\mathbf{P}$ is also a fixed compo., we take a closer look at any one of the reducible fibers of $\widetilde{f} : \widetilde{Z} \to \mathbb{CP}^1$.



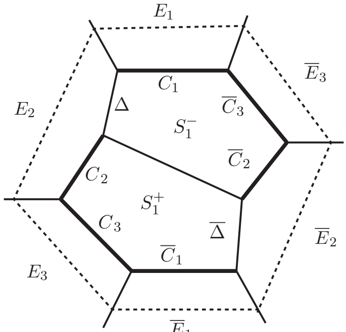
- Supposing C consists of 6 components for simplicity, we write

$$m_0\mathbf{P} = \sum_{1\le i\le 3} m_i E_i + \sum_{1\le i\le 3} m_i \overline{E}_i$$

Outline of a proof of Theorem

- Wlog, we may suppose $m_1 > m_2$.
- Then $m_0\mathbf{P}.\Delta = m_1 - m_2\ (>0)$. $\therefore m_0\mathbf{P}.C_2 = m_2 - m_1\ (<0)$ $(\because m_0\mathbf{P}|_C \in \mathrm{Pic}^0 C)$
- Hence again as $m_0\mathbf{P}|_C \in \mathrm{Pic}^0 C$,

$$(C_2 \cup C_3 \cup \overline{C}_1) \cup (\overline{C}_2 \cup \overline{C}_3 \cup C_1) \subset \mathrm{Bs}\left|\widetilde{f^*\mathcal{O}(km_0) + rm_0\mathbf{P}}\right|$$



A key lemma

The next lemma is a key to prove Theorem.

Lemma

$H^0\left(\widetilde{E}, \widetilde{f}^*\mathcal{O}(km_0) + rm_0\mathbf{P}\right) = 0$ *for any* $k, r \in \mathbb{N}$. *So* $\widetilde{E}$ *is a fixed compo. of* $\left|\widetilde{f}^*\mathcal{O}(km_0) + rm_0\mathbf{P}\right|$ *for any* $k, r \in \mathbb{N}$.

In fact, once this is proved, we have

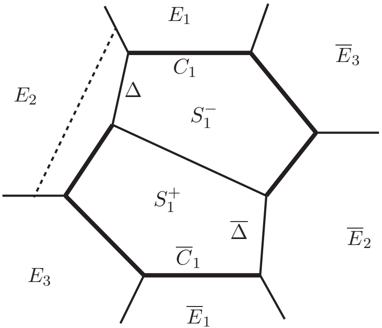
$$\begin{aligned}
\left|\widetilde{f}^*\mathcal{O}(km_0) + rm_0\mathbf{P}\right| &\simeq \left|\widetilde{f}^*\mathcal{O}(km_0) + rm_0\mathbf{P} - \widetilde{E}\right| \\
&\simeq \left|\widetilde{f}^*\mathcal{O}(km_0) + (r-1)m_0\mathbf{P}\right| \\
&\simeq \left|\widetilde{f}^*\mathcal{O}(km_0) + (r-1)m_0\mathbf{P} - \widetilde{E}\right| \\
&\simeq \left|\widetilde{f}^*\mathcal{O}(km_0) + (r-2)m_0\mathbf{P}\right| \\
&\simeq \cdots \\
&\simeq \left|\widetilde{f}^*\mathcal{O}(km_0)\right| \quad \text{as desired.}
\end{aligned}$$

A proof of the lemma

We show that any

$$s \in H^0\big(\widetilde{E}, \widetilde{f}^*\mathcal{O}(km_0) + rm_0\mathbf{P}\big)$$

vanishes along the cycle $C := (S_1^+ \cup S_1^-) \cap \widetilde{E}$ in arbitrary order.



- If $s|_\Delta \neq 0$, $s|_{E_1}$ defines a curve through the point $\Delta \cap C_1$, which intersects fibers of $\widetilde{f} : E_1 \to \mathbb{CP}^1$. Since $m_0\mathbf{P}|_C \in \mathrm{Pic}^0(C)$, this means $s|_{E_1} = 0$, which is a contradiction.

A proof of the lemma

- So $s|_\Delta = 0$ and hence $s|_C = 0$. Noting that C is a fiber of $\widetilde{f} : \widetilde{E} \to \mathbb{CP}^1$, this implies

$$\left|\widetilde{f}^*\mathcal{O}(km_0) + rm_0\mathbf{P}\,|_{\widetilde{E}}\right| \simeq \left|\widetilde{f}^*\mathcal{O}(km_0 - 1) + rm_0\mathbf{P}\,|_{\widetilde{E}}\right|$$

- We can apply the same argument to any section of the ingredient of RHS. Hence we can decrease the degree of the pull-back term as many as we want.

This finishes a proof for the proposition, and hence also for the proof of Theorem.