On the coarse geometry of Weil-Petersson's metric on Teichmüller space

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Surface complexity

- Σ is a compact, connected, and orientable surface.
- $g(\Sigma)$ the genus of Σ.
- $\#\partial\Sigma$ the number of boundary components of Σ.
- $\kappa(\Sigma) := 3g(\Sigma) + \#\partial\Sigma - 3$, the *complexity of* Σ.

Curves

- Two sets $C_0, C_1 \subseteq \Sigma$ are *freely homotopic* if there exists a continuous map $H : C_0 \times [0,1] \to \Sigma$ such that $H(C_0 \times \{i\}) = C_i$ for $i \in \{0,1\}$.
- A *curve* on Σ is the free homotopy class of an essential (i.e. does not bound a disc), non-peripheral (i.e. is not parallel to a single component of $\partial\Sigma$) simple closed loop.
- For two curves α and β, the *(geometric) intersection number* $\iota(\alpha,\beta)$ is defined equal to $\min\{|a \cap b| : a \in \alpha,\ b \in \beta\}$.
- We say a pair of curves is *disjoint* if it has zero intersection number. Any curve is disjoint from itself.

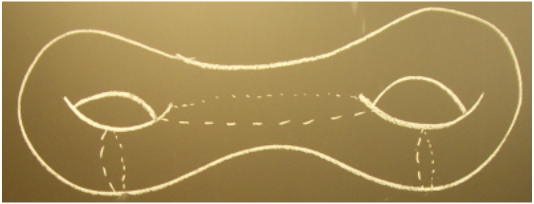
Multicurves

- A *multicurve* on Σ is a (possibly empty) set of pairwise distinct and pairwise disjoint curves.
- The intersection number of a pair of multicurves is defined additively.
- For a multicurve ω, its *corank* is defined equal to $\kappa(\Sigma) - |\omega|$. e.g. the empty multicurve has corank $\kappa(\Sigma)$.

Pants decompositions

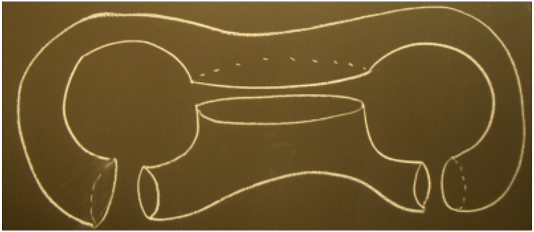
- A *pants decomposition* on Σ is a multicurve maximal subject to inclusion among all multicurves.
- The complement of a pants decomposition in Σ is the disjoint union of non-compact 3-spheres ("pants").
- Pants decompositions have $\kappa(\Sigma)$ curves and corank 0.
- Two pants decompositions are equal if and only if their intersection number is 0.
- We denote the set of all pants decompositions of Σ by $X(\Sigma)$.

A pants decomposition



[from Pat Hooper's page]

A pants decomposition's complement



[from Pat Hooper's page]

Pants graph

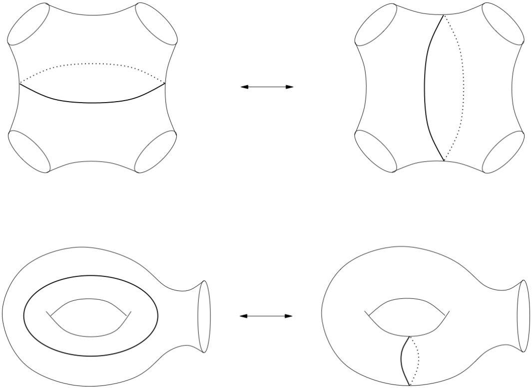
Definition (Hatcher-Thurston, 1980)

The *pants graph* $\mathcal{P}(\Sigma)$ of Σ is the graph with vertex set $X(\Sigma)$, where two vertices $x, y \in X(\Sigma)$ span an edge if and only if:

- the multicurve $x \cap y$ has corank 1, and
- the two curves $x \setminus y$ and $y \setminus x$ intersect once (filling a 1-holed torus) or intersect twice and fill a 4-holed sphere.

We say x and y are related by an *elementary move*, or a *flip*.

Examples of elementary moves

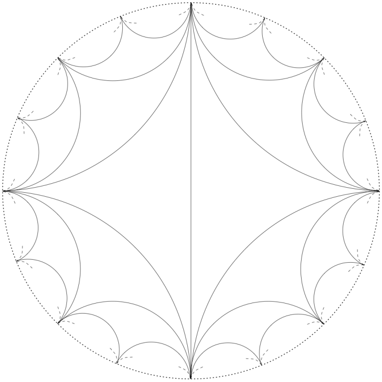


Pants graph

Hereon, we shall assume $\kappa(\Sigma) \geq 1$.

- (Aramayona) $\mathcal{P}(\Sigma) \cong \mathcal{P}(\Sigma')$ if and only if $\Sigma = \Sigma'$ or $\Sigma, \Sigma' \in \{1\mathrm{HT}, 4\mathrm{HS}\}$. e.g. $\mathcal{P}(2\mathrm{HT}) \ncong \mathcal{P}(5\mathrm{HS})$.
- (Series,...) $\mathcal{P}(1\mathrm{HT}) \cong \mathcal{P}(4\mathrm{HS}) \cong$ Farey graph, the only (non-empty) planar pants graph.

A Farey graph



Metric on the pants graph

[H-T] *The pants graph is path-connected for all surfaces.*

We introduce to $\mathcal{P} = \mathcal{P}(\Sigma)$ the canonical path metric

$$d = d_{\Sigma}$$

assigning length 1 to each edge. *Geodesics* are those paths of minimal length.

Metric on the pants graph

- The metric pants graph $(\mathcal{P}(\Sigma), d_\Sigma)$ is unbounded.
- $\mathcal{P}(\Sigma)$ is everywhere locally infinite, i.e. all balls of radius 1 are infinite. [This makes analysis challenging.]
- $\mathcal{P}(\mathrm{2HT}) \simeq_{\mathrm{qi}} \mathcal{P}(\mathrm{5HS})$.

Hereon, we shall assume $\kappa(\Sigma) \geq 2$.

Curvature

[Masur-Minsky, Brock-Farb] $\mathcal{P}(\Sigma)$ *is not hyperbolic in the sense of Gromov if* $\kappa(\Sigma) \geq 3$.

Conversely,

[Brock-Farb, Brock] $\mathcal{P}(\Sigma)$ *is hyperbolic in the sense of Gromov if* $\kappa(\Sigma) = 2$.

Motivation

Our motivation for studying the metric pants graph, and a whole source of open questions, is the following important theorem.

Theorem (Brock)

For any surface Σ*, the pants graph* $(\mathcal{P}(\Sigma), d_\Sigma)$ *is quasi-isometric to Weil-Petersson's metric on Teichmüller space (and its completion). Moreover, a family of such quasi-isometries arises by mapping a point in Teichmüller space to any one of its Bers-short pants decompositions.*

Metaquestion

The geometry of the pants graph coarsely models the geometry of the completion of Weil-Petersson's metric.

How strong is the analogy between the two geometries? Any geometric property of the completion of the Weil-Petersson metric has a coarse analogue, but does it also have a geometric analogue?

WP properties

- (Masur-Wolf-Farb, Wolpert) $\widehat{d}_{WP}$ is CAT(0), in particular uniquely geodesic.
- (Masur, Daskolopoulos-Wentworth, Wolpert) the frontier of $\widehat{d}_{WP}$ is the union of convex faces ("strata"), each a lower-dimensional Teichmüller space with Weil-Petersson's metric.

How could we interpret this in $\mathcal{P}$?

Natural subgraphs

Definition

For ω a multicurve on Σ, define $\mathcal{P}_\omega = \mathcal{P}_\omega(\Sigma)$ to be the subgraph of $\mathcal{P}(\Sigma)$ spanned by all pants decompositions containing ω. We refer to the subgraphs of this type as the natural subgraphs.

- $\mathcal{P}_\omega(\Sigma) \cong \mathcal{P}(\Sigma - \omega)$, e.g.:
- $\mathcal{P}_\emptyset \cong \mathcal{P}(\Sigma)$, and
- if $x \in X(\Sigma)$ then $\mathcal{P}_x = \{x\}$.

Characterization of Farey subgraphs

If ω is a corank 1 multicurve, then $\Sigma - \omega$ is the union of non-compact 3-holed spheres and a single non-compact 1HT or 4HS. Thus, $\mathcal{P}_\omega \cong$ Farey graph.

(Margalit) Conversely, if $\mathcal{F}$ is a Farey subgraph of $\mathcal{P}(\Sigma)$ then there exists a unique corank 1 multicurve on Σ such that $\mathcal{F} = \mathcal{P}_\omega$.

$$\begin{array}{ccc}\{\text{Farey subgraphs}\} & \overset{1-1}{\longleftrightarrow} & \{\text{corank 1 multicurves}\} \\ \mathcal{F}_\omega & & \omega\end{array}$$

Open questions

[Brock] $\exists$ explicit finite time algorithm taking $x, y \in X(\Sigma)$ and returning

$\Uparrow$

$\exists$ finite timefinite timefinite time algorithm returning all geodesicgeodesicgeodesic pat

$\Uparrow^{[S2]}$ $\Downarrow$

[APS] Is $\mathcal{P}_\omega$ *totally geodesic* for all ω? $\Longleftrightarrow^{[L]}$ Is $(\mathcal{P}(\Sigma), d_\Sigma)$ *finitely*

Extrinsic geometry

A complete affirmative answer for $\kappa(\Sigma) = 2$, where there are very few types of multicurve (empty, curve, pants decomposition), is implied by the following.

Theorem (Aramayona-Parlier-S)

Every Farey sugbraph of every pants graph is totally geodesic.

Complexity 2

Hereon Σ is the 5-holed sphere.

$\mathcal{P} = \mathcal{P}(\Sigma)$ is Gromov hyperbolic but everywhere locally infinite.

- Is $\partial\mathcal{P}$ visual? i.e. can any two points of the bordification $\overline{\mathcal{P}} := X \sqcup \partial\mathcal{P}$ be connected by a geodesic path?
- Do any (non-zero) powers of pseudo-Anosov mapping classes leave invariant a geodesic axis?

Note: $\exists$ connected path-metric graphs $Z \simeq_{\mathrm{qi}} \mathbb{R}$ such that Z admits a hyperbolic isometry and has no geodesic rays.

Subsurface projections

We require Masur-Minsky's notion of a subsurface projection.

Definition (Subsurface projection)

- For two distinct curves α, β and Y the complement of α on Σ, we define $\pi_\alpha(\beta)$ to be the set of all pants decompositions containing α and a second curve disjoint from a component of $\beta \cap Y$. We define $\pi_\alpha(\alpha) := \emptyset$.
- For a curve α and pants decomposition $x = \{\beta_1, \beta_2\}$, we define $\pi_\alpha(x) := \pi_\alpha(\beta_1) \cup \pi_\alpha(\beta_2)$.

Projections

We note some elementary properties of subsurface projections.

- π_α restricts to the identity on $\mathcal{F}_\alpha$: if $\alpha \in x$ then $\pi_\alpha(x) = \{x\}$.
- $\pi_\alpha(\beta) \subset \mathcal{F}_\alpha$ for all α, β.

- (distance non-increasing) If $\iota(\beta, \gamma) = 0$ and $\alpha \notin \{\beta, \gamma\}$ then for all $z \in \pi_\alpha(\beta)$ and for all $w \in \pi_\alpha(\gamma)$, $d(z, w) \leq 1$.
- (like nearest-point) For $x \in X, z \in \pi_\alpha(x)$ and $z' \in \mathcal{F}_\alpha$ a nearest point to x, we have $d(z, z') \leq d(x, z')$.

Projections

Lemma (S)

Let x_0, x_1, x_2, x_3 be a path in $\mathcal{P}$ and α a curve not contained in any x_i. Then, for all $z_0 \in \pi_\alpha(x_0)$ there exist $j \in \{1,2,3\}$ and $z_j \in \pi_\alpha(x_j)$ such that $d(z_0, z_j) \leq j-1$.

Proof.

We suppose $x_0 \cap x_1 \cap x_2 = \emptyset$ and $x_1 \cap x_2 \cap x_3 = \emptyset$. Let $\beta_0 \in x_0$ be any curve, and β_i be the curve from $x_i \cap x_{i-1}$ for each $i \in \{1,2,3\}$. Then, $\iota(\beta_0, \beta_3) = 2$ and $\exists!$ curve δ s.t. $\iota(\delta, \beta_0) = \iota(\delta, \beta_3) = 0$. (Say $\delta \neq \alpha$.) $\forall z_0 \in \pi_\alpha(\beta_0), w \in \pi_\alpha(\delta)$, $z_3 \in \pi_\alpha(\beta_3)$ we have $d(z_0, w) \leq 1$ and $d(w, z_3) \leq 1$. Thus, $d(z_0, z_3) \leq 2$. ☐

Projections

It follows subsurface projections contract geodesics by at least $\frac{1}{3}$.

Corollary (S)

Let $x, y \in X$ be two pants decompositions connected by a geodesic path whose every vertex does not contain the curve α. Then, for all $z \in \pi_\alpha(x)$ there exists $w \in \pi_\alpha(y)$ such that

$$3d(z, w) \leq 2d(x, y) + 4.$$

Main results

Recall Σ is the $5\mathrm{HS}$.

Theorem (S)

Suppose π *is a geodesic ray in* $\mathcal{P}(\Sigma)$ *remaining within a bounded distance of a Farey subgraph. Then,* π *is eventually contained in this Farey subgraph.*

Corollary (S)

For any Farey subgraph $\mathcal{F}$ *of* $\mathcal{P}(\Sigma)$*, the bordification* $\overline{\mathcal{F}}$ *is totally geodesic: Any geodesic connecting any two points of* $\overline{\mathcal{F}}$ *is entirely contained in* $\mathcal{F}$*.*

A local finiteness

Definition

For all $x, y \in \overline{\mathcal{P}}$, we define $\mathcal{G}(x, y)$ to be the union of all geodesic paths connecting x and y.

- $\forall x, y$, $\mathcal{G}(x, y) = \mathcal{G}(y, x)$.
- if $y \in \partial P$, is $\mathcal{G}(x, y) = \emptyset$?

A local finiteness

Theorem (S)

$\forall x, y \in \overline{\mathcal{P}}$ and $\forall B$ a finite radius ball, $|B \cap \mathcal{G}(x, y)| < \infty$.

Note: No explicit uniform bound as yet, as the proof is non-constructive. Computable?

A local finiteness

We extend our notation $\mathcal{G}$ to accept subsets of X, defining $\mathcal{G}(A,B) := \bigcup_{x\in A,\ y\in B} \mathcal{G}(x,y)$.

Theorem (S)

There exists a constant k s.t. for all B_0, B_1, B_2 three balls of radius $\leq r$ and s.t. $d(B_0, B_i) \geq 12(2r+k)+7$ for $i \in \{1,2\}$, we have $|B_0 \cap \mathcal{G}(B_1, B_2)| < \infty$.

Geodesic rays and lines

A standard diagonal subsequence argument now yields:

Theorem (S)

Any two points of $\overline{\mathcal{P}}$ *can be connected by a geodesic path.*

Pseudo-Anosov axes

An argument of Delzant's for hyperbolic groups can be adapted to give an analogue of Daskolopoulos-Wentworth's theorem.

Theorem (S)

$\forall \phi \in \mathrm{Map}(\Sigma)$ *pseudo-Anosov,* $\exists N \in \mathbb{N}_+$ *and a bi-infinite geodesic axis invariant under* ϕ^N.

Aside: Must N depend on the conjugacy class of ϕ?

Hierarchies

We close with some remarks concerning the relationship between the curve graph and the pants graph of the 5-holed sphere.

By $\mathcal{C} = \mathcal{C}(\Sigma)$ we denote *Harvey's curve graph*, the graph with vertex set all curves on Σ and a pair of distinct curves spans an if and only if they are disjoint. We give each edge length 1 to induce the canonical path-metric.

The vertex set of any link corresponds naturally to the vertex set of a Farey graph, from which we pull-back a second metric.

Hierarchies

We recall a notion of a hierarchy, or “wheel path”.

Definition

A hierarchy *in* $\mathcal{C}(\Sigma)$ *is the union of a finite geodesic path* π *and a geodesic in the link of* π^i *connecting* π^{i-1} *and* π^{i+1} *for each* i*.*

Edges in the curve graph correspond to pants decompositions, any hierarchy in $\mathcal{C}$ induces a path in $\mathcal{P}$. ([Masur-Minsky] In fact it is a uniform quasi-geodesic.)

Unions of Farey graphs

Theorem (S)

If $\mathcal{F}$ and $\mathcal{F}'$ are two distinct Farey subgraphs of $\mathcal{P}$ that intersect, then $\mathcal{F} \cup \mathcal{F}'$ is totally geodesic.

Corollary (S)

There exists a constant h such that if $(\mathcal{F}_i)_1^n$ is a sequence of distinct Farey subgraphs of $\mathcal{P}$ such that $\mathcal{F}_i \cap \mathcal{F}_{i+1} \neq \emptyset$ and where $d(\mathcal{F}_{i-1} \cap \mathcal{F}_i, \mathcal{F}_i \cap \mathcal{F}_{i+1}) \geq 20hn + 15$ for each i, then $\bigcup_1^n \mathcal{F}_i$ is totally geodesic.

Note: $\exists$ distinct Farey subgraphs $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ of $\mathcal{P}$ s.t. $\mathcal{F}_1 \cap \mathcal{F}_2 \neq \emptyset$, $\mathcal{F}_2 \cap \mathcal{F}_3 \neq \emptyset$ and $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$ is not convex.

Hierarchies and pants geodesics

It follows large-link hierarchies induce pants geodesics.

Corollary (S)

Let π *be a hierarchy in* $\mathcal{C}(\Sigma)$ *such that* $d_{\pi^i}(\pi^{i-1}, \pi^{i+1}) \geq 20h.l(\pi) + 15$ *for each* i*. Then,* π *induces a geodesic in* $(\mathcal{P}(\Sigma), d_\Sigma)$*.*

Links

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Reference:

[S] *Geodesic axes in the pants complex of the five holed sphere* : online preprint.

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