A VERY SIMPLE INTRODUCTION TO THE BROWNIAN MOTION AND THE ITÔ FORMULA

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The Brownian motion is very irregular movement of pollens in water, caused by collisions with molecules of water. The Wiener measure is a mathematical model of the movement and is a probability measure on the continuous path space over the Euclidean space.

It is almost clear that, for a (piecewise)-C^1 path w,

$$f(w_t) - f(w_0) = \int_0^t f'(w_s)dw_s, t > 0$$

(For simplicity, we consider one-dimensional case. Here, f is a nice function on $\mathbf{R}$.)

But, for a general continuous path w, the line integral "dw_s" on the right hand side never makes sense. Moreover, the set of all the (piecewise)-C^1 paths are negligible with respect to th Wiener measure. So, we consider the following problem:

What is $f(w_t) - f(w_0)$ for the Brownian motion w ?

The Itô formula answers this question. Although, the line integral cannot be defined for all w, it can be defined as a "stochastic integral" (which is introduced by Itô) and well-defined a.s. and the following formula holds:

$$f(w_t) - f(w_0) = \int_0^t f'(w_s)dw_s + (1/2)\int_0^t f''(w_s)ds$$

This formula enables us to do calculus along a stochstic process. The second term on the right hand side plays a very important role, and because of this term, the Brownian motion has very wide applications in many fields of analysis in which the Laplacian appears (e.g., the heat equations, the elliptic equations, etc.) A legend says that Itô got so disappointed when he found that the sencond term does not vanish. (But, I don't know if it's true.)

In the end, as an application of this formula, we will give a probabilistic proof of a famous theorem in the function theory, namely, the Casoratti-Weierstrass theorem (and Picard's little theorem if time permits).

(This talk is for non-experts and no new results will be presented. No particular knowledge is required except for the definition of a probability space.)